Miloš Kössler
On a generalization of the Lagrange series

Proc. Lond. Math. Soc. 20 (1922), No. 2, pp. 365–373

Persistent URL: http://dml.cz/dmlcz/501222

ON A GENERALISATION OF LAGRANGE'S SERIES

By M. Kössler.

[Read January 13th, 1921.—Received February 5th, 1921.]

The solution due to Lagrange of equation (2.1) of this paper gives only one root of the equation. By forming the slightly modified equations (2.3), (3.1), and (4.2), we get other roots, and, in some cases, *all* the roots of the original equation.

One of the consequences of this result is that we are thereby enabled to solve any given algebraic equation by means of series of polynomials; I therefore hope that the contents of this paper are of some interest. The methods are a novel application of the method of variable parameters, which has proved to be a powerful weapon in attacking the theory of integral equations.

2. Lagrange's solution of the equation

$$(2.1) \qquad x-a-uf(x)=0$$

is given by the formula

$$(2.2) \qquad x=a+\sum_{m=1}^{\infty} a_m u^m, \qquad a_m=\frac{1}{m!}\left[\frac{d^{m-1}}{dx^{m-1}}\{f(x)\}^m\right]_{x=a},$$

when $f(x)$ is a function of x, which is analytic at the point $x=a$, such that $f(a)\neq 0$. The radius of convergence of the series may be determined without difficulty.

Two generalisations of this expansion are possible. In the case of the first, we take the equation to be

$$(2.3) \qquad (x-a)^n-uf(x)=0,$$

or

$$u=\frac{(x-a)^n}{f(x)}.$$

By writing $\qquad f(x)=f(a)+(x-a)f'(a)+\ldots,$

we get
$$u = (x-a)^n \left[\frac{1}{f(a)} + \wp(x-a)\right],$$

where $\wp$ denotes a power series. When this expansion is reverted, $x-a$ is expressed as a function of u with a branch-point at $u=0$. We thus obtain n values of x, say $x_0, x_1, \ldots, x_{n-1}$, where

$$(2.4) \qquad x_k - a = \sum_{m=1}^{\infty} a_m u_k^m \quad (k = 0, 1, 2, \ldots, n-1),$$

$$u_k = u^{1/n} e^{2k\pi i/n},$$

and
$$u^{1/n} = |u^{1/n}| e^{i\phi} \quad (0 < \phi < 2\pi/n).$$

To evaluate the coefficients a_m, we write equation (2.3) in the form

$$x_k - a - u_k f^{1/n}(x_k) = 0,$$

whence, by (2.2), we have

$$(2.5) \qquad a_m = \frac{1}{m!}\left[\frac{d^{m-1}}{dx^{m-1}} \{f(x)\}^{m/n}\right]_{x=a}.$$

3. It is now possible to solve the equation

$$(3.1) \qquad \phi(x) - uf(x) = 0,$$

where $\phi(x)$ and $f(x)$ are functions of x which are both analytic in a well-defined region of the x-plane, if the roots of the equation $\phi(x) = 0$ are supposed known. If these roots are $a_1, a_2, \ldots, a_n$, of multiplicities $r_1, r_2, \ldots, r_n$ respectively, and if the functions $\phi(x)$ and $f(x)$ have no common zeros, then we transform equation (3.1) into

$$(x-a_k)^{r_k} - u\frac{(x-a_k)^{r_k}}{\phi(x)} f(x) = 0.$$

In this equation, the coefficient of u is analytic at a_k, and it does not vanish at that point. Hence, by the formula of § 2, we obtain r_k roots of the equation, and then, by putting $k = 1, 2, \ldots, n$, we get n sets of roots of equation (3.1).

The radii of convergence of the series (2.2) and (2.4) are given by the distance of the point $u=0$ from the nearest singularity of the functions inverse to

$$u = \frac{x-a}{f(x)}, \qquad u = \frac{(x-a)^n}{f(x)},$$

respectively.

When, as is frequently the case, we can solve the equation

$$\frac{du}{dx} = 0,$$

the radius of convergence is obtained by taking the roots $\beta_1, \beta_2, \ldots$ of this equation and constructing the set of expressions

$$u_l = \frac{\beta_l - a}{f(\beta_l)}, \qquad u_l^{1/n} = \frac{\beta_l - a}{\{f(\beta_l)\}^{1/n}},$$

in the respective cases, and selecting that one which has the smallest modulus; the modulus in question is the radius of convergence.

We apply to the expansions now obtained the well known theorem, due to Mittag-Leffler,* by which the power series

$$F(u) = a_0 + a_1 u + a_2 u^2 + \ldots$$

is transformed into a series of polynomials

$$\text{(M)} \qquad F(u) = \sum_{k=1}^{\infty} P_k(u),$$

where the coefficients in the polynomials P_k are linear functions of the coefficients $a_0, a_1, a_2, \ldots$. This series is convergent throughout Mittag-Leffler's *star* (étoile).

The application of Mittag-Leffler's transformation to the generalisations of Lagrange's series leads directly to the solution of the algebraic equation.

4. Let $f(x, y)$ be an analytic function of both of the variables x, y, and suppose that there exists a constant a such that the roots of the equation in x,

$$f(x, a) = 0,$$

are known; let these roots be $a_1, a_2, \ldots, a_n$.

Suppose also that the roots of the equation in x,

$$(4\,.\,1) \qquad f(x, y) = 0,$$

are not independent of the variable y.

To solve the last equation we consider the modified equation

$$(4\,.\,2) \qquad f(x, a) - u\left[f(x, a) - f(x, y)\right] = 0,$$

* *Acta Mathematica*, Vol. 23 (1899), pp. 43 *et seq.*

which reduces to (4.1) when $u = 1$. By (2.2), a solution of the last equation, valid near $u = 0$, is

$$x_k = a_k + \sum_{m=1}^{\infty} a_m u^m, \tag{4.3}$$

$$a_m = \frac{1}{m!}\left[\frac{d^{m-1}}{dx^{m-1}}\left(\frac{x-a_k}{f(x,a)}\right)^m \{f(x,a)-f(x,y)\}^m\right]_{x=a_k},$$

provided that a_k is a simple zero of $f(x, a)$; the modification to be made in the case of a multiple zero is evident.

The circle of convergence of (4.3) either does or does not contain the point $u = 1$. If it does, we may calculate n roots of the equation (4.1) by putting $u = 1$. If it does not, we must transform the power series by using the formula (M).

This transformation cannot fail by reason of the point $u = 1$ being a summit of the star of convergence, provided that equation (4.2) has no multiple roots in x, for the system

$$u \equiv \frac{f(x,a)}{f(x,a)-f(x,y)} = 1, \qquad \frac{du}{dx} = 0,$$

which forms the conditions that $u = 1$ should be a summit of the star, is equivalent to the system

$$f(x, y) = 0, \qquad f_x(x, y) = 0,$$

and this system is not satisfied if there is no multiple root.

5. Now take any trinomial equation

$$x^n - u(ax+1) = 0, \tag{5.1}$$

in which n is a positive integer.

The formulæ (2.4), (2.5) give immediately all the roots of the equation in the form

$$x_k = \frac{1}{a}\sum_{m=1}^{\infty}\frac{1}{m}\binom{m/n}{m-1} a^m u^{m/n} e^{2km\pi i/n}, \tag{5.2}$$

where $\quad u^{1/n} = |u^{1/n}|\, e^{i\phi}, \quad 0 \leqslant \phi < 2\pi/n, \quad k = 1, 2, \ldots, n.$

The roots are algebraic functions of u whose only singularities are at the branch-points, which are given as the solutions of the system

$$u = \frac{x^n}{ax+1}, \qquad \frac{du}{dx} \equiv \frac{(n-1)\,ax^n + nx^{n-1}}{(ax+1)^2} = 0.$$

The only value of u besides zero which satisfies this system is

$$u = -\frac{(-n)^n}{a^n (n-1)^{n-1}}.$$

Hence the series (5 . 2) is convergent when

$$|u| < \frac{n^n}{|a|^n (n-1)^{n-1}} = \rho.$$

If u does *not* satisfy this inequality, we put

$$x = \frac{1}{y}, \quad u = \frac{1}{v},$$

so that (5 . 1) transforms into

$$y^{n-1}(y+a)-v = 0.$$

When $v = 0$, the roots of this equation are 0 and $-a$, the former having multiplicity $n-1$. Hence in the neighbourhood of $v = 0$ we obtain the solutions

$$(5.3) \qquad y_k = a \sum_{m=1}^{\infty} \binom{-m/(n-1)}{m-1} \frac{v^{m/(n-1)}}{m a^{mn/(n-1)}} e^{2km\pi i/(n-1)}$$

$$(k = 1, 2, \ldots, n-1),$$

$$y_n = -a + \sum_{m=1}^{\infty} \binom{-m(n-1)}{m-1} \frac{(-1)^{mn-1} v^m}{m a^{mn-1}}.$$

It is easy to verify that these series converge when

$$|v| < \frac{|a|^n (n-1)^{n-1}}{n^n} = \frac{1}{\rho},$$

i.e. when $|u| < \rho$.

We have thus obtained the fundamental theorem:

The roots of the trinomial equation (5 . 1) *are given by* (5 . 2) *when** $|u| \leqslant \rho$, *and they are given by* (5 . 3) *when* $|u| \geqslant \rho$, *if* $x_k = 1/y_k$.

The only case of exception occurs when

$$u = -\frac{(-n)^n}{a^n(n-1)^{n-1}},$$

* It has been proved by Riesz, *Palermo Rendiconti*, t. 30 (1910), pp. 339–345, that such a series is convergent on the circumference of the circle of convergence.

but, in this case, the equation has a repeated root

$$x = \frac{-n}{a(n-1)},$$

and the degree is reducible by elementary methods.

The convergence is sufficiently rapid for numerical applications whenever $|u|$ is appreciably less than or greater than ρ.

The special case in which $n = 5$, $u = -1$, gives the solution of the general quintic equation when reduced to the trinomial form by the method of Bring and Jerrard. The method just described is obviously simpler than Hermite's well known solution of the quintic equation.

6. Now take the general algebraic equation in the form

$$x^n - u(c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_n) = 0, \tag{6.1}$$

where n is a positive integer, and $c_1, c_2, \dots, c_n$ are constants of which c_n is not zero. By formulæ (2 . 4) and (2 . 5), the solution is

$$x_k = \sum_{m=1}^{\infty} a_m u^{m/n} e^{2km\pi i/n} \quad (k = 1, 2, \dots, n), \tag{6.2}$$

$$a_m = \frac{1}{m!} \left[\frac{d^{m-1}}{dx^{m-1}} (c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_n)^{m/n} \right]_{x=0}.$$

To determine the radius of convergence, we have to solve the equation

$$nf(x) - xf'(x) = 0, \tag{6.3}$$

where $$f(x) = c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_n,$$

and construct the set of expressions

$$u = x^n / f(x), \tag{6.4}$$

where x is given the values of these roots in turn; we then select that value of u which has the smallest modulus; and the modulus in question is the radius of convergence.

This procedure evidently involves the solution of an algebraic equation of degree $n-1$.

The values of u which are determined by (6 . 3) and (6 . 4) are the only singularities of the functions x_k defined by the series. It is therefore possible to construct the star for each of the functions x_k, and then transform the power series into the expansions of polynomials (M), which are convergent at all points of the star with the exception of points on the boundary. But it has been shown by Painlevé* that it is possible to effect a transformation of the expansions (M), such that the transformed expansions converge at all points of the star, including points on the boundary, with the sole exception of the summits of the star.

For values of u which correspond to one of the summits, the equation (6.1) has a repeated factor, and it is consequently reducible.

Hence, for all values of u, the equation (6.1) *has been solved by an expression of the form*

$$x_k = \sum_{m=1}^{\infty} P_m(u^{1/n} e^{2k\pi i/n}) \quad (k = 1, 2, \ldots, n), \tag{6.5}$$

where the coefficients in the polynomials P_m are linear functions of the coefficients a_m of equation (6.2).

The formation of the expansion (6.5) does not depend upon the critical values of u. Hence, if the variable u is so chosen that equation (6.1) has no repeated roots, the form of the solution given by (6.5) is independent of the solution of an equation of lower degree. If the coefficients $c_1, c_2, \ldots, c_n$ in the equation are not constants, but functions of a variable, the same remark holds good.

It is evident that this solution of the general algebraic equation is complicated and it is not adapted for numerical applications, though it is simple and short in comparison with the solution due to F. Lindemann.†

7. The application of the general formulæ to equations involving integral transcendental functions leads to interesting results, but in this paper I shall confine myself to stating two formal examples.

(I) Let $f(x)$ be an integral function with simple zeros, none of which has any of the values $0, \pm 1, \pm 2, \ldots$. The equation

$$\sin \pi x - u[\sin \pi x - f(x)] = 0 \tag{7.1}$$

is of the form (2.1). We thus obtain the solution

$$x_k = k + (-1)^{k+1} \frac{f(k)}{\pi} u + \frac{(-1)^k \pi - f'(k)}{2\pi^2} u^2 + \ldots$$

$$(k = 0, \pm 1, \pm 2, \ldots).$$

If the radii of convergence of these power series are different from zero,‡

* Cf. Borel, *Leçons sur les fonctions de variables réelles* (1905), Note 1, pp. 140–145.

† *Nachrichten der k. Ges. der Wiss. Göttingen*, 1884, p. 245.

‡ This is by no means an essential restriction.

we can transform them into polynomial expansions (M) which are valid at the point $u = 1$; we have thus calculated an infinite set of zeros of the equation

$$f(x) = 0.$$

(II) Consider the equation

$$P(x) - ue^{Q(x)} = 0, \tag{7.2}$$

where

$$P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n,$$

$$Q(x) = b_0 x^p + b_1 x^{p-1} + \ldots + b_p.$$

When $u = 0$, this equation has n roots $a_1, a_2, \ldots, a_n$, and therefore, by (2.2),

$$x_k = a_k + \sum_{m=1}^{\infty} a_m^{(k)} u^m \quad (k = 1, 2, \ldots, n). \tag{7.3}$$

The radii of convergence and the Mittag-Leffler stars of these series can be constructed by solving the equation

$$\frac{du}{dx} = 0,$$

which, when written in the form

$$P'(x) - P(x)\, Q'(x) = 0,$$

is obviously algebraic, and substituting the roots in

$$u = \frac{P(x)}{e^{Q(x)}}.$$

But the n roots (7.3) are, of course, not all of the roots of the proposed equation. We therefore form from (7.2)

$$Q(x) = \log P(x) - \log u \pm 2k\pi i = \log P(x) - v,$$

where $\log P(x)$ denotes any definite branch of the multiform function.

We know n roots of the last equation when $v = 0$, and hence we can find a set of n roots for every value of k (by putting $v = \log u \pm 2k\pi i$) by using the expansion (2.2).

The equation

$$P(\sin x, \cos x) - ue^x = 0,$$

where P is a polynomial in both variables, may be treated in a similar manner, and many similar equations which are soluble by these methods can be constructed without difficulty.

The solution of the last equation is of some theoretical interest, though it is of little use in numerical applications. But a slightly modified method is effective in the asymptotic calculation of zeros of functions of types discussed by G. H. Hardy.* I hope to return to this topic in a subsequent paper.

In conclusion I have to express my thanks to Prof. G. H. Hardy for his kind help, and to Prof. G. N. Watson for the trouble he has taken by revising the equations and my imperfect English.

* *Proceedings*, Ser. 2, Vol. 2 (1905), pp. 1–7, 401–431.