John Robert Whiteman; Robert E. Barnhill

Finite element methods for elliptic mixed boundary value problems containing singularities

In: Miloš Ráb and Jaromír Vosmanský (eds.): Proceedings of Equadiff III, 3rd Czechoslovak Conference on Differential Equations and Their Applications. Brno, Czechoslovakia, August 28 - September 1, 1972. Univ. J. E. Purkyně - Přírodovědecká fakulta, Brno, 1973. Folia Facultatis Scientiarum Naturalium Universitatis Purkynianae Brunensis. Seria Monographia, Tomus I. pp. 261--267.

Persistent URL: http://dml.cz/dmlcz/700073

FINITE ELEMENT METHODS FOR ELLIPTIC MIXED BOUNDARY VALUE PROBLEMS CONTAINING SINGULARITIES

by J. R. WHITEMAN. and R. E. BARNHILL*

1. INTRODUCTION

This paper is concerned with Galerkin solutions to two dimensional mixed boundary value problems for Poisson's equation in which the function $u(x, y)$ satisfies

$$-\Delta[u(x, y)] = g(x, y), \qquad (x, y) \in \Omega, \tag{1.1}$$

$$u(x, y) = f_1(x, y), \qquad (x, y) \in \partial\Omega_1, \tag{1.2}$$

$$\frac{\partial u(x, y)}{\partial v} = f_2(x, y), \qquad (x, y) \in \partial\Omega_2, \tag{1.3}$$

where $\Omega \subset E^2$ is a simply connected open bounded domain with polygonal boundary $\partial\Omega$ satisfying a restricted cone condition, $\partial\Omega_1$ is the union of some of the sides of $\partial\Omega$ and $\partial\Omega_2 = \partial\Omega - \partial\Omega_1$, $f_1 \in L_2(\partial\Omega_1)$, $f_2 \in L_2(\partial\Omega_2)$ and $\partial/\partial v$ is the derivative in the direction of the outward normal to the boundary.

In recent years many finite element solutions with theoretical error bounds have been derived using both the Rayleigh–Ritz and Galerkin methods for Dirichlet problems for Poisson's equation, that is, the problem (1.1)–(1.3) with $\partial\Omega_2 = \emptyset$; see e.g. [1]–[7], [9], [10], [12], [18]–[20]. Much less work has been done on finite element methods for mixed problems of the above type, and correspondingly fewer results are available.

2. THE GALERKIN METHOD

In finite element analysis the region Ω is divided into *geometric elements*, having some generic length h, m internal nodes and n boundary nodes, over which piecewise approximation is used. We consider the Sobolev space $W^l_2(\Omega)$, l a non-negative integer, of functions with all l^{th} order generalized derivatives existing and in $L_2(\Omega)$. A norm on $W^l_2(\Omega)$ is

$$\| v \|_{W^l_2(\Omega)} = \{ \sum_{|\alpha| \leqq l} (\| D^\alpha v \|_{L_2(\Omega)})^2 \}^{1/2}, \tag{2.1}$$

.) This invited paper was presented by J. R. Whiteman at EQUADIFF 3 — the Czechoslovak Conference on Differential Equations and Their Applications — Brno, Czechoslovakia, August 28 — September 1, 1972.

*) The research of R. E. Barnhill was supported by the National Science Foundation with Grant GP 20293 to the University of Utah, by the Science Research Council with Grant B/SR/9652 at Brunel University, and by a N.A.T.O. Senior Fellowship in Science.

where $\alpha = (\alpha_1, \alpha_2)$, $D^\alpha = \partial^\alpha / \partial x^{\alpha_1} \partial y^{\alpha_2}$ and $|\alpha| \equiv \alpha_1 + \alpha_2$. The space of functions from $W_2^l(\Omega)$ which together with all their derivatives of order $\leqq l-1$ are identically zero on $\partial\Omega$ is $\mathring{W}_2^l(\Omega)$, and a norm on $\mathring{W}_2^l(\Omega)$ is

$$\| v \|_{\mathring{W}_2^l(\Omega)} = \{ \sum_{|\alpha|=l} (\| D^\alpha v \|_{L_2(\Omega)})^2 \}^{1/2}. \tag{2.2}$$

We define the bilinear functional $a(u, v)$ to be

$$a(u, v) = \iint_\Omega \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y \quad \text{for all} \quad u, v \in W_2^1(\Omega). \tag{2.3}$$

Then the weak problem corresponding to (1.1)–(1.3) is that of finding $u \in W_2^1(\Omega)$ satisfying (1.2) and (1.3) (i.e. $u \in W_2^1(\Omega) + f_1 + f_2$) such that

$$a(u, v) = (g, v) \quad \text{for all} \quad v \in \mathring{W}_2^1(\Omega). \tag{2.4}$$

The solution u of (2.4) is called the *generalized* solution of (1.1)–(1.3), and it is this that is approximated with the Galerkin technique.

Following [1] we let $\tilde{u}$ interpolate u, where the interpolation conditions are

$$\begin{aligned} L_i^h(\tilde{u}) &= L_i^h(u), \quad i = 1, \dots, m, \\ M_j^h(\tilde{u}) &= M_j^h(u), \quad j = 1, \dots, n, \end{aligned} \tag{2.5}$$

and L_i^h and M_j^h are interpolation functionals such that the $L_i^h(u)$ are unknown and the $M_j^h(u)$ are known from the boundary data (1.2) and (1.3). Let V^h be an $(m + n)$-dimensional subspace of $W_2^1(\Omega)$ such that the L_i^h and M_j^h are linearly independent over V^h. Then V^h has a basis of functions $\{B_i(x, y)\}_{i=1}^m$ and $\{C_j(x, y)\}_{j=1}^n$ that are biorthonormal with respect to L_i^h and M_j^h [8]. Let S^h be the subset of $W_2^1(\Omega)$ which consists of functions v of the form

$$v(x, y) = \sum_{i=1}^m a_i B_i(x, y) + \sum_{j=1}^n M_j^h(u) \, C_j(x, y),$$

where the a_i are constants, and let S_o^h be the m-dimensional subspace of $\mathring{W}_2^1(\Omega)$ generated by the B_i. In the Galerkin method we find $U \in S^h \subset W_2^1(\Omega) + f_1^h + f_2^h$ such that

$$a(U, v) = (g, v) \quad \text{for all} \quad v \in S_o^h, \tag{2.6}$$

where f_i^h means f_i at the boundary nodes, $i = 1, 2$.

Theorem. *The Galerkin approximation U is the best approximation from S^h to u in the energy norm induced by the inner product $a(u, v)$. That is,*

$$a(u - U, u - U) \leqq a(u - \tilde{u}, u - \tilde{u}) \quad \textit{for all } \tilde{u} \in S^h. \tag{2.7}$$

Now

$$U(x, y) = \sum_{i=1}^m A_i B_i(x, y) + \sum_{j=1}^n M_j^h(u) \, C_j(x, y),$$

and the equations used to calculate the A_i are

$$\sum_{i=1}^{m} A_i a(B_i, B_k) = (g, B_k) - \sum_{j=1}^{n} M_j^h(u)\, a(C_j, B_k), \qquad k = 1, 2, \dots, m. \tag{2.8}$$

The *normal equations* for the best approximation U are obtained from (2.8) with the substitution

$$a(U, B_k) = (g, B_k), \quad k = 1, 2, \dots, m$$

resulting from (2.6).

Let $\tilde{u}$ be an interpolant from S^h to u. Since from (2.7)

$$\| u - U \|_{\mathring{W}_2^1(\Omega)} \leqq \| u - \tilde{u} \|_{\mathring{W}_2^1(\Omega)}, \tag{2.9}$$

an upper bound on the interpolation remainder yields an upper bound on the Galerkin remainder. This is the relationship between remainder theory and finite element analysis.

Error bounds of the form

$$\| u - \tilde{u} \|_{W_2^1(\Omega)} \leqq K h^{l-1} \| u \|_{W_2^l(\Omega)}, \qquad l \geqq 2, \tag{2.10}$$

follow from the Bramble–Hilbert Lemma (see e.g. [7]). Numerical values for the constants K have been found for certain interpolants $\tilde{u}$ and regions Ω in [1]–[5].

3. BOUNDARY SINGULARITIES

The Galerkin error bound following from (2.9) and (2.10) involves a norm on the function u and certain of its derivatives. For the bounds to be meaningful and to imply convergence with decreasing mesh size of the Galerkin solution to the solution of (2.4), it is necessary for the norm of the function and derivatives to be finite. If $\partial\Omega$ is sufficiently smooth, this condition is satisfied. However, if $\partial\Omega$ contains a *re-entrant* corner with internal angle $\varphi = \alpha\pi/\beta$, $\alpha/\beta > 1$, then $u \notin W_2^2(\Omega)$. Suppose there is such a corner at a point 0 on $\partial\Omega_2$ and $f_2(x, y) \equiv 0$ on $\partial\Omega_2$. Then in terms of local polar co-ordinates (r, ϑ) with origin at 0 and zero angle along one of the arms of the corner, the asymptotic form of u may be written as

$$u(r, \vartheta) = \Sigma a_i \varphi_i(r, \vartheta), \tag{3.1}$$

see Lehman [11]. An interesting case is that for which $\varphi = 2\pi$, and (3.1) then becomes

$$u(r, \vartheta) = a_0 + a_1 r^{1/2} \cos \vartheta/2 + a_2 r \cos \vartheta + a_3 r^{3/2} \cos 3\vartheta/2 + \dots \tag{3.2}$$

From (3.1) it is clear that $\partial u/\partial r$ is unbounded at $r = 0$, that $u \in W_2^1(\Omega)$ but $u \notin W_2^2(\Omega)$, and so the error bound (2.10) is not applicable. The problem thus contains a *boundary singularity* at 0.

Following [3] and [5] we outline a method whereby some of the dominant part of the singularity in u is subtracted off in a neighbourhood $N(r_1) \subset \bar{\Omega}$ of 0, where for some fixed r_1 and $0 < r_0 < r_1$,

$$N(r_1) = \{(r, \vartheta) : 0 \leqq r \leqq r_1,\ 0 \leqq \vartheta \leqq \varphi\}.$$

We form the functions

$$w_i(r, \vartheta) = \begin{cases} \varphi_i(r, \vartheta), & 0 \leqq r \leqq r_0, \\ g_i(r)\, h_i(\vartheta) & r_0 \leqq r \leqq r_1, \\ 0 & r_1 < r, \end{cases} \tag{3.3}$$

$i = 1, 2, \ldots, N$, where N is discussed below and the φ_i are as in (3.1). The $g_i(r)$ are Hermite polynomials of degree 1 chosen so that each function $w_i(r, \vartheta)$ is in $W_2^1(\Omega)$, i.e. $g_i(r)$ is linear in $[r_0, r_1]$. The $h_i(\vartheta)$ are appropriate functions so that the w_i all satisfy the homogeneous boundary conditions on the arms of the corner. Using (3.3) we form the function

$$w = u - \sum_{i=1}^{N} c_i w_i(r, \vartheta), \tag{3.4}$$

and choose N so that w would be in $W_2^2(\Omega)$ if the c_i were known exactly. It is the function w that is approximated throughout Ω by the finite element solution U, and clearly if the c_i are known, making $w \in W_2^2(\Omega)$, the error bound (2.10) would then apply.

For the case $\varphi = 2\pi$ and the expansion of u as in (3.2), in order that w in (3.4) may be in $W_2^2(\Omega)$ the minimal sufficient N is 1 and only the function $\varphi_1(r, \vartheta) = r^{1/2} \cos \vartheta/2$ need be considered. The choice of trial functions affects only the left hand side of (2.10), that is $\| w - U \|_{W_2^1(\Omega)}$.

However, the c_i can unfortunately not be calculated exactly. In practice approximations are calculated by the method of augmenting with singular functions the trial function spaces in the Galerkin procedure. This was first suggested for the Rayleigh–Ritz method by Fix [9]. We discretize $\bar{\Omega}$ into triangular elements and let the space S^h of piecewise linear trial functions be augmented with singular functions. Thus in each element the trial functions are of the form

$$a + bx + cy + \sum_{i=1}^{N} c_i w_i(r, \vartheta).$$

By (3.3) these are the usual trial functions of S^h for elements in $\Omega - N(r_1)$. Extra equations are added to the linear system which when solved gives the finite element solution, and so only approximations $\tilde{c}_i$ to the c_i in (3.4) are obtained from the same numerical calculation as that which gives the values of U at the nodal points. Thus although we would like $u - \sum_i c_i w_i$ to be in $W_2^2(\Omega)$, we actually have $u - \sum_i \tilde{c}_i w_i \in W_2^1(\Omega) - W_2^2(\Omega)$ and, instead of having

$$\| (u - \sum_i c_i w_i) - U \|_{\mathring{W}_2^1(\Omega)} = \| w - U \|_{W_2^1} \leqq Kh \| w \|_{W_2^2(\Omega)},$$

we have on the left hand side $\| u - (\sum_i \tilde{c}_i w_i + U) \|_{\mathring{W}_2^1(\Omega)}$.

Hence the error bounds again do not apply since $w = u - \sum_i \tilde{c}_i w_i$ is in the same space as u. However, in a qualitative way by calculating good approximations to the c_i we are able to subtract off most of the singularity, and hence w is almost in $W_2^2(\Omega)$. The best approximation theorem of Section 2 applies to an arbitrary finite set of linearly independent functions in $W_2^1(\Omega) + f_1^h + f_2^h$, and so in particular to the space S^h augmented with singular functions. Thus the approximation $(U + \sum_i \tilde{c}_i w_i)$ is the *best approximation* to u in the $\mathring{W}_2^1$ norm from the augmented S^h.

4. MODEL MIXED PROBLEM

A much studied mixed problem (see [13]–[17]) of the type (1.1)–(1.3) which contains a boundary singularity is that in which the function $u(x, y)$ satisfies

$$-\Delta[u(x, y)] = 0,$$

in the square $-\pi/2 \leqq x, y \leqq \pi/2$ with the slit $y = 0, 0 \leqq x \leqq \pi/2$, and the boundary conditions

$$\frac{\partial u}{\partial y}(x, \pm\pi/2) = 0, \qquad -\pi/2 < x < \pi/2,$$

$$u(\pi/2, y) = \begin{cases} 1000, & 0+ \leqq y \leqq \pi/2, \\ 0, & -\pi/2 \leqq y \leqq 0-, \end{cases}$$

$$\frac{\partial u}{\partial x}(-\pi/2, y) = 0, \qquad -\pi/2 \leqq y \leqq \pi/2,$$

$$\frac{\partial u}{\partial y}(x, 0\pm) = 0, \qquad 0 < x < \pi/2.$$

This has a re-entrant angle ($\varphi = 2\pi$) at the origin, and the asymptotic form of u near the origin is as in (3.2). From the antisymmetry of the problem it suffices to consider only the upper region $\bar{\Omega} = \{(x, y) : |x| \leqq \pi/2, 0 \leqq y \leqq \pi/2\}$, and to add the boundary condition $u(x, 0) = 500, -\pi/2 \leqq x \leqq 0$.

For this problem WAIT and MITCHELL [13] use the Fix approach with rectangular elements and bilinear trial functions and augment first with one and then with two singular functions from (3.2). We have repeated this approach, but with right triangular elements using linear trial functions and augmenting with the second term of (3.2). As can be seen from the results of [3] there is a definite improvement in the Galerkin solution in the neighbourhood of the singularity through the inclusion of the singular term.

The advantage of using triangular rather than rectangular elements with the corresponding trial functions is that the computation is much simpler for comparable accuracy. This saving is even more valuable for higher order boundary value problems.

Acknowledgment

The authors wish to acknowledge helpful discussions with J. A. GREGORY.

REFERENCES

[1] BARNHILL R. E., and GREGORY J. A.: *Sard kernel theorems on triangular and rectangular domains with extensions and applications to finite element error bounds.* TR/11, Department of Mathematics, Brunel University, July 1972.

[2] BARNHILL R. E., GREGORY J. A., and WHITEMAN J. R.: *The extension and application of Sard kernel theorems to compute finite element error bounds.* Proceedings of O.N.R. Regional Symposium "Mathematical foundations of the finite element method with applications to partial differential equations", University of Maryland, Baltimore County, June 1973.

[3] BARNHILL R. E., and WHITEMAN J. R.: *Error analysis of finite element methods with triangles for elliptic boundary value problems.* In J. R. Whiteman (ed.), The Mathematics of Finite Elements and Applications, 83—112, Academic Press, London, 1972.

[4] BARNHILL R. E., and WHITEMAN J. R.: *Computable error bounds for the finite element method for eliptic boundary value problems.* Proceedings of Symposium "Numerische Methoden bei Differentialgleichungen", Mathematisches Forschungsinstitut, Oberwolfach, Deutschland, June 1972.

[5] BARNHILL R. E., and WHITEMAN J. R.: *Singularities due to re-entrant boundaries in elliptic problems.* Proceedings of Symposium "Numerische Methoden bei Differentialgleichungen", Mathematisches Forschungsinstitut, Oberwolfach, Deutschland, June 1972.

[6] BIRKHOFF G., SCHULTZ M. H., and VARGA R. S.: *Piecewise Hermite interpolation in one and two variables with applications to partial differential equations.* Numer. Math, 11, 232—256, 1968.

[7] BRAMBLE J. H., and ZLAMAL M.: *Triangular elements in the finite element method.* Math. Comp. 24, 809—820, 1970.

[8] DAVIS P. J.: *Interpolation and Approximation.* Blaisdell, New York, 1963.

[9] FIX G.: *Higher-order Rayleigh—Ritz approximations.* J. Math. Mech. 18, 645—657, 1969.

[10] KOLÁŘ V., KRATOCHVÍL J., ZLÁMAL M., and ŽENÍŠEK A.: *Technical, Physical and Mathematical Principles of the Finite Element Method.* Rozpravy Československé Akad. Věd. Řada Techn. Věd. 81 (1971)-2,

[11] LEHMAN R. S.: *Developments at an analytic corner of solutions of elliptic partial differential equations.* J. Math. Mech. 8, 727—760, 1959.

[12] VARGA R. S.: *The role of interpolation and approximation theory in variational and projectional methods for solving partial differential equations.* IFIP Congress 71, 14—19, North Holland, Amsterdam, 1971.

[13] WAIT R., and MITCHELL A. R.: *Corner singularities in elliptic problems by finite element methods.* J. Comp. Phys. 8, 45—52, 1971.

[14] WHITEMAN J. R.: *Treatment of singularities in a harmonic mixed boundary value problem by dual series methods.* Q. J. Mech. Appl. Math. 21, 41—50, 1968.

[15] WHITEMAN J. R.: *Numerical solution of a harmonic mixed boundary value problem by the extension of a dual series method.* Q. J. Mech. Appl. Math. 23, 449—455, 1970.

[16] WHITEMAN J. R.: *Finite-difference techniques for a harmonic mixed boundary problem having a re-entrant boundary.* Proc. Roy. Soc. Lond. A. 323, 271—276, 1971.

[17] WHITEMAN J. R., PAPAMICHAEL N., and MARTIN Q. W.: *Conformal transformation methods for the numerical solution of harmonic mixed boundary value problems.* Proc. Conf. Applications

of Numerical Analysis, Dundee. Lecture Notes in Mathematics No. 228, Springer-Verlag, Berlin, 1971.

[18] ŽENÍŠEK A.: *Interpolation polynomials on the triangle,* Numer. Math. 15, 283–296, 1970.

[19] ZLÁMAL M.: *On the finite element method.* Numer. Math. 12, 394–409, 1968.

[20] ZLÁMAL M.: *Some recent advances in the mathematics of finite elements.* In J. R. Whiteman (ed.), The Mathematics of Finite Elements and Applications, 59–81, Academic Press, London, 1973.

Author's address:

John R. Whiteman
Department of Mathematics
Brunel University
Uxbridge
Middlesex
England