J. H. Bramble

On the approximation of eigenvalues of non-selfadjoint operators

In: Miloš Ráb and Jaromír Vosmanský (eds.): Proceedings of Equadiff III, 3rd Czechoslovak Conference on Differential Equations and Their Applications. Brno, Czechoslovakia, August 28 - September 1, 1972. Univ. J. E. Purkyně - Přírodovědecká fakulta, Brno, 1973. Folia Facultatis Scientiarum Naturalium Universitatis Purkynianae Brunensis. Seria Monographia, Tomus I. pp. 15--19.

Persistent URL: http://dml.cz/dmlcz/700077

ON THE APPROXIMATION OF EIGENVALUES OF NON-SELFADJOINT OPERATORS

by J. H. BRAMBLE

In this lecture I am going to present some general theorems on the approximation of compact operators in a Hilbert space. The work presented is due to JOHN E. OSBORN and myself and detailed proofs may be found in [1] and [2]. I will then show how these general results may be applied to obtain estimates for the errors in the eigenvalues and generalized eigenfunctions obtained by various Galerkin type approximations to eigenvalue problems for second order strongly elliptic partial differential operators.

GENERAL RESULTS

Let H^0 be a Hilbert space and $T : H^0 \to H^0$ a given compact operator. Suppose that we have a family of operators $\{T_h\}_{0 < h < 1}$ and a family of finite dimensional subspaces of H^0, $\{S_h\}_{0 < h < 1}$ such that for each h, $T_h : H^0 \to S_h$. Finally we tie the operators together by requiring that

$$\lim_{h \to 0} \| T - T_h \| = 0$$

where $\| \,.\, \|$ here denotes the operator norm.

It is known [cf. 4] that if μ is an eigenvalue of T with algebraic multiplicity m then for h sufficiently small there are exactly m eigenvalues of T_h (counting according to algebraic multiplicities) $\mu_1(h), \ldots, \mu_m(h)$, such that

$$\lim_{h \to 0} \mu_j(h) = \mu, \qquad j = 1, \ldots, m.$$

Let $M(\mu)$ denote the subspace of generalized eigenvectors of T corresponding to μ and $M_h(\mu)$ the direct sum of the subspaces of generalized eigenvectors of T_h corresponding to the eigenvalues $\mu_1(h), \ldots, \mu_m(h)$.

For our theorem we want to consider H^0 as part of a certain scale of Hilbert spaces. Suppose we have given Hilbert spaces H^s, $s \geqq 0$ such that

1. H^s is a dense subset of H^0, and
2. for $0 \leqq s_1 \leqq s_2$, $H^{s_2} \subset H^{s_1}$ with a compact injection. Let $\langle .,. \rangle$ denote the inner product in H^0. Then for $s < 0$ we define

$$\| \psi \|_{H^s} = \sup_{\varphi \in H^{-s}} \frac{| \langle \psi, \varphi \rangle |}{\| \varphi \|_{H^{-s}}},$$

where $\| \,.\, \|_{H^{-s}}$ is the norm in H^{-s}. The space H^s, $s < 0$, is defined to be the completion of H^0 with respect to the norm $\| \,.\, \|_{H^s}$.

We need some further notation. For an operator $A : H^0 \to H^0$ we define for $s_0, s_1 \geqq 0$

$$\| A \|_{-s_0, s_1} = \sup_{\varphi \in H^{s_1}} \frac{\| A\varphi \|_{H^{-s_0}}}{\| \varphi \|_{H^{s_1}}};$$

i.e., $\| A \|_{-s_0, s_1}$ is the operator norm on A considered as $A : H^{s_1} \to H^{-s_0}$.

We require finally that T be a "*smoothing*" operator in the sense that T is also defined on H^s for $s < 0$ and for some $\varepsilon > 0$

$$T : H^s \to H^{s+\varepsilon} \text{ for all } s.$$

This means that T is a compact operator when considered on any H^s.

In order to simplify the statement of the theorems we set for $s, s_1 \geqq 0$,

$$E_h(s, s_1) = \| T - T_h \|_{-s,s_1} + \| T - T_h \|_{-s,0} \| T - T_h \|_{0,s_1} + \\ + \| T - T_h \|^2_{0,s_1}.$$

We state now the two main results.

Theorem 1: *Let $\{u_j\}_{j=1}^m$ be an orthonormal basis (in H^0) for $M(\mu)$ and let $0 \leqq s \leqq s_0$, $0 \leqq s_1$. Then there exist constants C_s and h_1 and an orthonormal basis $\{w_j\}_{j=1}^m$ for $M_h(\mu)$ such that*

$$\max_{1<j<m} \| u_j - w_j \|_{H^{-s}} \leqq C_s E_h(s, s_1),$$

for all $0 < h \leqq h_1$.

As an approximation to the eigenvalue μ we take the average

$$\hat{\mu}(h) = \frac{1}{m} \sum_{j=1}^m \mu_j(h).$$

Theorem 2: Let $s_1 \geqq 0$. *Then there exist constants* C and h_1 *such that for* $0 < h \leqq h_1$

$$| \mu - \hat{\mu}(h) | \leqq C E_h(s_0, s_1).$$

In [2] theorems on the convergence of individual eigenvalues and eigenvectors are given but will be omitted here. In general the convergence of the individual eigenvalues $\mu_j(h)$ to μ will not be as fast as the convergence of $\hat{\mu}(h)$ to μ.

APPLICATIONS OF THE PRECEDING THEOREMS

Let Ω be a bounded domain in E^N with smooth boundary $\partial\Omega$ and denote by $H^s(\Omega)$ and $H^s(\partial\Omega)$ the usual Sobolev spaces [cf. 1].

Let $\{S_h\}_{0<h<1}$ be a one parameter family of finite dimensional subspaces of $H^0(\Omega) = \mathscr{L}_2(\Omega)$. For given integers k and r with $0 \leqq k \leqq r$ we say that $\{S_h\}_{0<h\leqq 1}$

is of class $\mathscr{S}_{k,r}$ if $S_h \subset H^k(\Omega)$ for each h and there is a constant C independent of h such that for any $v \in H^t$ with $k \leqq t \leqq r$

$$\inf_{\chi \in S_h} \sum_{j=0}^{k} h^j \| v - \chi \|_j \leqq C h^t \| v \|_t .$$

Here $\| . \|_j$ denotes the norm on $H^j(\Omega)$.

Let L be a uniformly strongly elliptic second order operator with complex coefficients; i.e.,

$$L\varphi = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial \varphi}{\partial x_j}\right) + \sum_{j=1}^{N} b_j \frac{\partial \varphi}{\partial x_j} + c\varphi,$$

where a_{ij}, b_j and c are in $C^\infty(\bar{\Omega})$, $a_{ij} = a_{ji}$, and $\operatorname{Re} \sum_{i,j=1}^{N} a_{ij}(x)\, \xi_i \xi_j \geqq a_0 \sum_{i=1}^{N} \xi_i^2$ for some constant $a_0 > 0$, $x \in \Omega$ and all real $\xi_1, \ldots, \xi_N$. We assume that the corresponding quadratic form B is coercive over $H^1(\Omega)$; i.e., there is a positive constant C such that

$$\operatorname{Re} B(\varphi, \varphi) \geqq C \| \varphi \|_1^2 .$$

THE GENERALIZED FREE MEMBRANE EIGENVALUE PROBLEM

Let the complex number λ and $u \in H^2(\Omega)$, $u \neq 0$, satisfy

$$Lu = \lambda u \quad \text{in} \quad \Omega$$

$$\frac{\partial u}{\partial v} = 0 \quad \text{on} \quad \partial\Omega,$$

where $\dfrac{\partial}{\partial v}$ denotes the conormal derivative on $\partial\Omega$. Define $T : H^0(\Omega) \to H^1(\Omega)$ by $Tf = v$ where $f \in H^0(\Omega)$ and v is the unique solution in $H^1(\Omega)$ of

$$B(v, \varphi) = (f, \varphi) \quad \forall \varphi \in H^1(\Omega)$$

where $(.,.)$ is the inner product in $H^0(\Omega)$. Then $\lambda \neq 0$ and

$$Tu = \frac{1}{\lambda} u .$$

Now we define for each h with $0 < h \leqq 1$, $T_h : H_0(\Omega) \to S_h$ as $T_h f = v_h$ where $f \in$ $\in H^0(\Omega)$ and v_h is the unique solution in S_h of

$$B(v_h, \varphi) = (f, \varphi), \quad \forall \varphi \in S_h .$$

This is the standard Galerkin approximation to v. Now if λ has algebraic multiplicity m then for h sufficiently small we obtain $\lambda_1(h), \ldots, \lambda_m(h)$ such that $\lambda_j(h) \to \lambda$ as

$h \to 0, j = 1, \ldots, m$. Now Theorems 1 and 2 apply. It is shown in [1], for example, that if $k = 1$ and $r \geqq 2$,

$$\left| \lambda - \left(\frac{1}{m} \sum_{j=1}^{m} 1/\lambda_j(h) \right)^{-1} \right| \leqq Ch^{2r-2}.$$

The right hand side is obtained by estimating $E_h(s_0, s_1)$ for appropriate values of s_0 and s_1. This quantity involves norms of $T - T_h$ and the problem of estimating eigenvalue errors is reduced to estimating errors in the corresponding non-homogeneous boundary value problem. Details may be found in [1].

THE GENERALIZED FIXED MEMBRANE PROBLEM

Let the complex number λ and $u \in H^2(\Omega)$, $u \neq 0$, satisfy

$$Lu = \lambda u \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on} \quad \partial\Omega.$$

In this case the operator T is the solution operator in the corresponding Dirichlet problem.

Several methods for constructing the operator T_h are described in [1]. As shown there, T_h may be constructed by the standard Galerkin method, a method due to NITSCHE or a LAGRANGE multiplier method due to BABUŠKA. In each case the result is the same as that for the Neumann problem. As it has been noted in [1] this result is the same as that obtained in the selfadjoint case using the variational characterization of eigenvalues through the Rayleigh quotient.

Finally we mention that T_h may be taken to be the leastsquares solution operator introduced in [3]. In this case we obtain for $k \geqq 2$ and $r \geqq 4$

$$\left| \lambda - \left(\frac{1}{m} \sum_{j=1}^{m} (1/\lambda_j(h)) \right)^{-1} \right| \leqq Ch^{2r-4}.$$

In this case the approximate eigenvalue problem is non-selfadjoint even if the original problem were selfadjoint. Hence even in this case this result does not follow from standard arguments.

THE STEKLOV PROBLEM

Let the complex number λ and $u \in H^2(\Omega)$, $u \neq 0$, satisfy

$$Lu = 0 \quad \text{in} \quad \Omega$$
$$\frac{\partial u}{\partial \nu} = \lambda u \quad \text{on} \quad \partial\Omega.$$

Define $A : H^0(\partial\Omega) \to H^{3/2}(\Omega)$ by $Ag = v$ where v is the unique solution of

$$B(v, \varphi) = \langle g, \varphi \rangle \quad \forall \varphi \in H^1(\Omega).$$

Here $\langle .,. \rangle$ denotes the inner product of $\mathscr{L}_2(\partial\Omega) = H^0(\partial\Omega)$. We then define $Tg = (Ag)'$ where the prime denotes the restriction to $\partial\Omega$. It follows that $T : H^0(\partial\Omega) \to H^1(\partial\Omega)$.

Now let A_h be the Galerkin solution operator for the inhomogeneous Neumann problem; i.e., $A_h g = v_h$ where v_h is the unique solution in S_h of

$$B(v_h, \varphi) = \langle g, \varphi \rangle \quad \forall \varphi \in S_h.$$

Now take $T_h g = (A_h g)'$.

The operators T and T_h satisfy the conditions set forth in the general results and it is shown in [2] that, for example, for $k \geqq 1$ and $r \geqq 2$

$$\left| \lambda - \left(\frac{1}{m} \sum_{j=1}^{m} (1/\lambda_j(h)) \right)^{-1} \right| \leqq C h^{2r-2}.$$

We remark that this result may be obtained in the case L is formally selfadjoint by using the variational characterization of the eigenvalues through the Rayleigh quotient.

REFERENCES

[1] BRAMBLE, J. H. and OSBORN, J. E.: *Rate of convergence estimates for non-selfadjoint eigenvalue approximations,* Math. Comp. (in print.)

[2] BRAMBLE, J. H.: *Approximation of Steklov eigenvalues of non-selfadjoint second order elliptic operators,* The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A. K. Aziz, Editor, Academic Press, 1972.

[3] BRAMBLE, J. H. and SCHATZ, A. H.: *Rayleigh—Ritz—Galerkin methods for Dirichlet's problem using subspaces without boundary condition,* Comm. Pure Appl. Math., 23 (1970), pp. 653—675.

[4] DUNFORD, N. and SCHWARTZ, J. T.: *Linear Operators,* Part II, *Spectral Theory,* Interscience, New York, 1963.

Author's address:

J. H. Bramble
Cornell University
Ithaca, New York 14850
USA