A MODEL FOR PHENOMENA OF INSTABILITY

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1. INTRODUCTION

The analysis of some phenomena of instability in hydrodynamics (in particular of the simpler forms of unstable behaviour for the Couette flow of a viscous fluid between two rotating coaxial cylinders) is well developed (see, e.g., [1]). For instance, the important contributions of VELTE particularly [2], [3], and then of Kirchgässner [4] and others have strengthened the significance of earlier results of linear analyses. Extensive numerical experiments have also been carried out [5], [6]. The peculiar difficulty of the case treated by VELTE in [3] is in part connected with the fact that the differential operators involved are not gradient operators, as happens in the case of the analytical schemes for phenomena of instability in elastic systems, for instance. Again, partially as a consequence of this fact, there are still many open questions even within the relatively narrow field of study of branching phenomena. Analysis and numerical work cannot go much beyond indications of behaviour of solutions in the neighbourhood of branching points and actually it is usual to consider in detail only one of the branching points. Therefore it was thought worthwhile to study a special differential problem which, though relatively simple, nevertheless maintains essential features occurring in some of the much more difficult problems quoted above.

In itself the problem belongs to a class which has direct relevance to the description of certain natural phenomena. In fact, the best way to introduce the problem is to recall, even if rather superficially, certain developments contained in a recent paper devoted to the study of a biophysical question [7].

Let $\mathscr{U}$ and $\mathscr{V}$ be two sets of functions $u(P)$ and $v(P)$, respectively; assume that the domain of both u and v is a certain set $\boldsymbol{D}$ of $\boldsymbol{E}_2$. Let F be a mapping defined over $\mathscr{U}$ and with values in $\mathscr{V}$; then, for each choice of $v \in \mathscr{V}$, let us look for functions $u \in \mathscr{U}$ such that

$$F(u) = v. \tag{1.1}$$

Naturally, to decide upon the significance of the problem thus set, regularity properties of $\mathscr{U}$, $\mathscr{V}$ and F are needed; we refer to [7] for a statement of these properties. We recall here only the mathematical-physical hypothesis which is *essential* for the developments which follow: we assume that the relationship (1.1) is invariant to orthogonal changes of coordinates in $\boldsymbol{E}_2$. Then we proceed to examine the following question of approximation. Let $\{v_\alpha\}$ be a class of functions of $\mathscr{V}$, dependent upon the values of a real parameter α in a neighbourhood of zero; and such that v tends, in appropriate sense, to a constant function when $\alpha \to 0$. Then, again in an appropriate sense, (and we refer again to [7] for details), an approximate solution of the problem

$$F(u) = v_\alpha, \quad \text{for} \quad \alpha \sim 0$$

must be sought among the functions w such that

$$a\,\Delta w + b\,|\operatorname{grad} w|^2 + c = v_\alpha, \quad w \in \mathscr{U}, \tag{1.2}$$

where a, b, c are certain functions of w.

This result is not surprising: the orthogonal invariants which can be formed starting from a function w (defined over a set of $\boldsymbol{E}_2$) and its first and second derivatives are (w apart) the modulus of the gradient and the invariants of the hessian matrix; the "simplest" of the latter is the laplacian.

Our problem involves a special case of eqn (1.2). Other special cases have been already amply studied; apart from the case $b \equiv 0$, we can quote the case $a \equiv 0$, which is the fundamental equation in geometrical optics (see, for instance, [8], p. 88 and ff., p. 369 and ff.). It is of interest to remark here that, when b is not identically zero, the left-hand side of eqn (1.2) is not a gradient operator.

2. STATEMENT OF THE PROBLEM

We must now specify the domain $\boldsymbol{D}$, the functions a, b, c, v, and the class $\mathscr{U}$. In this task we are guided by some general requirements:

(i) Our equation must be of the second order and without singularities; hence $a \neq 0$.
(ii) $w \equiv 0$ must be a solution of the problem; hence $c(0) - v_\alpha \equiv 0$, and the boundary conditions must be homogeneous. These conditions focus all the attention on cases of lack of unicity as a consequence of the existence of "non trivial" solutions.
(iii) The interest lies exclusively in real solutions.
(iv) For the associated "linearized" problem eigenvalues and eigenfunctions can be given explicitly.

Obviously these requirements do not lead to a unique problem; reasons of simplicity and of analogy with cases quoted in the Introduction suggest the choice made below. Other relevant cases will be dealt with in later papers.

The special case of (1.2) studied here is the following one

$$\Delta w - w\,|\operatorname{grad} w|^2 + \lambda w = 0, \tag{2.1}$$

(λ, a real parameter). $\mathscr{U}$ is the set of real continuous functions $w(x, y)$ defined in the strip $\bar{\boldsymbol{S}}$

$$\bar{\boldsymbol{S}}\colon\ 0 \leqq x \leqq 1, \quad -\infty < y < +\infty,$$

even in y and periodic in y of period 2; w has partial derivatives up to order two, square integrable over the rectangle $\boldsymbol{R}$

$$\boldsymbol{R}\colon\ 0 \leqq x \leqq 1, \quad -1 \leqq y \leqq 1;$$

w vanishes along the boundary of the strip. A strong solution of our problem, which we will call henceforth *problem* A, is a function of $\mathscr{U}$ which satisfies eqn (2.1). We consider only functions even in y to avoid certain disturbing ambiguities in our analysis.

It is now appropriate to remark that the associated linear problem, which we will call problem $\mathbf{A}_l$, admits in $\mathscr{U}$ the set of eigenfunctions

$$w_{h,k}(x, y) = \sin h\pi x \cos k\pi y, \quad h \in N,\ k \in N_0, \tag{2.2}$$

which correspond to eigenvalues

$$\lambda = (h^2 + k^2)\,\pi^2, \quad h \in N,\ k \in N_0. \tag{2.3}$$

In the analysis which follows we must consider two other auxiliary problems, which we specify now.

The first of these, problem $\mathbf{B}$, involves an ordinary differential equation

$$\frac{\mathrm{d}^2 s}{\mathrm{d}x^2} - s\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2 + \lambda s = 0, \qquad x \in (0, 1), \tag{2.4}$$

$$s(0) = 0, \quad s(1) = 0. \tag{2.5}$$

The linear problem $\mathbf{B}_l$ associated with problem $\mathbf{B}$ is elementary; the eigenvalues are simple and form that part of the sequence (2.3) which corresponds to the choice $k = 0$.

In the third problem, problem $\mathbf{C}(h, k)$, $(h, k \in N)$ we seek a function $z(x, y)$ defined over the rectangle

$$\boldsymbol{R}(h, k)\,;\, 0 \leqq x \leqq \frac{1}{2h}, \qquad 0 \leqq y \leqq \frac{1}{2k},$$

having in $\boldsymbol{R}(h, k)$ first and second derivatives, the latter square integrable in $\boldsymbol{R}(h, k)$, and satisfying eqn (2.1) and the boundary conditions

$$\begin{aligned} z(0, y) &= 0, & z\left(x, \frac{1}{2k}\right) &= 0, \\ \left(\frac{\partial z}{\partial x}\right)_{x=1/2h} &= 0, & \left(\frac{\partial z}{\partial y}\right)_{y=0} &= 0. \end{aligned} \tag{2.6}$$

The linearized problem $\mathbf{C}_l(h, k)$ associated with problem $\mathbf{C}(h, k)$ admits the functions

$$\sin h\pi(2m + 1)x \cos k\pi(2n + 1)y, \quad m, n \in N_0,$$

as eigenfunctions; the corresponding eigenvalues are

$$[h^2(2m + 1)^2 + k^2(2n + 1)^2]\,\pi^2, \quad m, n \in N_0.$$

For us it is important to remark here that the lowest eigenvalue $(h^2 + k^2)\,\pi^2$ is simple for problem $\mathbf{C}_l(h, k)$; notice that the same value is also (perhaps multiple) eigenvalue for problem $\mathbf{A}_l$.

3. EXISTENCE OF SOLUTIONS OF PROBLEM A

We remark first of all that a strong non trivial solution $\hat{w}$ of problem **A** might exist only if $\lambda \geqq \pi^2$.

In fact from eqn (2.1) we get by partial integration over the rectangle $\boldsymbol{R}$

$$\lambda = \frac{\int_R |\operatorname{grad} \hat{w}|^2 \, \mathrm{d}R}{\int_R \hat{w}^2 \, \mathrm{d}R} + \frac{\int_R \hat{w}^2 |\operatorname{grad} \hat{w}|^2 \, \mathrm{d}R}{\int_R \hat{w}^2 \, \mathrm{d}R}. \tag{3.1}$$

On the other hand, for any non null function in $\mathscr{U}$ the first term in the right-hand side of (3.1) is not less than the lowest eigenvalue of problem $\mathbf{A}_l$ (see for instance [8]); hence the property.

It follows that, instead of looking for solutions of problem **A**, we could seek that function f of $\mathscr{U}$ that satisfies the equation

$$\Delta f - \lambda f |\operatorname{grad} f|^2 + \lambda f = 0, \quad \lambda > 0, \tag{3.2}$$

and then take $w = \sqrt{\lambda} f$. There is the following advantage of eqn (3.2) over eqn (2.1): let $g(x, y)$ be a function defined over $\bar{S}$, even in y and of period 2; consider the problem

$$\Delta f = g, \quad f \in \mathscr{U}, \tag{3.3}$$

and the relevant Green's function $G(P, Q)$ such that

$$f(P) = \int_R G(P, Q)\, g(Q) \, \mathrm{d}R; \tag{3.4}$$

then eqn (3.2) can be substituted by the integral equation

$$\frac{1}{\lambda} f(P) = \int_R G(P, Q) \{f(Q) [|\operatorname{grad} f|^2 - 1]\} \, \mathrm{d}R, \tag{3.5}$$

which we will write briefly, using a standard notation,

$$\mu f = Tf, \qquad \mu = \frac{1}{\lambda}.$$

We turn now to questions of existence of solutions of our problem. Consider again the set of functions $\mathscr{U}$, introduce the norm

$$\| w \| = \left(\int_R w^2 \, \mathrm{d}R\right)^{\frac{1}{2}} + \left(\int_R |\operatorname{grad} w|^2 \, \mathrm{d}R\right)^{\frac{1}{2}} + \left[\int_R \sum \left(\frac{\partial^2 w}{\partial x_i \, \partial x_j}\right)^2 \mathrm{d}R\right]^{\frac{1}{2}}, \tag{3.6}$$

and let $\bar{\mathscr{U}}$ be the closure of $\mathscr{U}$ with respect to this norm. Notice that $\bar{\mathscr{U}}$ is a Sobolev space and also a Hilbert space.

If $g(P)$ is a function $\boldsymbol{L}^2$ in $\boldsymbol{R}$, then eqn (3.3) admits a solution in $\bar{\mathscr{U}}$; i.e. there exists a Green's function such that formula (3.4) defines a function $f(P)$ of $\bar{\mathscr{U}}$. Furthermore if $f(P)$ belongs to $\bar{\mathscr{U}}$, the expression within curly brackets in the right-hand side of (3.5) is in $\boldsymbol{L}^2$.

Moreover the mapping $\mathscr{T} : f \to f'$

$$\mu f' = Tf$$

is a compact mapping of $\bar{\mathscr{U}}$ into itself. A non-trivial fixed point of $\mathscr{T}$, if it exists, is a solution in a generalized sense of our problem; actually, known regularization theorems assure then the existence also of a strong solution.

Similarly one can consider the mapping $\mathscr{T}_l : f \to f'$ where

$$\mu f' = T_l f, \quad f, f' \in \bar{\mathscr{U}}.$$

Its fixed points are solutions of problem $\mathbf{A}_l$.

Now the mapping $\mathscr{T}_l$ is the Fréchet differential of the mapping $\mathscr{T}$ at the null element of $\bar{\mathscr{U}}$; one can use then the theorem of Leray–Schauder on bifurcation of solutions of functional equations to reach the proof of the existence of non trivial fixed point for the mapping $\mathscr{T}$ for λ in the neighbourhood of all eigenvalues of $\mathbf{A}_l$ which have odd multiplicity, see for instance [9] and references quoted there, in particular [10].

Many eigenvalues of problem $\mathbf{A}_l$ (such as π^2, $2\pi^2$, $4\pi^2$, etc.) are in fact simple, but many others are double (such as $5\pi^2$, $10\pi^2$, etc.). Even when triple eigenvalues are considered (such as $25\pi^2$), there is an interest in deciding if they correspond to a multiple branching. These questions may be settled through an analysis of the auxiliary problems **B** and **C**.

4. EXISTENCE AND PROPERTIES OF THE SOLUTIONS OF PROBLEM B

For problem **B** detailed properties of solutions can be found. We do not enter here into details of proofs; we only quote results.

Consider the function

$$F(q) = \int_0^q \frac{\mathrm{d}\xi}{(1 - e^{\xi^2 - q^2})^{\frac{1}{2}}}$$

for $q \geqq 0$; $F(q)$ has the following properties

$$\lim_{q \to 0} F(q) = \frac{\pi}{2}; \qquad F(q) > q;$$

$$F'(q) > 0 \quad \text{for} \quad q > 0, \quad \lim_{q \to 0} F'(q) = 0;$$

$$\lim_{q \to +\infty} \frac{F(q)}{q} = 1.$$

For each positive integer h, consider the real function $q_h(\lambda)$ defined implicitly by the relation

$$F(q_h) = \frac{\sqrt{\lambda}}{2h};$$

$q_h(\lambda)$ has values only for $\lambda \geqq h^2\pi^2$, vanishes for $\lambda = h^2\pi^2$, is always less than $\sqrt{\lambda}/2h$ and is such that $\lim_{\lambda\to\infty}\left[q_h(\lambda) - \frac{\sqrt{\lambda}}{2h}\right] = 0$.

Then, the problem **B** has a sequence of non trivial solutions $s_h(x, \lambda)$, $h \in N$. $s_h(x, \lambda)$ exists only for $\lambda \geqq h^2\pi^2$ and has $h - 1$ zeros for $x \in (0, 1)$, its values over the whole interval $(0, 1)$ can be easily defined when its values over the interval $\left(0, \frac{1}{2h}\right)$ are known, in exactly the same way as the values of $\sin h\pi x$ in the whole interval $(0, 1)$ can be defined by the values in $\left(0, \frac{1}{2h}\right)$.

In $\left(0, \frac{1}{2h}\right)$, $s_h(x, \lambda)$ is defined by the inverse relation

$$\sqrt{\lambda}x = \int_0^{s_h(x,\lambda)} \frac{d\sigma}{[1 - e^{\sigma^2 - q^2_h(\lambda)}]^{\frac{1}{2}}}.$$

$s_h(x, \lambda)$ reaches its maximum for $x = \frac{1}{2h}$ and

$$s_h\left(\frac{1}{2h}, \lambda\right) = q_h(\lambda).$$

For large values of λ, $s_h(x, \lambda)$ is equal approximately to

$$s_h(x, \lambda) \cong \sqrt{\lambda}\, x$$

in $\left(0, \frac{1}{2h}\right)$, so that $s_h(x, \lambda)$ approaches, as $\lambda \to \infty$, a continuous function with jumps in the first derivative amounting in absolute value to $2\sqrt{\lambda}$ at all points $\frac{2m+1}{2h}$ $(m = 0, \dots h - 1)$.

We conclude then, in particular, that our problem **A** has at least a solution branching away from the trivial one at the values of λ belonging to the sequence $h^2\pi^2$, irrespective of the fact if these values are or not eigenvalues of odd multiplicity for $\mathbf{A}_l$.

5. EXISTENCE OF SOLUTIONS OF PROBLEM C(h, k).

The existence of branching points for solutions of each problem **C**(h, k) can be ascertained with the same techniques which have been used in dealing with problem **A**. We are interested in particular in the following facts:

(i) The lowest eigenvalue of $\mathbf{C}_l(h, k)$ is simple; hence it corresponds to a branching point for solutions of $\mathbf{C}(h, k)$ (which we will call $z_{hk}(x, y)$).
(ii) Only the trivial solution exists for $\mathbf{C}(h, k)$ when $\lambda < (h^2 + k^2)\,\pi^2$.
(iii) A solution of problem **A** branching away from the trivial one at the eigenvalue $\lambda \doteq (h^2 + k^2)\,\pi^2$ can be found once $z_{hk}(x, y)$ is known. This statement is based on the following circumstance: suppose a function $z(x, y)$ of class C_2 is given over $R(h, k)$, so that it satisfies eqn (2.1) and the boundary conditions (2.6); then a function $w(x, y)$ of class C_2 can be defined over the whole rectangle $\boldsymbol{R}$ so that it coincides with z over $\boldsymbol{R}(h, k)$ and has over the rest of $\boldsymbol{R}$ the symmetry properties of the function $\sin h\pi x\, \cos k\pi y$.

The doubts expressed at the end of Sect. 3 are thus completely settled. A number of open questions remains; in particular the following conjectures require proof:

(i) $z_{h,k}(x, y)$ is defined for all $\lambda > (h^2 + k^2)\,\pi^2$;
(ii) the maximum of $z_{h,k}(x, y)$ increases with increasing λ, with the order 1/2;
(iii) as λ tends to infinity $z_{h,k}(x, y)$ tends to a "roof function" $r_{h,k}(x, y)$:

$$r(x, y) = \begin{cases} \sqrt{\lambda}\, x & \text{for } y \leqq \dfrac{1}{2k} - x, \\ \sqrt{\lambda}\left(\dfrac{1}{2k} - y\right) & \text{for } y \geqq \dfrac{1}{2k} - x. \end{cases}$$

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