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Some Aspects of Products of Derivatives

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1. INTRODUCTION. In 1921, Wilcosz [**W**] showed that the function $f(x) = \cos 1/x$ ($f(0) = 0$) is a derivative, but the function f^2 is not. (Saying f "is," rather than "has," a derivative means that there is a differentiable function F such that $F'(x) = f(x)$ for all x.) The Wilcosz example shows simultaneously that the class of derivatives is not closed under multiplication nor under outside composition with continuous functions. As the title suggests, this article deals primarily with the first consequence. However, concerning the second, it is natural to seek functions φ such that for each derivative f the composition $\varphi \circ f$ is again a derivative. It is obvious that linear functions φ have this property. However, it is not difficult to prove that there are no other possibilities; every such function φ is linear.

The Wilcosz example has other consequences as well. It is well known that a function f is continuous if and only if each of its associated sets, i.e., sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$, where a is any real number, is open. One might be tempted to find a similar characterization for derivatives; in other words, to prove that a function is a derivative if and only if each of its associated sets has a certain property. The Wilcosz example can be used to show that there is no such theorem. Namely, if f is that function and if $F = f + 1$, then, since $F \geqslant 0$, F and F^2 have the same system of associated sets while F is a derivative but F^2 is not.

Incidentally, the theorem about the outside composition mentioned above yields another way, although a little less elementary, of showing that such a characterization of derivatives is not possible. Namely, if φ is any nonlinear, continuous, increasing function on $\mathbb{R}$ with range $\mathbb{R}$, then, for some derivative f, the composition $\varphi \circ f$ is not a derivative while, obviously, f and $\varphi \circ f$ have the same system of associated sets. A more complete treatment of this associated sets problem and a discussion of the topological character of the class of derivatives together with some applications can be found in [**B**, pp. 135–144].

The fact that the class of derivatives is not an algebra raises a number of interesting questions, some of which have been studied only in the past few years. The purpose of this article is to state these questions, to try to impart some of the flavor of the subject to the reader, and to indicate applications of some of the results. We shall try to present the material in a nontechnical, expository manner.

2. FOUR QUESTIONS. We shall denote by Δ the class of differentiable functions on $\mathbb{R}$ and by Δ' the class of derivatives. Thus, $f \in \Delta'$ if and only if there exists $F \in \Delta$ such that $F'(x) = f(x)$ (finite) for all $x \in \mathbb{R}$. The Wilcosz example immediately raises the following two questions.

Question 1. If f and g are in Δ', what else should be required of one or both of them to conclude that $fg \in \Delta'$?

Question 2. Given that the product of derivatives need not be a derivative, what functions f admit a representation of the form $f = f_1 f_2 \cdots f_n$ ($f_1, f_2, \ldots, f_n$ all in Δ')?

These two questions lead to the next two.

Question 3. What other algebraic representations of functions by derivatives are of interest?

Question 4. What functions are in Alg Δ', the algebra generated by the derivatives?

These questions are the obvious ones to ask, but attempts to solve them have led to some surprisingly deep mathematics. The first one has the longest history; we discuss it in Section 4. The other three have been investigated only recently and we treat them in Sections 5, 6 and 7. The next section contains necessary information which may not be known to some readers.

3. SOME NEEDED FACTS. First, we recall that every continuous function is in Δ'. Of course not every function in Δ' is continuous, but every member of Δ' has the Intermediate Value (or Darboux) Property. It should be emphasized that a derivative can behave rather "unreasonably." For example, a derivative need not be locally summable (that is, locally Lebesgue integrable). We will soon see examples of functions $f \in \Delta'$ that are continuous on $(0, 1]$ such that $\int_0^1 |f| = \infty$.

When we deal with derivatives we often come across an essential, but not well-known concept, namely, approximate continuity which is defined next.

Let m be Lebesgue measure. Saying "a function f is approximately continuous at a point x" means that there is a Lebesgue measurable set E such that

$$\lim_{h \to 0} + m(E \cap (x - h, x + h))/2h = 1 \tag{1}$$

and that $\lim_{y \to x,\, y \in E} f(y) = f(x)$. So ordinary continuity is weakened by requiring that $f(y)$ converges to $f(x)$ only as y approaches x through a subset E; one that is "dense" enough at x so that, among other things, the limit is unique (does not depend on the choice of E). The set E is also dense enough to guarantee that the sum and the product of two functions approximately continuous at x are again approximately continuous at x. In what follows, "a function is approximately continuous" will mean that it is approximately continuous everywhere (i.e. at each point in $\mathbb{R}$). For the purpose of this article it is important to know that every bounded, approximately continuous function is in Δ'. (Every such function is the derivative of its indefinite Lebesgue integral.) Finally, we state the following two important facts. The second is somewhat deeper than the first.

Fact 1. If F is differentiable and monotone, then its derivative F' is locally summable. Consequently, if f is nonnegative and not locally summable, then $f \notin \Delta'$.

Fact 2. If $F \in \Delta$ and if F' is summable on $[a, b]$, then $\int_a^b F' = F(b) - F(a)$. Therefore, if a locally summable function is a derivative, it is the derivative of its indefinite integral.

4. MULTIPLIERS FOR Δ'. The Wilcosz example shows that the product of two derivatives f and g need not be a derivative. What happens if we require more of one of the factors, say f? Can we then conclude that $fg \in \Delta'$ for all $g \in \Delta'$? If so, we would call f a "multiplier" for the class of derivatives. What sort of regularity conditions would imply that a function is a multiplier for Δ'? It is not hard to see that continuity is not enough. So let us suppose more; for example, that the first factor, now denoted by F, is differentiable (i.e., $F \in \Delta$). Does this imply that $Fg \in \Delta'$ for all $g \in \Delta'$? If one believes that differentiability provides enough regularity, then, in view of Fact 2, one would perhaps try to prove that if $H(x) = \int_0^x Fg$, then $H'(x) = F(x)g(x)$ for all x. This may seem a plausible approach, but one immediately encounters difficulties involving the summability of the integrand (even g need not be summable). This difficulty, together with Fact 1, actually provides a clue toward obtaining a counter-example. We need only construct F and g so that the product Fg is nonnegative and not locally summable. No function H could meet the requirement that H' is nonnegative (everywhere) and not locally summable. Such combinations of functions F and g are easy to find using properties of functions of the form $x^n \sin x^{-m}$ and $x^n \cos x^{-m}$. For example, if

$$F(x) = x^2 \sin x^{-5} \qquad (F(0) = 0)$$

and

$$G(x) = \tfrac{1}{5}x^2 \cos x^{-5} - \tfrac{2}{5}\int_0^x t \cos t^{-5}\,dt \qquad (G(0) = 0),$$

then the function $g = G'$ fulfills the relations $g(x) = x^{-4} \sin x^{-5}$, $g(0) = 0$ and $(Fg)(x) = x^{-2} \sin^2 x^{-5}$ $((Fg)(0) = 0)$. This product is nonnegative and (as can be easily shown) not summable in any neighborhood of the origin. Thus Fg cannot be a derivative.

What happens if we remove the apparent problem in our example? That is, if we require Fg to be locally summable, can we conclude that $Fg \in \Delta'$? According to Fact 2, we must then try to prove that if $H(x) = \int_0^x Fg$, then $H'(x) = F(x)g(x)$ for all x. After some unsuccessful attempts we may arrive at the following example: Let $F(x) = x^2 \sin x^{-3}$, $G(x) = x^2 \cos x^{-3}$ $(F(0) = G(0) = 0)$. We verify easily that FG' and GF' are bounded and therefore summable on any bounded interval. Straightforward calculations show that

$$F(x)G'(x) - F'(x)G(x) = \begin{cases} 3, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

If either of the functions FG' or GF' were a derivative, then the other would be also since $FG' + GF' = (FG)' \in \Delta'$, and the same would be true of their difference. But it is not, since derivatives have the Darboux property.

So we are forced to assume even more about F. Suppose that F' is continuous. Let G be a primitive of g (i.e. $G' = g$). Then, obviously, $Fg = (FG)' - F'G$. The function $F'G$ is continuous and $(FG)' \in \Delta'$. Thus $Fg \in \Delta'$ and we have our first positive result!

(A_1) If $g \in \Delta'$ and F' is continuous, then $Fg \in \Delta'$.

More generally, one can prove

(A_2) If $g \in \Delta'$ and F' is locally summable, then $Fg \in \Delta'$.

Thus, such functions F are multipliers for Δ'. So we see that local summability is relevant—but for F' rather than for Fg.

Getting to the essence of (A_2), if F' is locally summable, then, as is easily proved, F is the difference of two continuous nondecreasing functions. We see that (A_2) follows from the next assertion:

(A'_2) If $g \in \Delta'$ and if F is continuous and nondecreasing, then $Fg \in \Delta'$.

(If $G' = g$ and if $H(x) = F(x)G(x) - \int_0^x G\,dF$, then $H(x+h) - H(x) = F(x+h)(G(x+h) - G(x)) - \int_x^{x+h}(G - G(x))\,dF$ which easily implies that $H' = Fg$.)

Using the product formula we obtain from (A_2) a companion theorem:

(A_3) If $F \in \Delta$, $g \in \Delta'$ and if g is locally summable, then $Fg \in \Delta'$.

The assertion (A_3), however, is not of the same type as (A_1), (A_2), and (A'_2). In (A_1), (A_2) and (A'_2) we impose conditions only on F whereas in (A_3) we require also local integrability of g.

It is natural to ask whether we can improve (A_3) by weakening the requirement that $F \in \Delta$ to simply that F be continuous. The following example shows that we cannot.

Let

$$F(x) = \sqrt{x}\cos\frac{1}{x}, \qquad g(x) = \frac{1}{\sqrt{x}}\cos\frac{1}{x} \qquad (x > 0),$$

$$F(x) = g(x) = 0 \qquad (x \leqslant 0).$$

Then F is continuous and one can calculate that g is a locally summable derivative. Yet

$$(Fg)(x) = \begin{cases} \cos^2\dfrac{1}{x} & (x > 0) \\ 0 & (x \leqslant 0), \end{cases}$$

a function which, according to Wilcosz, is not a derivative.

If, however, we require g to be nonnegative (which, by Fact 1, implies that g is locally summable), then we can conclude that $Fg \in \Delta'$:

(A_4) If $g \in \Delta'$, $g \geqslant 0$ and if F is continuous, then $Fg \in \Delta'$.

It is easy to see that the zero function in (A_4) can be replaced by any nonpositive derivative. In this way we obtain the following generalization of (A_4):

(A_5) If $g, h \in \Delta'$, $g \geqslant h$, $h \leqslant 0$ and if F is continuous, then $Fg \in \Delta'$.

It is also easy to see that the following three properties of a function $g \in \Delta'$ are equivalent:

(i) There is an $h \in \Delta'$ such that $h \leqslant 0$ and $h \leqslant g$.

(ii) There are $h_1, h_2 \in \Delta'$ such that $h_1 \geqslant 0$, $h_2 \geqslant 0$ and $g = h_1 - h_2$.

(iii) There is an $h \in \Delta'$ such that $|g| \leqslant h$.

These conditions suggest obvious modifications of (A_5).

The preceding results may be formulated also in another way, if we speak about multipliers for subclasses of Δ'. A function f is said to be a multiplier for such a subclass S, in symbols $f \in M(S)$, if and only if $fg \in \Delta'$ for all $g \in S$. Using this terminology we get the following:

(A'_2) A continuous, nondecreasing function is a multiplier for Δ'.

(A_3) A differentiable function is a multiplier for locally summable derivatives.

(A_4) A continuous function is a multiplier for nonnegative derivatives.

We have also seen that a continuous function need not be a multiplier for locally summable derivatives (such a derivative need not be the difference of two nonnegative derivatives).

For certain subclasses of Δ' the multipliers have been completely characterized. For example, let S_0 be the set of all locally bounded derivatives. Then $M(S_0)$ is the set L of Lebesgue functions, i.e., functions f such that

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| \, dt = 0$$

for each $x \in \mathbb{R}$. It is easy to prove that each element of L is locally summable and approximately continuous, that $L \subset \Delta'$ and that each bounded approximately continuous function is in L. Surprisingly, the "dual" statement $M(L) = S_0$ is also valid.

It can be proved that $M(\Delta')$ is the class of all derivatives F such that

$$\limsup_{n \to \infty} \left(\operatorname{var}\left(F, \left[x + \frac{1}{n}, x + \frac{2}{n} \right] \right) + \operatorname{var}\left(F, \left[x - \frac{2}{n}, x - \frac{1}{n} \right] \right) \right) < \infty$$

$$\text{for each } x \in \mathbb{R}. \quad (2)$$

The multipliers for the class of all summable derivatives can be characterized in a similar way.

Our results and our examples give a sense of the delicacy of determining conditions on two functions $F, g \in \Delta'$ such that $Fg \in \Delta'$. We are looking for some regularity conditions that, when imposed on F, would imply that $Fg \in \Delta'$ for each $g \in \Delta'$, or for each $g \in S$, where S is a given class of derivatives. However, such conditions have sometimes surprisingly little to do with continuity or differentiability of F. It is easy to construct a discontinuous derivative F fulfilling (2); thus continuity is not a necessary condition for being a member of $M(\Delta')$. On the other hand, we have seen that differentiability is not sufficient.

A different notion of multipliers has also been studied. A function f is sometimes called a multiplier for S if and only if $fg \in S$ for each $g \in S$. In this setting, the multipliers of locally bounded derivatives consist of the locally bounded approximately continuous functions.

It is obvious that these two definitions of multipliers yield the same result, if $S = \Delta'$; an analogous assertion holds also, if S is the class of locally summable derivatives.

The interested reader may wish to consult Fleissner [**F**]. This survey article was current at the time it was written.

Some of the results we have mentioned can be found in [**Mi** 1–4]; the proof of the relation $M(S_0) = L$ and the characterization of $M(S)$ for some other classes S have yet to be published.

5. REPRESENTATIONS AS PRODUCTS OF DERIVATIVES. Since the product of two or more derivatives need not be a derivative, it is natural to ask what functions admit such a representation. Now any $f \in \Delta'$ is in B_1, the first class of Baire (that is, it is the pointwise limit of a sequence of continuous functions) and it has the Darboux property as has already been mentioned. Since B_1 is an algebra, any product of derivatives must also be in B_1. What can we say about the Darboux property for the product? The fact that the product of two functions with the Darboux property need not have that property suggests that the product of two derivatives need not have the Darboux property. On the other hand, in spite of the fact that the *quotient* of two functions with the Darboux property need not have that property, the quotient of two derivatives will have the Darboux property (if the denominator is never zero) [**Hr**]. This suggests that products of derivatives may have the Darboux property.

Let us first try to settle this question by considering the simplest sort of function without the Darboux property. Let

$$h(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

That is, h is the characteristic function of the origin, $\chi_{\{0\}}$. If we wish to express h in the form $h = fg$ ($f, g \in \Delta'$), then f must be zero whenever g is not (except at $x = 0$). It is not difficult to construct two differentiable functions F and G, both of whose graphs are trapped between the curves $y = x$ and $y = x^2 + x$ such that for every $x \neq 0$ there is an interval containing x on which F or G is constant. Then $F'(x)G'(x) = 0$, if $x \neq 0$. Clearly $F'(0) = G'(0) = 1$. This provides the desired construction, $h = F'G'$.

Another (more "arithmetical") such representation is the following: Let

$$f(x) = \max\left(\pi \sin\frac{1}{x}, 0\right), \qquad g(x) = \max\left(-\pi \sin\frac{1}{x}, 0\right) \qquad (x \neq 0),$$

$$f(0) = g(0) = 1.$$

It is not difficult to prove that $f, g \in \Delta'$ and that $fg = \chi_{\{0\}}$. So the product of two derivatives need not have the Darboux property.

Using refined versions of either of these two arguments, one can actually prove that if K is any closed set, then χ_K is the product of two derivatives.

What other simple non-Darboux functions are the product of two derivatives? What about χ_U, where U is an open set, say $U = (0, \infty)$?

Suppose $h = \chi_U$ and $h = fg$, $f, g \in \Delta'$. Then f and g have the same sign on $(0, \infty)$. Let $x > 0$. Since f and g are in Δ', both are summable on $[0, x]$ according to Fact 1. It follows from the Cauchy-Schwarz inequality that

$$x^2 = \left(\int_0^x \sqrt{fg}\right)^2 \leqslant \left(\int_0^x f\right)\left(\int_0^x g\right) = (F(x) - F(0)) \cdot (G(x) - G(0)),$$

where $F' = f$ and $G' = g$. Hence $f(0)g(0) = F'(0) \cdot G'(0) \geqslant 1$, a contradiction.

This shows not only that h is not the product of two derivatives, but also that if h were redefined at 0 to be such a product, it would have to satisfy $h(0) \geqslant 1$. Similar arguments show that h is not the product of any number of derivatives.

We have arrived at the following comparison: $\chi_{[0,\infty)}$ can be expressed as the product of two derivatives but $\chi_{[0,\infty)}$ cannot be expressed as the product of any number of derivatives. Yet these two functions, in addition to differing at only one point, are closely related by various identities; for example, $\chi_{[0,\infty)}(x) + \chi_{[0,\infty)}(-x) = 1$ for each x.

The mentioned results concerning characteristic functions are special cases of Corollary 3.7, page 33 of [**BMW**]. Also see [**Mi5**]. A more general (but still not-too-technical) special case is the following:

Theorem. *Let $u > 0$ on $[0, \infty)$, let u be continuous on $(0, \infty)$ and constant on $(-\infty, 0]$. There exist nonnegative numbers $q_2 \geqslant q_3 \geqslant q_4 \geqslant \cdots$ such that if $u(0) \geqslant q_k$, then u can be expressed as the product of k derivatives but if $u(0) < q_k$, no such representation is possible.*

Explicit values of the numbers q_k are given in [**MW**].

As an illustration of this theorem let us consider a function u with the following properties: Let $0 < a < b$, $\alpha, \beta > 0$, $\alpha + \beta = 1$. Let u be continuous on $(0, \infty)$, constant on $(-\infty, 0]$ and let

(i) $a \leqslant u \leqslant b$ on $(0, \infty)$,

(ii) $$\lim_{h \to 0^+} \frac{m\{x \in (0,h) : u(x) = a\}}{h} = \alpha,$$

(iii) $$\lim_{h \to 0^+} \frac{m\{x \in (0,h) : u(x) = b\}}{h} = \beta.$$

(It is not difficult to construct such a function u.) Then one can calculate (using Prop. 5.3 and Remark 1 on page 367 of [**MW**]) that $q_n = (\alpha a^{1/n} + \beta b^{1/n})^n$. An elementary application of L'Hôpital's Rule yields the result $q_n \to a^\alpha b^\beta$. (See also Prop. 6.6 of [**MW**].) For example, if $a = 1$, $b = 4$, $\alpha = \beta = 1/2$, then $q_2 = 9/4$ and $q_n \to 2$.

In this example, if $u = 5/2$ on $(-\infty, 0]$, then u is a derivative; if $u = 9/4$ on $(-\infty, 0]$, then u is not a derivative but can be expressed as the product of two derivatives, and as $u(0)$ decreases, the number of factors in a representation of u as a product of derivatives increases. When $u(0) \leqslant 2$, no such representation exists.

Let P be the set of all functions that can be expressed as the product of (finitely many) derivatives. How big is P in B_1? Let us equip B_1 with the topology of uniform convergence. Our function u with $u(0) = 2$ is not in P, but, obviously, is in its closure. Hence (as we could expect) P is not closed.

We have indicated that the characteristic function of a nonempty open set $G \neq \mathbb{R}$ is not in P. Similarly, the following can be proved: If $c \in \mathbb{R}$, $\varepsilon \in [0, \infty)$ and if f is a function such that $\varepsilon < \liminf f(x)$ $(x \to c+)$, then $f \notin P$. It is easy to see that such an f is not even in the closure of P. Using the fact that each Baire one function has points of continuity we now see that P is nowhere dense in B_1.

On the other hand, P contains some rather complicated functions. For example, every Baire 1 function that is zero almost everywhere (a.e.) is the product of two derivatives [**BMW**]. This fact provides a very simple solution to a problem which at one time baffled some of the leading mathematicians of the day. Let us discuss this problem briefly and then show how our theorem on products of derivatives provides a simple solution.

Over one hundred years ago DuBois-Reymond held the view that a differentiable function must be monotone on some interval. Dini, on the other hand, believed the existence of nowhere monotone differentiable functions highly probable. (See [**Ho**], page 412.) In 1887, Koepcke provided a construction of such a function [**K**]. In discussing Koepcke's work, Denjoy wrote in 1915 [**D1**], "In 1887, Koepcke gave in Math. Annalen an example of a function possessing at each point (or so he thought) a derivative which vanished and took both signs in every interval contained in its domain of definition. This geometer returned to this subject on several occasions, correcting each time the errors contained in the previous proofs." This question of differentiable nowhere monotone functions has also provoked many other works.

The Koepcke constructions Denjoy referred to were quite complicated. They were later simplified by Pereno and other mathematicians. Denjoy then gave four separate constructions of his own, which were also quite complicated.

Hobson modified Pereno's modification of Koepcke's construction in the second edition of his book [**Ho**]. This edition was published in 1921, about forty years after

Koepcke's first correction, thirty years after Pereno's modification and fifteen years after Denjoy's several developments. It required ten pages!

Today a number of faster proofs of the existence and constructions of differentiable nowhere monotone functions exist. Here is a quick one based on the result we mentioned; namely that each Baire 1 function that equals 0 a.e. is the product of two derivatives.

Let

$$h(x) = \begin{cases} \dfrac{1}{q} & \text{if } x = \dfrac{p}{q},\ q \text{ even} \\ -\dfrac{1}{q} & \text{if } x = \dfrac{p}{q},\ q \text{ odd,} \\ 0 & \text{elsewhere,} \end{cases}$$

where p and q are relatively prime integers and $q > 0$. Then h is continuous except on a denumerable set, and therefore in Baire class 1 [**N**]. Clearly h is zero a.e. According to the result alluded to, there exist $f, g \in \Delta'$ such that $h = fg$. If f takes both signs on each interval, then a primitive of f is the desired function. If not, then there is an interval I on which f is unsigned. But since h takes both signs on dense subsets of I, so does g and then a primitive of g is the desired function.

6. OTHER REPRESENTATIONS BY DERIVATIVES. We have seen that the characteristic function of a proper nonempty open subset of $\mathbb{R}$ cannot be expressed as the product of any number of derivatives. If we allow addition as well as multiplication, then such a function can be expressed in terms of derivatives as we shall now see. Recall that if $F(x) = x^2 \sin x^{-3}$ and $G(x) = x^2 \cos x^{-3}$ for $x > 0$ and $F(x) = G(x) = 0$ for $x \leqslant 0$, then $FG' - F'G = 3\chi_{(0,\infty)}$. Of course $(FG)' = FG' + F'G$. Thus $2FG' - (FG)' = 3\chi_{(0,\infty)}$; that is, there are functions $F, G, H \in \Delta$ such that $\chi_{(0,\infty)} = FG' + H'$. It will not surprise the reader to learn that the characteristic function of any open set can be written in the same fashion. It follows that the characteristic function of any closed set can also be thusly written.

Another representation of $\chi_{(0,\infty)}$ may interest the reader. In the previous section we have encountered bounded derivatives f and g with $fg = \chi_{\{0\}}$. Let us define functions f_1, g_1 setting $f_1 = g_1 = 1$ on $(-\infty, 0)$ and $f_1 = f$, $g_1 = g$ on $[0, \infty)$. It is easy to see that f_1 and g_1 are bounded derivatives and that $f_1 g_1 = \chi_{(-\infty,0]}$. Hence $\chi_{(0,\infty)} = 1 - f_1 g_1$. Our previous representation $\chi_{(0,\infty)} = FG' + H'$ is, in some sense, better; we multiply the derivative G' by a "more reasonable" function. However, the function G' is obviously unbounded. It is worth mentioning that the unboundedness of G' was not caused by our awkwardness. No matter how we represent the function $\chi_{(0,\infty)}$ in the form $FG' + H'$ with $F, G, H \in \Delta$, G' must be unbounded (in fact, not even locally integrable); because if it were bounded, we would have (see (A_3) in section 4) $FG' \in \Delta'$ and hence $\chi_{(0,\infty)} \in \Delta'$ which is impossible.

The association of open sets with functions admitting this type of a representation is intrinsic as the following theorem from [**ABBM**] shows.

Theorem. *Let* $\Phi: \mathbb{R} \to \mathbb{R}$. *The following two conditions are equivalent*:

1. *There are* $F, G, H \in \Delta$ *such that* $\Phi = FG' + H'$.
2. *There is an open set* U, *a function* $K \in \Delta$, *and a function* L *differentiable on* U *such that* $\Phi = L'$ *on* U *and* $\Phi = K'$ *on* $\mathbb{R} \setminus U$.

It follows from (A_2) (section 4) that if $F \in \Delta$ has a summable derivative, then $FG' + H'$ is a derivative, which, along with condition 2 shows that functions of the form $FG' + H'$ $(F, G, H \in \Delta)$ are very close to being derivatives. This fact can be exploited to show that some desirable properties are possessed by certain classes of functions that arise naturally in differentiation theory. For example, every approximately continuous function satisfies conditions 1 and 2. The same is true of functions in O'Malley's class B_1^* (see **[O]**) which are also called generalized continuous functions. A function f is in B_1^* if to every nonempty closed set E there corresponds an open interval I intersecting E such that $f|I \cap E$ is continuous. It is well-known (see **[N]**) that a function f is in B_1 if and only if every nonempty closed set E contains a point x such that $f|E$ is continuous at x. So $B_1^* \subset B_1$. Actually the class B_1^* is much smaller than the class B_1. To see this, let us denote by V the system of all functions Φ with the property 1 (or 2). It is obvious that $V \supset \Delta'$ and (because B_1 is an algebra) that $V \subset B_1$. We can easily construct a derivative with a dense set of points of discontinuity and we have a function that is in V, but not in B_1^*. On the other hand, every function in V is a derivative on some interval. It follows that no increasing function with a dense set of points of discontinuity is in V. Thus we see that the inclusions $B_1^* \subset V \subset B_1$ are proper in a rather strong sense.

We know that there are functions $F, G \in \Delta$ such that $FG' \notin \Delta'$; we have, of course, $FG' \in V$. Moreover, it follows from 2 that V contains some functions that do not have the Darboux property. So V is also "much bigger" than Δ'.

Of particular interest is the fact that the so-called approximate derivatives are in V.

The approximate derivative is the most thoroughly studied generalized derivative. It serves as an excellent substitute for the ordinary derivative when the latter is not known to exist. To say that f is approximately differentiable at x with approximate derivative $f'_{ap}(x)$ means that there is a set E satisfying the same conditions as in the definition of approximate continuity such that

$$\lim_{y \to x,\, y \in E} \frac{f(y) - f(x)}{y - x} = f'_{ap}(x).$$

The reason that f'_{ap} is such a good substitute for the ordinary derivative is that it shares all the known desirable properties of ordinary derivatives. This fact was established, in pieces, by various authors **[D2]**, **[C]**, **[Mc]**, **[We1]**, **[We2]**, **[P1]**. Moreover, one has, for example, the result that any monotonicity theorem valid for differentiable functions has a complete analogue for approximately differentiable functions **[OW]**.

Much of this good behavior of f and f'_{ap} can be understood by the fact that f'_{ap} satisfies conditions 1 and 2 above. For example, one sees immediately that $f \in B_1^*$ and that f is differentiable on a dense open set.

When dealing with a class S of functions, one often wonders whether the members of S must remain in S when "perturbed" algebraically or topologically; that is, is S closed under the perturbations under consideration? For many classes the answer to specific questions of this type is often an unqualified "yes." For classes whose definitions involve the notion of derivative, the answer is usually "only in exceptional cases". The class Δ' is sensitive to algebraic and topological perturbations. We have seen, for example, that multiplication of a derivative by even a differentiable function can result in a function that is not a derivative. We have also seen that compositions of functions in Δ' with homeomorphisms may result in functions that are not derivatives. We mentioned in Section 1 that if $\varphi \circ f \in \Delta'$ for every $f \in \Delta'$, then φ is linear. As a further example, if $f \in \Delta'$, and $\varphi \circ f \in \Delta'$ for *some* strictly convex φ, then f is approximately continuous. (Thus, the reciprocal of a positive derivative is usually not a derivative.) For inner compositions we mention that if $f \in \Delta'$ and $f \circ h$ is a derivative for every homeomorphism h, then f is continuous. (These results can all be found in [**B**]).

Recent results involving the representation of functions by derivatives provide illustrations of a similar phenomenon. The general idea can be roughly described in the following way. If a well-behaved function is expressed algebraically in terms of several derivatives, then these derivatives are themselves well-behaved. (This statement is, of course, a vague one and shouldn't be taken too literally.) We present some illustrations. But first we remark that within the class of bounded derivatives, the class of approximately continuous (a.c.) functions is "small"; more precisely, it is a nowhere dense subset when the bounded derivatives are equipped with the sup norm.

We have seen that the product of several derivatives may be rather badly behaved. The Baire one functions that vanish almost everywhere can serve as an illustration. (We mentioned in Section 5 that every such function f is the product of two derivatives. If, moreover, $f \geqslant 0$, then both factors can be taken to be nonnegative.)

What happens if the product is well-behaved? It is clear that the approximate continuity alone would not help much; the product of two very wild functions can be identically zero. We have, however, the following result [**MW**]: If the product is a.c. and positive, then each factor is a.c. This result actually holds "pointwise": If $f_k \in \Delta'$ for all $k = 1, \ldots, n$, if the function $f = \prod_{k=1}^{n} f_k$ is a.c. at x_0 and if $f > 0$, then each f_k is a.c. at x_0.

It is natural to ask various analogous questions. For example: What can we say about derivatives f and g, if we know that the sum of their squares is well behaved? One possible answer is contained in the following theorem: Let $f, g, h \in \Delta'$ and let ε be a positive number such that (everywhere) $f^2 + g^2 = h^2 \geqslant \varepsilon$. Then both ratios f/h, g/h are a.c. If, in particular, h is a.c., then also f and g are a.c.

In a similar way it can be proved that derivatives f and g are in L (= Lebesgue functions) if and only if $(f^2 + g^2)^{1/2} \in L$. Or, equivalently: Let $h \in L$. Then the set of all pairs (f, g) of derivatives such that $f^2 + g^2 = h^2$ is identical with the set of all pairs (f, g) of Lebesgue functions fulfilling the equation.

Instead of squares we may, of course, investigate also other powers; the corresponding results are sometimes even better. For example, the following theorem holds: Let $f, g, h \in \Delta'$, $h \geqslant 0$ and let $f^4 + g^4 = h^2$. Then $f, g, h \in L$ (in particular, f, g, and h are all a.c.).

7. THE ALGEBRA GENERATED BY Δ'. We have already seen that many types of Baire 1 functions can be represented algebraically by derivatives. This leads naturally to Question 4: What functions are in Alg Δ', the algebra generated by the derivatives? Since $\Delta' \subset B_1$ and B_1 is an algebra, it is clear that Alg $\Delta' \subset B_1$. It is also not difficult to verify that the class B_1^* mentioned in section 6 is uniformly dense in B_1. One need only observe (see [**ABBM**], Lemma 5 and Proposition 3) that a Baire 1 function with isolated range is in B_1^*. Since each $f \in B_1^*$ admits the representation $f = gh' + k'$ $(g, h, k \in \Delta)$, it is clear that Alg Δ' is uniformly dense in B_1. This suggests that perhaps Alg $\Delta' = B_1$. On the other hand, there is a good deal of evidence that might cause one to believe that B_1 is much larger than Alg Δ'. For one thing, Baire 1 functions can exhibit a great deal more pathology than can any derivative. For another, Δ' is closed with respect to uniform convergence from which it follows without much difficulty that Δ' is nowhere dense in B_1 (in the topology of uniform convergence). Finally, if it is true that Alg $\Delta' = B_1$, then there is an integer N such that each $f \in B_1$ can be represented algebraically in terms of no more than N derivatives. To see this one need only observe that if this were not the case one could construct $f \in B_1$ with the property that on the interval $[n, n+1]$ at least n derivatives are needed to represent f algebraically. Then no algebraic representation of f in terms of finitely many derivatives would be possible. If one believes that Alg $\Delta' \neq B_1$, one may attempt to prove this by showing that for each n there exists $f \in B_1$ which cannot be expressed algebraically in terms of fewer than n derivatives.

During the beginning of this decade, one of the authors used this approach (unsuccessfully) while another obtained several classes of functions whose members admitted a representation of the form $f = g'h' + k'$ $(g, h, k \in \Delta)$. For example, each function of bounded variation admits such a representation (but not necessarily representations as products of derivatives or representations of the form $gh' + k'$ $(g, h, k \in \Delta)$). In fact, no matter what approach was used, no examples of Baire 1 functions which didn't admit such a representation were forthcoming. Eventually, this led to the conjecture that *every* Baire 1 function admits such a representation. Various attempts to prove this conjecture seemed promising—but none worked. The problem was a very elusive one.

Finally, in 1982, David Preiss **[P2]** succeeded in proving the conjecture. In fact he was able to impose additional conditions on the derivatives appearing in the representation. We state his result as a Theorem.

Theorem (Preiss **[P2]**). *Let $f \in B_1$. There exist functions g, h and k such that $f = g'h' + k'$, g' is bounded and k is a Lebesgue function. If f is bounded, one can choose g', h' and k' all bounded.*

The representation in Preiss' theorem may be compared with the representation $\Phi = FG' + H'$ discussed in the previous section. Functions admitting the latter representation have many desirable properties. Yet replacing the function $F \in \Delta$ by a function $f \in \Delta'$ may result in a function with no specific properties (beyond the obvious one of membership in the algebra B_1). This contrast may be viewed as another indication of the unstable nature of derivatives.

We close by returning to Question 2. Preiss' remarkable theorem provides an indication of the difficulty inherent in attempting to answer this question. We have seen that the class of functions whose members are representable as the product of two or more derivatives is quite restricted. Yet because of Preiss' theorem we see that each $f \in B_1$ differs from a product of two derivatives by a derivative!

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