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SUPERCONVERGENCE RESULTS FOR LINEAR TRIANGULAR ELEMENTS

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The aim of the paper is to present several superconvergence phenomena which have been observed and analyzed when employing the standard linear elements to second order elliptic problems. We shall illustrate them in their simplest form solving the model problem:

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \subset R^2 , \\ u &= 0 \quad \text{on} \quad \partial\Omega , \end{aligned} \tag{1}$$

where Ω is a convex polygonal domain and u is supposed to be smooth enough.

Let $\{T_h\}$ be a regular family of triangulations of $\bar{\Omega}$, i.e., Zlámal's condition on the minimal angle of triangles is fulfilled. The discrete analogue of (1) will consist in finding $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h)_{0,\Omega} = (f, v_h)_{0,\Omega} \quad \forall v_h \in V_h , \tag{2}$$

where

$$V_h = \{v_h \in H_0^1(\Omega) \mid v_{h|T} \in P_1(T) \quad \forall T \in T_h\} .$$

It is known [15,39] that the error estimates

$$||u - u_h||_{0,p,\Omega} \leq \begin{cases} C_p h^2 ||u||_{2,p,\Omega} & \text{if} \quad p \in [2,\infty) , \\ Ch^2 |\ln h| \; ||u||_{2,\infty,\Omega} & \text{if} \quad p = \infty , \end{cases} \tag{3}$$

$$||\nabla u - \nabla u_h||_{0,p,\Omega} \leq Ch ||u||_{2,p,\Omega} \quad \text{if} \quad p \in [2,\infty] , \tag{4}$$

are optimal. Nevertheless, we can improve the order convergence (in some norm $|||\cdot|||$ which is close to $||\cdot||_{0,p,\Omega}$) by a suitable post-processing $\sim$, and this we call the superconvergence. The post-processing $\sim$ should be easily computable and the norm $|||\cdot|||$ may be e.g. a discrete analogue of $||\cdot||_{0,p,\Omega}$, or $|||\cdot||| = ||\cdot||_{0,p,\Omega_0}$ for $\Omega_0 \subset\subset \Omega$ (i.e. $\bar{\Omega}_0 \subset \Omega$), or $|||\cdot||| = ||\cdot||_{0,p,\Omega}$, etc. We introduce several examples where $\sim$ is a restriction operator to some subset of Ω , an averaging and an integral smoothing operator. Let us emphasize that many superconvergence phenomena are very sensitive to the mesh geometry (therefore, uniform, quasiuniform or piecewise uniform triangulations are mostly employed). In this paper, we assume for brevity that each T_h is uniform, i.e., any two adjacent triangles of T_h form a parallelogram.

Let N_h be the set of nodal points of T_h . Then the use of the expansion theorem for linear elements by [32] yields (cf. (3))

$$\max_{x \in N_h} |u(x) - u_h(x)| \leqq Ch^4 ||u||_{C^4(\bar{\Omega})} , \tag{5}$$

provided T_h consists of equilateral triangles. We mention that the (stiffness) matrix arising from (2), when taking the standard Courant basis functions, is the same as for the well-known 7-point finite difference scheme (see e.g. [35], p. 91)

$$\frac{2}{3} (6u_0 - u_1 - u_2 - u_3 - u_4 - u_5 - u_6) = h^2 f_0 + h^4 \Delta f_0/16$$

with the rate of convergence $O(h^4)$.

Remark 1. Using (1), (2), (5), and the affine one-to-one mapping F between any uniform triangulation $\hat{T}_h$ and a triangulation T_h consisting of equilateral triangles, one easily obtains an analogue of (5) for $\hat{T}_h = F^{-1}(T_h)$, indeed, but for other equation. For instance, the triangulation sketched in Fig. 1 guarantees the nodal superconvergence for the equation $- \Delta\hat{u} + \partial^2\hat{u}/\partial x\partial y = \hat{f}$.

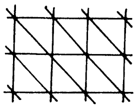

Fig. 1

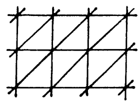

Fig. 2

Remark 2. A convenient combination of linear and bilinear elements may give the $O(h^4)$-superconvergence at nodes for the problem (1) on triangulations consisting of right-angled triangles. Let $\{u^i\}$ and $\{v^i\}$ be the Courant piecewise linear basis functions over the triangulation of Fig. 1 and 2, respectively, and let $\{t^i\}$ be the standard basis functions for bilinear rectangular elements. Put

$$w^i = t^i/2 + u^i/4 + v^i/4$$

and denote by W_h the linear hull of $\{w^i\}$ ($\dim W_h = \dim V_h$). Now, the matrix arising from (2), if we replace V_h by W_h , is the same as for the 9-point difference scheme over square meshes [35], p. 90; and it is thus easy to derive the rate $O(h^4)$ at nodes employing the basis $\{w^i\}$ The next table shows the values of the maximum error over all nodes for various choices of basis functions when $u(x,y) = y(y - 1) \sin \pi x$ is the exact solution of (1) on the unit square $\Omega = (0,1) \times (0,1)$.

h^{-1}	v^i	$(v^i+u^i)/2$	t^i	$(t^i+v^i)/2$	w^i
4	1.2069 E-2	1.2069 E-2	1.2962 E-2	6.0703 E-4	1.6832 E-4
8	3.1027 E-3	3.1027 E-3	3.1589 E-3	1.3156 E-4	1.0307 E-5
16	7.8126 E-4	7.8126 E-4	7.8478 E-4	3.5250 E-5	6.4092 E-7
32	1.9567 E-4	1.9567 E-4	1.9589 E-4	8.7640 E-6	4.0006 E-8

Further we present superconvergence results for the gradient of $u_h \in V_h$. According to [1,26], the tangential component of ∇u_h is a superconvergent approximation to the tangential component of ∇u at midpoints of sides. Denoting by M_h the set of these midpoints, we may then define a recovery operator for both the components of the gradient by the relation (see [4,8,9,11,26,28,30,31,33,40])

$$\tilde{\nabla} u_h(x) = \frac{1}{2}(\nabla u_h|_{T_1} + \nabla u_h|_{T_2}), \quad x \in M_h \cap \Omega, \tag{6}$$

where $T_1, T_2 \in \mathcal{T}_h$ are those adjacent triangles for which $x \in T_1 \cap T_2$ (note that $\nabla u|_{T_i}$ is constant). As shown in [11,30],

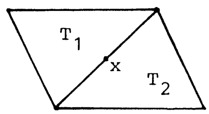

Fig. 3

$$\max_{x \in M_h \cap \Omega} ||\nabla u(x) - \tilde{\nabla} u_h(x)|| \leqq Ch^2 |\ln h| \; ||u||_{3,\infty,\Omega}$$

or even $O(h^2)$ for the discrete L^2-norm [26] (cf. (4)). For a three-dimensional analogue of (6), see [5].

Note that the sampling at centroids of the bilinear elements leads to the superconvergence of the gradient [24]. This is not true for the linear elements. However, a weighted averaging scheme between four elements,

$$\tilde{\nabla} u_h(x) = \frac{1}{6}(3\nabla u_h|_T + \sum_{i=1}^{3} \nabla u_h|_{T_i}), \quad x \in C_h \cap \Omega_0,$$

yields [26]

$$h\Big(\sum_{x \in C_h \cap \Omega_0} ||\nabla u(x) - \tilde{\nabla} u_h(x)||^2\Big)^{\frac{1}{2}} \leqq Ch^2 ||u||_{3,\Omega}.$$

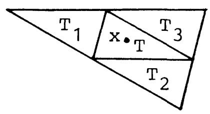

Fig. 4

Here C_h is the set of centroids of all $T \in \mathcal{T}_h$, $\Omega_0 \subset\subset \Omega$, and $T_1, T_2, T_3 \in \mathcal{T}_h$ are the triangles adjacent to that triangle $T \in \mathcal{T}_h$ for which $x \in T$. Using (6), one can define a discontinuous piecewise linear field $\tilde{\nabla} u_h$ which recovers the gradient of u even at any point of $\Omega_0 \subset\subset \Omega$ (see [36]). By the following averaging at nodes $x \in N_h$ we may determine a continuous piecewise linear field $\tilde{\tilde{\nabla}} u_h$ over the whole domain $\overline{\Omega}$:

$$\tilde{\tilde{\nabla}} u_h(x) = \begin{cases} \frac{1}{6} \sum_{T \cap \{x\} \neq \emptyset} \nabla u_h|_T, & x \in N_h \cap \Omega, \\ 0, & x \in Y, \\ \frac{1}{2}(\sum_{i=1}^{3} \nabla u_h|_{T_i} - \nabla u_h|_{T_0}), & x \in N_h \cap (\partial\Omega - Y), \end{cases} \tag{7}$$

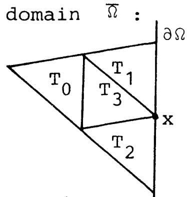

Fig. 5

where Y is the set of vertices of $\bar{\Omega}$, T_i and T_3 form a parallelogram for every $i = 0,1,2$, and $T_1 \cap T_2 \cap T_3 = \{x\}$ when $x \in N_h \cap (\partial\Omega - Y)$ - see Fig.5. In this case the global superconvergence estimate reads [23]:

$$||\nabla u - \widetilde{\nabla u}_h||_{0,p,\Omega} \leqq Ch^2 |\ln h|^{1-2/p} ||u||_{3,p,\Omega} \, , \; p \in \{2,\infty\} \, . \qquad (8)$$

For the generalization of the scheme (7) to elliptic systems with nonhomogeneous boundary conditions of several types, we refer to [20]. If $\partial\Omega$ is smooth then a local $O(h^{3/2})$-superconvergence in $\Omega_0 \subset\subset \Omega$ can be achieved [20,21] in the L^2-norm (T_h are not uniform near the boundary $\partial\Omega$).

Consider now triangulations as marked in Fig. 1 or 2 and the smoothing post-processing operator

$$\tilde{u}_h(x) = \tfrac{1}{4}h^{-2} \int_{D_h} u_h(x + y) \, dy \, ,$$

where $D_h = (-h,h) \times (-h,h)$. If $\Omega_0 \subset\subset \Omega$ and $\partial\Omega$ is again smooth then (see [37,38])

$$||u - \tilde{u}_h||_{1,\Omega_0} \leqq Ch^{3/2} ||u||_{3,\Omega} \, ,$$

which is, in fact, a superconvergent estimate for the gradient.

Another type of an integral smoothing operator which yields a superconvergent approximation for ∇u as well as for u even on irregular meshes is presented in [3]. In [19] a least squares smoothing of ∇u_h is proposed to obtain a better approximation to ∇u . Related papers with superconvergence of linear elements further include [2,6,7,12,13,17,18, 25,29,34], see also the survey papers [10,22,27].

Let us now turn to superconvergent approximations to the boundary flux $q = \frac{\partial u}{\partial n}|_{\partial\Omega}$ (n is the outward unit normal to $\partial\Omega$). Setting

$$\tilde{q}_h = n \cdot \widetilde{\nabla u}_h|_{\partial\Omega} \, ,$$

where $\widetilde{\nabla u}_h$ is given by (7), we immediately get from (8) that

$$||q - \tilde{q}_h||_{0,\infty,\partial\Omega} \leqq Ch^2 \; |\ln h| \; ||u||_{3,\infty,\Omega} \, ,$$

i.e., the continuous piecewise linear function $\tilde{q}_h$ approximates q better than the piecewise constant function $q_h = n \cdot \nabla u_h|_{\partial\Omega}$.

Another continuous piecewise linear approximation $\tilde{q}_h$ to the boundary flux q can be defined with the help of Green's formula

$$\int_{\partial\Omega} \tilde{q}_h \, v_h \, ds = (\nabla u_h, \nabla v_h)_{0,\Omega} - (f, v_h)_{0,\Omega} \quad \forall v_h \in U_h \, ,$$

where

$$U_h = \{v_h \in H^1(\Omega) \mid v_h|_T \in P_1(T) \quad \forall T \in T_h\} \, .$$

This technique suggested by [16], p.398, is based on some ideas of [14].

Numerical tests of the presented superconvergent schemes can be found in [3,6,11,19,21,23,24,26,36].

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