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SPATIALLY NONHOMOGENEOUS PATTERN GENERATED BY HOMOCLINIC/EQUILIBRIUM BIFURCATIONS

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ABSTRACT. Assume that an ODE system has a homoclinic solution asymptotic to a hyperbolic equilibrium E. Breaking of the homoclinic solution creates stable period solutions [8]. After adding diffusion, E becomes nonhyperbolic, and stable spatially nonhomogeneous (SN) periodic solutions can be generated. When Neumann boundary conditions are imposed, simple or double SN periodic solutions can be generated depending on the twistedness of the homoclinic solution. Systems with spatially periodic boundary conditions are also studied.

Consider a diffusively perturbed system

$$U' = DU_{xx} + F(U,k), \qquad 0 < x < 1, \quad U_x(0,t) = U_x(1,t) = 0\,, \tag{1}$$

where $D = \operatorname{diag}\{d_1, d_2\}$. When d_1 and d_2 are large, all the solutions of (1) approach spatially homogeneous (SH) solutions as $t \to \infty$, [2] and [4]. We show when decreasing d_1 and d_2, spatially nonhomogeneous (SN) stable periodic solutions can be generated from a SH homoclinic solution.

For the unperturbed ODE system, assume that $F\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is C^∞. When the parameter $k = k_0$, the unperturbed ODE has a homoclinic solution $q(t)$ asymptotic to a hyperbolic equilibrium E. The Jacobian matrix at E

$$J = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

satisfies $ad - bc < 0$ and $a + d > 0$. Let $M(k)$ be the gap between $W^u(E)$ and $W^s(E)$, measured on a cross section of $q(t)$. Assume that the Melnikov

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function $\frac{dM(k)}{dk} \neq 0$. Breaking of the homoclinic orbit creates a periodic orbit on one side of k_0, [8]. Assume that for $k \in (k_0 - \varepsilon, k_0)$, a long period solution $p(t,k)$ bifurcates from $q(t)$. All these hypotheses are satisfied by an example of Freedman and Wolkowicz [9] which motivates this study. See [7].

The diffusively perturbed system is studied in intermediate spaces $D_A(\theta)$, $0 < \theta < 1$, where $X = [L^2(0,1)]^2$, $D_A = \{[H^2(0,1)]^2$ with boundary conditions$\}$. The existence of a solution $U \in C^1([0,\tau]\colon D_A(\theta)) \cap C([0,\tau]\colon D_A(\theta+1))$ that depends C^r on (d_1, d_2, k) and the initial condition $U_0 \in D_A(\theta+1)$ for any $r > 0$ is known [3]. We also can show the existence of a center stable (unstable) manifold and an invariant foliation of it by strongly stable (unstable) fibers. If the equilibrium is hyperbolic with a simple unstable eigenvalue, then the linearization around $q(t)$ has exponential dichotomies in $(-\infty, -\omega]$ and $[\omega, \infty)$, [6]. Melnikov–Silnikov type function $G(d_1, d_2, k, T)$ can be constructed which measures the jump $U(T) - U(0)$ of a piecewise continuous period T solution. The zero of G corresponds to a true periodic solution of period T. The function G is a continuation of M. Therefore $dM/dk \neq 0$ implies $\partial G/\partial k \neq 0$. Thus for each $T > t^*(d_1, d_2)$, there exists a unique $k = k^*(d_1, d_2, T)$ such that there is a unique period T solution near $q(t)$. The above theorem was proved for ODE and delay equations [5]. The proof for parabolic equations is almost identical. Since the SH periodic solution persists under diffusive perturbations, we have proved the following theorem.

THEOREM 1. *If E is hyperbolic in $D_A(\theta+1)$ with one simple unstable eigenvalue, then for each $T > t^*(d_1, d_2)$, the SH period T solution is the only period T solution near $q(t)$.*

The loss of hyperbolicity of E occurs in the first Fourier mode

$$\{(u \cos \pi x, v \cos \pi x) \mid (u, v) \in \mathbb{R}^2\},$$

when $(a + \pi^2 d_1)(d + \pi^2 d_2) = bc$, $d_1 > 0$, $d_2 > 0$. In the (d_1, d_2)–plane, the above defines a curve Γ. The bifurcation to stable SN periodic solutions occurs when (d_1, d_2) is near Γ. Let λ be the eigenvalue with the real part closest to zero. We make a change of variable in a neighborhood of Γ: $(d_1, d_2) \to (\ell, m)$. Let $m = \lambda$. Let ℓ be the arc length on Γ when $m = 0$, and $\ell = C$ be an orthogonal family of curves to $\lambda = C$. This change of variable is valid in a neighborhood of Γ, since we can show that $\nabla \lambda = \left(\frac{\partial \lambda}{\partial d_1}, \frac{\partial \lambda}{\partial d_2}\right) \neq 0$ when $\lambda = 0$.

To understand the bifurcation when $m \approx 0$, we need two more notions:

i) The weak stability of E on $W^c(E)$.

ii) Twistedness of the homoclinic orbit $q(t)$.

When $m = 0$, the linearization at E has one simple positive eigenvalue, one simple zero eigenvalue and the rest of the eigenvalues are stable. Write $D_A(\theta + 1) = X \times Y \times Z$, where X, Y, Z are the unstable, stable and center eigenspaces respectively. In a special coordinates, we assume that the flow on $W^c(E)$ is

$$x = 0, \quad y = 0, \quad z' = mz - cz^3 + \text{ h.o.t.}, \, c > 0 .$$

The assumption $c > 0$ means that E is weakly stable on $W^c(E)$. When $m > 0$, a pair of SN equilibria E_1, E_2 bifurcates from E. Notice that the solutions of (1) are invariant under a reflection of the domain: $RU(t, x) = U(t, 1 - x)$. This causes the flow on $W^c(E)$ is odd in z.

There is a solution $\phi(t)$ to the linear variational equation around $q(t)$ such that $\phi(t) \to \phi_c$ as $t \to -\infty$, [7]. Here ϕ_c is a unit eigenvector corresponding to the zero eigenvalue. One can show that $\phi(t) \to c^* \phi_c$ as $t \to \infty$. c^* is a function of ℓ. We say that the homoclinic solution is nontwisted, twisted, or degenerate if $c^* > 0$, < 0, or $= 0$. In fact, $\phi(t)$ is tangent to $W^{cu}(E)$, and is transverse to $q'(t)$. We are in fact talking about the twistedness of a strip of $W^{cu}(E)$ around the orbit of $q(t)$.

We say that $U(t, x)$ is a simple periodic solution if it stays near the homoclinic orbit and hits a cross section Σ once. We say that $U(t, x)$ is a double periodic solution if it hits Σ twice and $U(t + T, x) = U(t, 1 - x)$, where $2T$ is the period.

To find a simple or double period solution, we construct inner and outer mappings similar to the method of S i l n i k o v 's. The inner mapping is defined in a neighborhood of E and spends time t_0. It is written as a boundary value problem that generates Silnikov's method to nonhyperbolic equilibria [1]. The outer mapping is near the outer loop of $q(t)$ and spends time t_1. $T = t_0 + t_1$. Using the hyperbolicity in the (x, y) direction, by a Lyapunov–Schmidt reduction, we are led to two bifurcation equations and two variables k and z. The first equation $G_1(\ell, m, k, T, z) = 0$ is the continuation of the gap condition in $D_A(\theta + 1)$, and its solution is $k = k^*(\ell, m, T, z)$. The second equation

$$z = \quad G_2(\ell, m, T, z), \quad \text{for a simple period T solution,}$$
$$z = -G_2(\ell, m, T, z), \quad \text{for a double period 2T solution,}$$

asserts that the z variable has to match after cycling around a periodic solution. The z variable cannot be reduced by the Lyapunov–Schmidt method since E is not hyperbolic in the z direction. Observe that the outer mapping is almost linear with $z(t_1)/z(0) \approx c^*(\ell)$. If we choose $z = \varepsilon$, $m \ll \varepsilon^2$, the inner mapping satisfies $z(t_0)/z(0) \ll 1$. Thus $|G_2| < |z|$. The greatest ratio of stretching in z occurs when $z \approx 0$, and is denoted by $H(\ell, m, T)$. We can see that $z = G_2(\ell, m, T, z)$ (or $-G_2(\ell, m, T, z)$) has a solution if $H(\ell, m, T) > 1$ (or $H(\ell, m, T) < -1$).

THEOREM 2. *Assume that $c^*(\ell_0) \neq 0$. We can show*

$$\frac{\partial}{\partial m} H(\ell, m, T) \neq 0\,,$$
$$H(\ell, m, T) \simeq e^{mT} c^*(\ell)\,.$$

There exist two families of curves $H(\ell, m, T) = \pm 1$ in the (ℓ, m)-plane near $m = 0$. For each $(\ell_0, m_0) \in \Gamma$, $m_0 = 0$, there is an open set $\mathcal{O} \subset \mathbb{R}^2$ containing (ℓ_0, m_0), the size of which depends on ℓ_0. $\mathcal{O}$ is divided by each curve into two parts — $|H(\ell, m, T)| < 1$ or > 1.

i) *If $c^*(\ell_0) > 0$, $\neq 1$, then there exist exactly two stable simple period T SN solutions if $H(\ell, m, T) > 1$; no such solution if $H(\ell, m, T) \leq 1$.*

ii) *If $c^*(\ell_0) < 0$, $\neq -1$, then there exists a unique stable double period $2T$ SN solution if $H(\ell, m, T) < -1$; no such solution if $-1 \leq H(\ell, m, T)$.*

iii) *If $c^*(\ell_0) = 1$, then*
$H(\ell, m, T) > 1 + \delta \Rightarrow$ there exist exactly two stable simple period T SN solutions;
$H(\ell, m, T) < 1 - \delta \Rightarrow$ no such solution;
$1 + \delta \geq H(\ell, m, T) > 1 \Rightarrow$ there exist at least two simple period T SN solutions. (The uniqueness and stability of such solutions are unknown.)

iv) *If $c^*(\ell_0) = -1$, then*
$H(\ell, m, T) < -1 - \delta \Rightarrow$ there exists exactly one stable double period $2T$ SN solution;
$H(\ell, m, T) > -1 + \delta \Rightarrow$ no such solution;
$-1 - \delta \leq H(\ell, m, T) < -1 \Rightarrow$ there exists at least one double period $2T$ SN solution. (The uniqueness and stability of such solutions are unknown.)

v) *The SH period T solution loses stability when the SN solutions are known to be stable.*

The above results also show the bifurcation to a pair of SN homoclinic solutions asymptotic to E_1 and E_2 if $c^*(\ell_0) > 0$ or a pair of heteroclinic solutions between E_1 and E_2 if $c^*(\ell_0) < 0$ when crossing Γ. They are special cases with $T = \infty$. Except for the stability result, the proof of Theorem 2 can be found in [7]. To show the uniqueness or nonexistence of solutions, consider $z > 0$ only. (The bifurcation function is odd.) Observe that the solution $\Phi(t, z_0)$ to the equation $z' = mz - cz^3 +$ h.o.t satisfies

$$\frac{\partial}{\partial z_0} \frac{\Phi(t_0, z_0)}{z_0} < 0\,.$$

The return map in the z direction is almost like $z(T) = \Phi(t_0, z_0) \cdot c^*(\ell)$. Thus $\frac{\partial}{\partial z_0} \frac{z(T)}{z_0} < 0$. This shows that if $\lim_{z_0 \to 0} \frac{z(t)}{z_0} > 1$, then $\frac{z(t)}{z_0} = 1$ admits a unique solution $0 < z_0 < \varepsilon$; otherwise there is no such solution. In the real situation the reduction has some error. Therefore the argument does not work well when $c^*(\ell_0) = 0$ or ± 1. See [7].

We now indicate how the stability of SN solutions can be proved. Consider $z_0 > 0$ only. Again the return map in the z direction satisfies $\frac{\partial}{\partial z_0} \frac{z(T)}{z_0} < 0$. Assume that $z(T)/z_0 = 1$. We have $\frac{\partial z(T)}{\partial z_0} z_0 - z(T) < 0$. Thus $0 < \frac{\partial z(T)}{\partial z_0} < \frac{z(T)}{z_0} = 1$. This shows that the return map is stable in the z direction. It is easy to show that the return map is stable in the other directions transverse to the periodic orbit using the roughness of exponential dichotomies.

The next theorem shows that for a given large T, one can move (d_1, d_2) across Γ along a narrow strip near $c^*(\ell_0) = 0$ without creating any simple period T or double period $2T$ SN solutions [7].

THEOREM 3. *Assume that $c^*(\ell_0) = 0$ and $\frac{d}{d\ell} c^*(\ell_0) \neq 0$. There exist constants $\varepsilon > 0$ and $\bar{t} > 0$ such that functions $\ell^*(m), |m| < \varepsilon$ and $\delta(T) = ce^{-mT}$, $T > \bar{t}$ for some $c > 0$ can be defined. If $|\ell - \ell^*(m)| < \delta(T)$, $|m| < \varepsilon$ and $T > \bar{t}$, then there is no simple period T or double period $2T$ SN solution to (2.1), inside a $(\delta(T))^{1/2}$ neighborhood of the orbit of $q(t)$.*

We now consider equation (1) for $x \in \mathbb{R}$ with spatially periodic boundary conditions $V(t, x+2) = V(t, x)$. Let $U(t, x)$, $0 < x < 1$, be a solution satisfying the Neumann boundary conditions at $x = 0,\ 1$. Define

$$V_0(t, x) = \begin{cases} U(t, x), & 0 < x < 1, \\ U(t, -x), & -1 < x < 1. \end{cases}$$

Then extend V_0 to $x \in \mathbb{R}$ periodically with period 2. Define $V_\xi(t, x) = V_0(t, x + \xi)$.

THEOREM 4. *All the simple period T and double period $2T$ solutions of the diffusively perturbed system with spatially periodic boundary conditions of period 2 have the form $V_\xi(t, x)$, where V_ξ is defined from U, which is a corresponding solution satisfying the Neumann boundary conditions at $x = 0,\ 1$.*

P r o o f. The eigen vectors corresponding to the zero eigenvalue span a two dimensional space that is invariant under the reflection $(r, \theta) \to (r, -\theta)$ and rotations $(r, \theta) \to (r, \theta + \xi)$ in polar coordinates. Let $V(t, x)$ be a solution satisfying periodic boundary conditions. Then after reflection, or rotation of the domain by ξ, $RV(t,x) = V(t,-x)$ and $R_\xi V(t,x) = V(t,x+\xi)$ are still solutions with the periodic boundary conditions. If V is a simple period T SN solution, then it is associated to a solution $z = (r,\theta)$ of the two dimensional bifurcation equation $z = G_2(\ell, m, T, z)$ which respects the $O(2)$ symmetry. The solution $R_{-\theta}V$ is then associated to $z = (r,0)$. We can see that $R(R_{-\theta}V) = R_{-\theta}V$ since $z = (r,0)$ is invariant with respect to the reflection. The restriction of $R_{-\theta}V$ to $[0,1]$ is a solution to the PDE with the Neumann boundary conditions at $x = 0,\ 1$. This proves the theorem. □

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