Chapter II

GENERAL METRIC SPACES

§ 6. Distance

6.1. Let P be a given set. Let ϱ be a finite function on the cartesian product $P \times P$ such that

[1] $\varrho(x, x) = 0;\ x \neq y \Rightarrow \varrho(x, y) > 0;$
[2] $\varrho(x, y) = \varrho(y, x);$
[3] $\varrho(x, y) + \varrho(y, z) \geqq \varrho(x, z).$

Then we say that ϱ is a *distance function* (or a *metric*) in P. The set P is said to be a *metric space*, if there is given a distance function ϱ in P. The elements of a metric space are, as a rule, called *points*. If a, b are two points, then by their *distance* is understood the number $\varrho(a, b)$.

A metric space P with a distance function ϱ is sometimes denoted more precisely by (P, ϱ), in particular when dealing with different distance functions in the same P. The letters P and ϱ will normally denote, throughout all this book, a metric space and its distance function.

The set $\mathbf{E}_1$ of all real numbers is a metric space, if we define $\varrho(x, y) = |x - y|$. In the following, unless otherwise stated, $\mathbf{E}_1$ will denote the metric space with the distance function just defined.

More generally, we denote by $\mathbf{E}_m$ (and call it the *m-dimensional euclidean space*) the set $\mathbf{E}_1 \times \mathbf{E}_1 \times \ldots \times \mathbf{E}_1$ (m factors in the product), where the distance function ϱ is defined by putting, for $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m)$,

$$\varrho(x, y) = \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2}.\ ^{*)}$$

The function ϱ just described obviously possesses properties [1] and [2]. Property [3] may be proved as follows: For $1 \leqq i < k \leqq m$ one has $(x_i y_k - x_k y_i)^2 \geqq 0$ and hence $2x_i y_i x_k y_k \leqq x_i^2 y_k^2 + x_k^2 y_i^2$; therefore

$$\left(\sum_{i=1}^{m} x_i y_i\right)^2 = \sum_{i=1}^{m} x_i^2 y_i^2 + 2 \sum_{i=1}^{m-1} \sum_{k=i+1}^{m} x_i y_i x_k y_k \leqq$$
$$\leqq \sum_{i=1}^{m} x_i^2 y_i^2 + \sum_{i=1}^{m-1} \sum_{k=i+1}^{m} (x_i^2 y_k^2 + x_k^2 y_i^2) = \sum_{i=1}^{m} x_i^2 \cdot \sum_{i=1}^{m} y_i^2,$$

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*) If a is a non-negative real number, $\sqrt{a}$ always denotes the *non-negative* b with $b^2 = a$.

and hence

$$\left|\sum_{i=1}^{m} x_i y_i\right| \leqq \sqrt{\sum_{i=1}^{m} x_i^2} \cdot \sqrt{\sum_{i=1}^{m} y_i^2}. \tag{1}$$

Since

$$\sum_{i=1}^{m} (x_i + y_i)^2 = \sum_{i=1}^{m} x_i^2 + \sum_{i=1}^{m} y_i^2 + 2\sum_{i=1}^{m} x_i y_i,$$

by (1) we have

$$\sum_{i=1}^{m} (x_i + y_i)^2 \leqq \left(\sqrt{\sum_{i=1}^{m} x_i^2} + \sqrt{\sum_{i=1}^{m} y_i^2}\right)^2,$$

and hence

$$\sqrt{\sum_{i=1}^{m} (x_i + y_i)^2} \leqq \sqrt{\sum_{i=1}^{m} x_i^2} + \sqrt{\sum_{i=1}^{m} y_i^2}. \tag{2}$$

Writing $x_i - y_i$ instead of x_i and $y_i - z_i$ instead of y_i we obtain $\varrho(x, z) \leqq \varrho(x, y) + \varrho(y, z)$.

Another important example of a metric space is the *Hilbert space*, which we shall denote by $\mathbf{H}$. It is the set of all sequences $x = \{x_i\}_{i=1}^{\infty}$ ($x_i \in \mathbf{E}_1$) such that the series $\sum_{i=1}^{\infty} x_i^2$ converges, endowed with the metric ϱ given by

$$\varrho(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}.$$

If $x \in \mathbf{H}$, $y \in \mathbf{H}$, the series $\sum_{i=1}^{\infty} x_i^2$, $\sum_{i=1}^{\infty} y_i^2$ converge and therefore [by (2), write $-y_i$ instead of y_i] the series on the right-hand side in (3) also converges. Properties [1] and [2] of the function ϱ are again evident. Formula (2) implies

$$\sqrt{\sum_{i=1}^{\infty} (x_i + y_i)^2} \leqq \sqrt{\sum_{i=1}^{\infty} x_i^2} + \sqrt{\sum_{i=1}^{\infty} y_i^2}.$$

Writing $x_i - y_i$ instead of x_i and $y_i - z_i$ instead of y_i, we obtain $\varrho(x, z) \leqq \varrho(x, y) + \varrho(y, z)$.

Remark: Let a, b, c be three points of a metric space P. Then there are points α, β, γ in $\mathbf{E}_2$ such that

$$\varrho(a, b) = \varrho(\alpha, \beta), \qquad \varrho(a, c) = \varrho(\alpha, \gamma), \qquad \varrho(b, c) = \varrho(\beta, \gamma),$$

(ϱ on the left-hand side designates the distance function in P, ϱ on the right-hand side the distance function in $\mathbf{E}_2$).

Proof: For brevity, we write $\varrho(a, b) = r$, $\varrho(a, c) = s$, $\varrho(b, c) = t$, so that the numbers $r + s + t$, $r + s - t$, $r - s + t$, $-r + s + t$ are greater than or equal to zero.

It is easy to see that the points

$$\alpha = \left(\frac{1}{2}r, 0\right), \quad \beta = \left(-\frac{1}{2}r, 0\right),$$

$$\gamma = \left(\frac{t^2 - s^2}{2r}, \frac{\sqrt{[(r + s + t)(-r + s + t)(r - s + t)(r + s - t)]}}{2r}\right)$$

have the required property.

6.2. Let P_1 and P_2 be two given metric spaces; let ϱ_1 and ϱ_2 be their distance functions. Let us define a function ϱ_{12} on $(P_1 \times P_2) \times (P_1 \times P_2)$ as follows: for $x = (x_1, x_2)$, $y = (y_1, y_2)$ put

$$\varrho_{12}(x, y) = \sqrt{[(\varrho_1(x_1, y_1))^2 + (\varrho_2(x_2, y_2))^2]}.$$

Properties [1] and [2] of the functions ϱ_1 and ϱ_2 imply the same properties of the function ϱ_{12}. We shall prove that the function ϱ_{12} also has property [3]. Let $z = (z_1, z_2)$. By the remark at the end of section 6.1, there exist real numbers a_1, a_2, b_1, b_2, c_1, c_2 such that

$$\varrho_1(x_1, y_1) = \sqrt{[(b_1 - a_1)^2 + (b_2 - a_2)^2]}, \quad \varrho_1(x_1, z_1) = \sqrt{[(c_1 - a_1)^2 + (c_2 - a_2)^2]},$$
$$\varrho_1(y_1, z_1) = \sqrt{[(c_1 - b_1)^2 + (c_2 - b_2)^2]},$$

and real numbers $a_3, a_4, b_3, b_4, c_3, c_4$ such that

$$\varrho_2(x_2, y_2) = \sqrt{[(b_3 - a_3)^2 + (b_4 - a_4)^2]}, \quad \varrho_2(x_2, z_2) = \sqrt{[(c_3 - a_3)^2 + (c_4 - a_4)^2]},$$
$$\varrho_2(y_2, z_2) = \sqrt{[(c_3 - b_3)^2 + (c_4 - b_4)^2]}.$$

Hence,

$$\varrho_{12}(x, y) = \sqrt{\sum_{i=1}^{4}(b_i - a_i)^2}, \quad \varrho_{12}(x, z) = \sqrt{\sum_{i=1}^{4}(c_i - a_i)^2},$$
$$\varrho_{12}(y, z) = \sqrt{\sum_{i=1}^{4}(c_i - b_i)^2}.$$

Since the distance function in $\mathbf{E}_4$ has property [3], we obtain $\varrho_{12}(x, y) + \varrho_{12}(y, z) \geqq \varrho_{12}(x, z)$.

If P_1 and P_2 are given metric spaces with distance functions ϱ_1 and ϱ_2, we shall understand in the following by their *cartesian product* $P_1 \times P_2$, the set $P_1 \times P_2$ with the distance function ϱ_{12} defined above.

Remark: Let $m, n = 1, 2, 3, \ldots$. By the remark at the end of section 2.1 we do not distinguish between $\mathbf{E}_m \times \mathbf{E}_n$ and $\mathbf{E}_{m+n}$. This is in accordance with the evident fact that the distance function in $\mathbf{E}_m \times \mathbf{E}_n$ derived from the usual distance functions in $\mathbf{E}_m$ and $\mathbf{E}_n$ is the same as the usual distance function in $\mathbf{E}_{m+n}$.

6.3. Let P be a metric space with a distance function ϱ. Let $M \subset P$. The partial function (see 2.4) $\varrho_{M \times M}$ is evidently a distance function in M. Consequently, every subset M of a metric space P may be taken for a metric space. We say that M is a *point set* embedded into the space P. Hence, a point set is a metric space M which is a subset of a metric space P, such that the distance function in M is the corresponding partial function of the distance function in P.

6.4. Let P and Q be metric spaces; let ϱ_1 and ϱ_2 be their distance functions. Let f be a mapping of P onto Q. We say that the mapping f is an *isometry*, if

$$x \in P, \quad y \in P \;\Rightarrow\; \varrho_2[f(x), f(y)] = \varrho_1(x, y).$$

Since the distance function has property [1], f is obviously *one-to-one*. Evidently, the inverse mapping f_{-1} is an isometry of Q onto P.

We say that spaces P and Q are *isometric* if there is an isometry of P onto Q (or of Q onto P).

A *metric property* of a space P is a property which is preserved on replacing P by an arbitrary isometric space, i.e. a property which depends only on the distances of points and not on the "concrete form" of the points. *We shall investigate only metric properties of metric spaces.*

6.5. Let P be a metric space (with a distance function ϱ). Let $A \subset P$, $B \subset P$. Let M be the set of all real numbers $\varrho(x, y)$ with $x \in A$, $y \in B$. The number $\inf M$ (see 4.10) will be denoted by $\varrho(A, B)$ and called the *lower distance* of the point sets A and B. The number $\sup M$ (see 4.10) will be denoted by $d(A, B)$ and called the *upper distanec* of the point sets A and B.*)
Evidently

$$\varrho(A, B) = \varrho(B, A), \qquad d(A, B) = d(B, A).$$

If either $A = \emptyset$ or $B = \emptyset$, we have $\varrho(A, B) = \infty$, $d(A, B) = -\infty$. If $A \neq \emptyset \neq B$ then $\varrho(A, B)$ is a non-negative real number, and $d(A, B)$ is either a non-negative real number or ∞. Evidently, for $a \in P$, $b \in P$

$$\varrho((a), (b)) = d((a), (b)) = \varrho(a, b).$$

If $A = (a)$ is a one-point set, we write

$$\varrho(a, B) = \varrho(B, a) = \varrho((a), B),$$
$$d(a, B) = d(B, a) = d((a), B).$$

and call $\varrho(a, B)$ the *lower distance***) (and $d(a, B)$ the *upper distance*) of the point a from the point set B.

*) The lower distance is much more important than the upper one. Therefore it is often called simply the *distance*.

**) Or simply the *distance* (see the previous footnote).

For $A \subset P$ we put $d(A) = 0$ provided $A = \emptyset$ and $d(A) = d(A, A)$ provided $A \neq \emptyset$. The number $d(A)$ is called the *diameter* of the point set A. A point set A is said to be *bounded* if $d(A) < \infty$, *unbounded*, if $d(A) = \infty$. In the case of $P = \mathbf{E}_1$ this definition is in accordance with the definition given in 4.10.

Exercises

6.1. Let P be a given set, let ϱ be a finite function with domain $P \times P$ such that [1] $x = y \Leftrightarrow \varrho(x, y) = 0$, [2] $\varrho(x, y) + \varrho(z, y) \geqq \varrho(x, z)$. Then ϱ is a distance function in P.

6.2. The statement of ex. 6.1 is not true, if we write $\varrho(y, z)$ instead of $\varrho(z, y)$.

In exercises 6.3 and 6.4 C denotes the set of all complex numbers.

6.3. Let $C_m = C \times C \times \ldots \times C$ (with m factors). For $x \in C_m$, $y \in C_m$, $x = (x_1, x_2, \ldots, x_m)$ $y = (y_1, y_2, \ldots, y_m)$ put $\varrho(x, y) = \sqrt{\sum_{i=1}^{m} |x_i - y_i|^2}$. Then ϱ is a distance function in C_m, and C_m with this distance function is isometric to the euclidean $\mathbf{E}_{2m}$.

6.4. Let $\mathbf{H}'$ be the set of all sequences $\{x_i\}_{i=1}^{\infty}$ $(x_i \in C)$, such that the series $\sum_{i=1}^{\infty} |x_i|^2$ converges. For $x \in \mathbf{H}'$, $y \in \mathbf{H}'$, $x = \{x_i\}$ $y = \{y_i\}$ put $\varrho(x, y) = \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2}$. Then ϱ is a distance function in $\mathbf{H}'$ and $(\mathbf{H}', \varrho)$ is isometric to the Hilbert space $\mathbf{H}$.

6.5. Let P be the set of all *bounded* sequences $\{x_i\}_{i=1}^{\infty}$ $(x_i \in \mathbf{E}_1)$. If $x \in P$, $y \in P$, $x = \{x_i\}$, $y = \{y_i\}$ put $\varrho(x, y) = \sup |x_i - y_i|$. Then ϱ is a distance function in P.

In the following exercises 6.6.—6.12, a and b are points of a metric space P, and A, B and C are non-void point-sets embedded into P.

6.6.* $|\varrho(a, A) - \varrho(b, A)| \leqq \varrho(a, b)$.

6.7. $\varrho(A, B) \leqq \varrho(a, A) + \varrho(a, B)$.

6.8. $d(A, C) \leqq d(A, B) + d(B, C)$.

6.9. The inequality $\varrho(A, C) \leqq \varrho(A, B) + \varrho(B, C)$ need not hold.

6.10. If $d(A) < \infty$, then for every point a there is a number δ $(0 < \delta < \infty)$ such that $x \in A$ implies $\varrho(a, x) < \delta$.

6.11. If there is a point a and a number δ $(0 < \delta < \infty)$ such that $x \in A$ implies $\varrho(a, x) < \delta$, then $d(A) < \infty$.

6.12. $d(A, B) < \infty$ if and only if both A and B are bounded.

6.13. Let P and Q be metric spaces; let $A \subset P$, $B \subset Q$. Then $d(A \times B) = \sqrt{(d(A))^2 + (d(B))^2}$

§ 7. Convergence

7.1. If $\{x_n\}$ is a sequence of real numbers and if x is a real number, then the symbol $x_n \to x$ indicates that for every $\varepsilon > 0$ there is an index $p(\varepsilon)$ such that $n > p(\varepsilon)$ implies $|x_n - x| < \varepsilon$. This is a particular case $(P = \mathbf{E}_1)$ of the following definition:

Let P be a metric space. Let $\{x_n\}$ be a *point sequence* in P, i.e. a sequence, the terms of which are points of the space P. Let x be a point of P. Then the symbol

$x_n \to x$ indicates that for every $\varepsilon > 0$ there is an index $p(\varepsilon)$ such that $n > p(\varepsilon)$ implies $\varrho(x_n, x) < \varepsilon$. In other words

$$x_n \to x \quad \text{if and only if} \quad \varrho(x_n, x) \to 0 .$$

We write also $\lim x_n = x$ instead of $x_n \to x$; more definitely we write $x_n \to x$ for $n \to \infty$, or $\lim\limits_{n\to\infty} x_n = x$. We say that x is a *limit* of the sequence $\{x_n\}$.

If $x_n \to x$ and $x_n \to y$, then $0 \leqq \varrho(x, y) \leqq \varrho(x_n, x) + \varrho(x_n, y) \to 0$, hence $\varrho(x, y) = 0$ and hence $x = y$. Thus, a sequence $\{x_n\}$ has at most one limit in the space P. A sequence $\{x_n\}$ having a limit is called *convergent* (in the space P); if it has no limit, it is said to be *divergent* (in the space P).

7.1.1. *If there is an index p such that $x_n = x$ for $n > p$ then $x_n \to x$.*

7.1.2. *If $x_n \to x$ and if $\{y_n\}$ is a subsequence* (see 3.1) *of $\{x_n\}$ then also $y_n \to x$.*

7.2. Let P be a given set. Let ϱ_1 and ϱ_2 be two distance functions in P, i.e. two finite functions on $P \times P$ having properties [1], [2] and [3] stated at the beginning of section 6.1. Due to the distance function ϱ_1, P is a metric space, which will be denoted for clarity (P, ϱ_1); due to the distance function ϱ_2, P is a metric space, which will be denoted by (P, ϱ_2).

We say that the distance functions ϱ_1, ϱ_2 are *equivalent* if

$$x_n \to x \text{ in } (P, \varrho_1) \quad \text{if and only if} \quad x_n \to x \text{ in } (P, \varrho_2) .$$

As an example consider the set $\mathbf{E}_m$. For $x = (x_1, x_2, \dots, x_m)$, $y = (y_1, y_2, \dots, y_m)$ put

$$\varrho_p(x, y) = \Big(\sum_{i=1}^{m} | x_i - y_i |^p\Big)^{1/p}$$

where p is a real number greater than 1. For $p = 2$ we obtain the distance function with which the set $\mathbf{E}_m$ was called the m-dimensional euclidean space. We shall prove that ϱ_p is a distance function for every $p > 1$.

Let us begin with the following remark: If α, a, b are real numbers, $0 < \alpha < 1$, $a \geqq 0$ and $b \geqq 0$ then

$$a^\alpha b^{1-\alpha} \leqq \alpha a + (1 - \alpha) b . \tag{1}$$

Proof: (1) is evident for $a = 0$ and for $b = 0$. Let $a > 0$, $b > 0$. The function $\varphi(t) = t^\alpha - \alpha t + \alpha - 1$ has the derivative $\varphi'(t) = \alpha[(1/t)^{1-\alpha} - 1]$ in the interval $\underset{t}{\mathrm{E}}[0 < t]$ and hence $0 < t < 1$ implies $\varphi'(t) > 0$, $t > 1$ implies $\varphi'(t) < 0$; as $\varphi(1) = 0$ we obtain the implication $0 < t \Rightarrow \varphi(t) \leqq 0$, consequently $\varphi(a/b) \leqq 0$ and hence (1).

Now, we are going to prove that for real x_i, y_i and $p > 1$ the so caled *Hölder inequality* holds:

$$\left|\sum_{i=1}^{m} x_i y_i\right| \leqq \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p} \cdot \left(\sum_{i=1}^{m} |y_i|^{p/(p-1)}\right)^{(p-1)/p}. \tag{2}$$

Of course, it is sufficient to prove this assuming $x_i \geqq 0$, $y_i \geqq 0$ with not every $x_i = 0$ and not every $y_i = 0$. Then (2) may be obtained by (1), putting

$$\alpha = \frac{1}{p}, \qquad a = \frac{x_k^p}{\sum_{i=1}^{m} x_i^p}, \qquad b = \frac{y_k^{p/(p-1)}}{\sum_{i=1}^{m} y_i^{p/(p-1)}}$$

and adding for $1 \leqq k \leqq m$.

Inequality (2) yields the so called *Minkowski inequality*

$$\left(\sum_{i=1}^{m} |x_i + y_i|^p\right)^{1/p} \leqq \left(\sum_{i=1}^{m} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{m} |y_i|^p\right)^{1/p}. \tag{3}$$

To prove (3) we may, again, assume $x_i \geqq 0$, $y_i \geqq 0$, $\sum_{i=1}^{m} x_i > 0$, $\sum_{i=1}^{m} y_i > 0$. Under this assumption let us write the formula V_1 obtained from (2) by replacing y_i by $(x_i + y_i)^{p-1}$ and preserving x_i; then, write the formula V_2 obtained from V_1 by changing the letters x and y. We obtain (3) by adding V_1 and V_2.

Now, we see easily that (for any $p > 1$) ϱ_p is a distance function in $\mathbf{E}_m$. The sole less obvious inequality was $\varrho_p(x, z) \leqq \varrho_p(x, y) + \varrho_p(y, z)$; this is, however, the inequality (3), where we write $x_i - y_i$ instead of x_i and $y_i - z_i$ instead of y_i. If we define the relation $\lim_{n\to\infty} (x_{n1}, x_{n2}, \ldots, x_{nm}) = (x_1, x_2, \ldots, x_m)$ by the distance function ϱ_p we verify easily that this relation holds if and only if $\lim_{n\to\infty} x_{ni} = x_i$ (in the ordinary sense) simultaneously for every $1 \leqq i \leqq m$. Thus, all the distance functions ϱ_p are equivalent.

7.3. If $\{x_i\}_1^\infty$ is a sequence of real numbers such that $|x_i| \leqq 1/i$ for every i, the series $\sum_{i=1}^{\infty} x_i^2$ converges; thus, $\{x_i\}$ is a point of the Hilbert space $\mathbf{H}$. We denote by $\mathbf{U}$ the set of all $\{x_i\} \in \mathbf{H}$ such that $|x_i| \leqq 1/i$ and call it the *Urysohn space*; the distance function in $\mathbf{U}$ is, by 6.3, determined by the inclusion $\mathbf{U} \subset \mathbf{H}$.

7.3.1. *If* $x_n = \{x_{ni}\}_{i=1}^\infty \in \mathbf{U}$, $y = \{y_i\}_{i=1}^\infty \in \mathbf{H}$ *then* $\lim_{n\to\infty} x_n = y$ *if and only if* $\lim_{n\to\infty} x_{ni} = y_i$ *for every index i.**)

Proof: I. Let $x_n \to y$. For every $\varepsilon > 0$ there is an index p such that $\varrho(y, x_n) < \varepsilon$

*) The last equality evidently yields $|y_i| \leqq 1/i$, i.e. $y \in \mathbf{U}$.

for $n > p$. We have $\varrho(y, x_n) = \sqrt{\sum_{i=1}^{\infty} (y_i - x_{ni})^2} \geqq |y_i - x_{ni}|$ and hence $n > p$ implies $|y_i - x_{ni}| < \varepsilon$; thus, $x_{ni} \to y_i$ for each index i.

II. Let $x_{ni} \to y_i$ for every index i; then, of course, $|y_i| = \lim |x_{ni}| \leqq 1/i$. Let us choose an $\varepsilon > 0$. Since the series $\sum_{i=1}^{\infty} 1/i^2$ converges, there is an index q such that $\sum_{i=q+1}^{\infty} 1/i^2 < \varepsilon^2/8$. As $1 \leqq i \leqq q$, there is an index p_i such that

$$n > p_i \Rightarrow |y_i - x_{ni}| < \frac{\varepsilon}{2q}.$$

Put $p = \max\limits_{1 \leqq i \leqq q} p_i$. Then

$$1 \leqq i \leqq q, \quad n > p \Rightarrow |y_i - x_{ni}| < \frac{\varepsilon}{2q},$$

and hence

$$n > p \Rightarrow \varrho(y, x_n) = \sqrt{\left[\sum_{i=1}^{q} (y_i - x_{ni})^2 + \sum_{i=q+1}^{\infty} (y_i - x_{ni})^2\right]} \leqq$$

$$\leqq \sqrt{\left[\sum_{i=1}^{q} (y_i - x_{ni})^2 + \sum_{i=q+1}^{\infty} \frac{4}{i^2}\right]} \leqq \sqrt{\left[q\left(\frac{\varepsilon}{2q}\right)^2 + \frac{\varepsilon^2}{2}\right]} < \varepsilon.$$

Exercises

7.1. For $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$ put $\varrho'(x, y) = \max\limits_{1 \leqq i \leqq m} |x_i - y_i|$, $\varrho''(x, y) = \sum_{i=1}^{m} |x_i - y_i|$. Then ϱ' and ϱ'' are distance functions in $\mathbf{E}_m$ equivalent with each other and equivalent with the ordinary distance function in $\mathbf{E}_m$.

7.2. Let p be a given real number, $p > 1$. Let $\mathbf{H}_p$ be the set of all sequences $\{x_i\}_1^\infty$ of real numbers such that the series $\sum_{i=1}^{\infty} |x_i|^p$ converges. For $x = \{x_i\}$, $y = \{y_i\}$ put $\varrho_p(x, y) = (\sum_{i=1}^{\infty} |x_i - y_i|^p)^{1/p}$. Then ϱ_p is a distance function in $\mathbf{H}_p$.

7.3. Let P and Q be metric spaces with distance functions ϱ_1 and ϱ_2. For $x_1 \in P$, $y_1 \in P$, $x_2 \in Q$, $y_2 \in Q$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ put $\varrho'_p(x, y) = [(\varrho_1(x_1, y_1))^p + (\varrho_2(x_2, y_2))^p]^{1/p}$ $(p \geqq 1)$, $\varrho''(x, y) = \max\limits_{i=1,2} \varrho_i(x_i, y_i)$. Then ϱ'_p and ϱ'' are equivalent distance functions in $P \times Q$ (ϱ'_2 is the distance function from 6.2). Regarding these distance functions

$$(x_n, y_n) \to (x, y) \quad \text{if and only if} \quad x_n \to x,\ y_n \to y.$$

7.4.* Let S be the set of all sequences $\{x_i\}_1^\infty$ with real terms. For $x = \{x_i\}$, $y = \{y_i\}$ put

$$\varrho'(x, y) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - y_n|}{1 + |x_n - y_n|},$$

$$\varrho''(x, y) = \inf\left(\frac{1}{n} + \max_{1 \leqq i \leqq n} |x_i - y_i|\right).$$

Then ϱ' and ϱ'' are equivalent distance functions in S. If $x_n = \{x_{ni}\}_{i=1}^{\infty}$ then, regarding these distance functions, we have $x_n \to x$ if and only if $\lim_{n\to\infty} x_{ni} = x_i$ (in the ordinary sense) simultaneously for every i.

§ 8. Closure of a point set. Open and closed sets

8.1. Let P be a given metric space. Let $A \subset P$. Put

$$\bar{A} = \mathop{\mathrm{E}}_{x}[x \in P, \varrho(x, A) = 0].$$

The set $\bar{A}$ is called the *closure* of the point set A, more precisely, the closure of A in the space P.

The following formulas are evident

$$\bar{\emptyset} = \emptyset \tag{1}$$

$$A \subset \bar{A} \tag{2}$$

Further, one has:

$$A \subset B \quad \text{implies} \quad \bar{A} \subset \bar{B} \tag{3}$$

Proof: As $A \subset B$ we have evidently $\varrho(x, A) \geqq \varrho(x, B)$ for every point x. Thus, we obtain the following sequence of implications

$$x \in \bar{A} \Rightarrow \varrho(x, A) = 0 \Rightarrow \varrho(x, B) = 0 \Rightarrow x \in \bar{B}.$$

Further, we have

$$\overline{A \cup B} = \bar{A} \cup \bar{B}. \tag{4}$$

Proof: By (3), $\bar{A} \subset \overline{A \cup B}$, $\bar{B} \subset \overline{A \cup B}$. Hence, if (4) does not hold, then there is a point $x \in \overline{A \cup B} - (\bar{A} \cup \bar{B})$. Since $x \in P - \bar{A}$, we have $\varrho(x, A) > 0$ and similarly $\varrho(x, B) > 0$; thus, there is an $\varepsilon > 0$ such that $\varrho(x, A) > \varepsilon$, $\varrho(x, B) > \varepsilon$. If $y \in A \cup B$, we have either $y \in A$ and hence $\varrho(x, y) \geqq \varrho(x, A)$, or $y \in B$ and hence $\varrho(x, y) \geqq \varrho(x, B)$. Consequently, $y \in A \cup B$ implies $\varrho(x, y) > \varepsilon$ and hence $\varrho(x, A \cup B) \geqq \varepsilon > 0$. This is a contradiction, since $x \in \overline{A \cup B}$.

Formula (4) yields by induction for $m = 1, 2, 3, \ldots$

$$\overline{\bigcup_{i=1}^{m} A_i} = \bigcup_{i=1}^{m} \bar{A}_i. \tag{5}$$

8.2. 8.2.1. *The closure $\bar{A}$ of a point set A is the set of all limits of convergent sequences, the terms of which are points of A.*

Proof: I. Let $x \in \bar{A}$; thus, $\varrho(x, A) = 0$ and hence for $n = 1, 2, 3, \ldots$, $\varrho(x, A) < 1/n$ and consequently, there is a point $x_n \in A$ such that $\varrho(x, x_n) < 1/n$. Obviously $x_n \to x$.

II. Let $x_n \in A$, $x_n \to x$. For every n, $\varrho(x, A) \leqq \varrho(x, x_n) \to 0$ and hence $\varrho(x, A) = 0$, i.e. $x \in \bar{A}$.

Further, we have

$$\varrho(x, A) = \varrho(x, \bar{A}) . \tag{6}$$

Proof: I. If $y \in A$, we have $y \in \bar{A}$ and hence $\varrho(x, y) \geqq \varrho(x, \bar{A})$. Hence, $\varrho(x, A) = \inf_{y \in A} \varrho(x, y) \geqq \varrho(x, \bar{A})$.

II. Let $y \in \bar{A}$. By the preceding theorem there is a sequence $\{y_n\}$ such that $y_n \in A$, $y_n \to y$; we have $\varrho(x, A) \leqq \varrho(x, y_n) \leqq \varrho(x, y) + \varrho(y_n, y) \to \varrho(x, y)$, hence $\varrho(x, A) \leqq \varrho(x, y)$. Consequently $\varrho(x, A) \leqq \inf_{y \in \bar{A}} \varrho(x, y) = \varrho(x, \bar{A})$.

By (6) and by the definition of closure we obtain

$$\bar{\bar{A}} = \bar{A}$$

or, in words: *the closure of the closure of a point set A coincides with the closure of A.*

8.3. A point set A (embedded into a space P) is said to be *closed*, more precisely, closed in P, if $A = \bar{A}$. Hence:

8.3.1. *$\emptyset$ and P are closed sets.*

By the definition, we obtain easily:

8.3.2. *Any one-point set is closed.*

By 8.2.1 it follows that

8.3.3. *A point set A is closed if and only if*

$$x_n \in A, \; x \in P, \; x_n \to x \quad \textit{imply} \quad x \in A$$

or, in words: *if and only if A contains the limit of every convergent sequence, the terms of which are points of A.*

By (5) it follows that

8.3.4. *The union of any finite number of closed sets is a closed set.*

8.3.5. *The intersection $\bigcap_{z \in \mathbf{C}} A(z)$ of closed sets $A(z)$ is a closed set, the number of the members $A(z)$ being finite or infinite.*

Proof: Put $B = \bigcap_{z \in \mathbf{C}} A(z)$. We have $B \subset A(z)$ for every $z \in \mathbf{C}$, hence, by (3), $\bar{B} \subset \overline{A(z)}$; as the sets $A(z)$ are closed, we have $\overline{A(z)} = A(z)$. Thus, $\bar{B} \subset A(z)$ for every $z \in \mathbf{C}$ and hence $\bar{B} \subset \bigcap_{z \in \mathbf{C}} A(z)$, i.e. $\bar{B} \subset B$. Consequently, by (2), $\bar{B} = B$.

8.4. *The closure $\bar{A}$ of a point set A is the least closed set containing the set A.*

Proof: I. $\bar{A} \supset A$ by (2); $\bar{A}$ is closed by (7).

II. Let F be a closed set, $A \subset F$. By (3), $\bar{F} \supset \bar{A}$; as $\bar{F} = F$, we obtain $F \supset \bar{A}$.

8.5. A point set A (embedded into a space P) is said to be *open*, more precisely, open in P, if $P - A$ is closed.

Consequently:

8.5.1. *$\emptyset$ and P are open sets.*

By analogous theorems of section 8.3, we obtain (see ex. 1.8) the following theorems.

8.5.2. *The intersection of any finite number of open sets is an open set.*

8.5.3. *The union $\bigcup_{z \in \mathbf{C}} A(z)$ of open sets $A(z)$ is an open set, the number of the members $A(z)$ being finite or infinite.*

8.6. Any *open* set $G \subset P$ such that $a \in G$ is called a *neighborhood of the point a* (more precisely, a neighborhood of a in P). A *neighborhood of a point set $A \subset P$* (more precisely a neighborhood of A in P) is any *open* set $G \subset P$ such that $A \subset G$. Thus, the neighborhoods of a point a coincide with the neighborhoods of the set (a).

Let $a \in P$. Let $r \in \mathbf{E}_1$, $r > 0$. The set

$$\mathop{\mathrm{E}}_{x}[x \in P, \varrho(a, x) < r]$$

will be denoted $\Omega(a, r)$, more precisely $\Omega_P(a, r)$. *This is an open set.*

Proof: Let $M = P - \Omega(a, r)$. We have to prove that M is a closed set. Let $x_n \in M$, $x_n \to x$. It suffices to prove that $x \in M$. Since $x_n \in M$, $\varrho(a, x_n) \geqq r$. We have $\varrho(a, x) + \varrho(x, x_n) \geqq \varrho(a, x_n)$. Since $\varrho(x, x_n) \to 0$, we obtain $\varrho(a, x) \geqq r$, i.e. $x \in M$.

Since $a \in \Omega(a, r)$, the set $\Omega(a, r)$ is a neighborhood of a. It is called the *spherical neighborhood* of the point a with *radius r*.

Let $A \subset P$. Let $r \in \mathbf{E}_1$, $r > 0$. The set

$$\mathop{\mathrm{E}}_{x}[x \in P, \ \varrho(x, A) < r]$$

will be denoted by $\Omega(A, r)$; more precisely, $\Omega_P(A, r)$. We evidently have $\Omega[(a), r] = \Omega(a, r)$, $A \subset \bar{A} \subset \Omega(A, r)$, $\Omega(\bar{A}, r) = \Omega(A, r)$ [see (6)]. We see easily that

$$\Omega(A, r) = \bigcup_{x \in A} \Omega(x, r),$$

consequently, the set $\Omega(A, r)$ is open, and hence it is a neighborhood of the set A. It will be termed the *spherical neighborhood* of the set A with *radius* r.

Let $A \subset P$. A point $a \in P$ will be called an *interior point* of A (with respect to the space P), if there is an $r > 0$ such that $\Omega(a, r) \subset A$ (then, of course, $a \in A$).

8.6.1. *A point set A is open if and only if each of its points is an interior point.*

Proof: I. Let A be open. Choose a point $a \in A$. The set $B = P - A$ is closed, i.e. $B = \bar{B}$, and hence $a \in P - \bar{B}$. Hence, the number $r = \varrho(a, B)$ is positive.*) Evidently, $\Omega(a, r) \subset A$.

II. Let every point $x \in A$ be an interior point. We may associate with every $x \in A$ a positive number $r(x)$ such that $\Omega[x, r(x)] \subset A$. We have

$$A = \bigcup_{x \in A} (x) \subset \bigcup_{x \in A} \Omega[x, r(x)] \subset A,$$

hence

$$A = \bigcup_{x \in A} \Omega[x, r(x)],$$

and consequently A is a union of open sets and hence open.

Many authors use the term "neighborhood of $a \in P$" for every $U \subset P$ (open or not open) such that a is its interior point. In this, more general, sense, the set

$$\underset{x}{\mathrm{E}}[x \in P, \varrho(a, x) \leqq r],$$

(r given, $a \in P$) is a neighborhood of the point a. It will be denoted by $\bar{\Omega}(a, r)$.

8.7. Let P be a metric space and let Q be a point set embedded into P. By 6.3, Q is also a metric space. A point set A embedded into Q is also embedded into P.

In the following, if $A \subset Q$, the symbol $\bar{A}$ denotes the closure of A in P.

8.7.1. *The closure of A in Q is equal to $Q \cap \bar{A}$.* Actually, this closure equals

$$\underset{x}{\mathrm{E}}[x \in Q, \varrho(x, A) = 0] = Q \cap \underset{x}{\mathrm{E}}[x \in P, \varrho(x, A) = 0] = Q \cap \bar{A}.$$

8.7.2. *The set $A \subset Q$ is closed in Q if and only if there is a closed set F in P such that $A = Q \cap F$.*

Proof: I. Let A be closed in Q. Then A coincides with its closure in Q, i.e. $A = Q \cap \bar{A}$. $\bar{A}$ is closed in P.

II. Let $A = Q \cap F$, $F = \bar{F}$. We have $A \subset F$ and hence $\bar{A} \subset F$ by (3). Thus, $A \subset Q \cap \bar{A} \subset Q \cap F = A$ and hence $A = Q \cap \bar{A}$.

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*) If $B = \emptyset$, we have $\varrho(a, B) = \infty$ and $\Omega(a, r) \subset A$ for every $r > 0$.

The following two corollaries follow easily by 8.7.2:

8.7.3. *If a set $A \subset Q$ is closed in P, then it is closed in Q.*

8.7.4. *If a set $A \subset Q$ is closed in Q and Q is closed in P, then A is closed in P.*

8.7.5. *A set $A \subset Q$ is open in Q if and only if there is a set G open in P such that $A = Q \cap G$.*

Proof: I. Let A be open in Q. Then $Q - A$ is closed in Q and hence there is a closed F in P such that $Q - A = Q \cap F$, hence $A = Q \cap (P - F)$. The set $P - F$ is open in P.

II. Let G be open in P and let $A = Q \cap G$. The set $P - G$ is closed in P and $Q - A = Q \cap (P - G)$. Thus, $Q - A$ is closed in Q and hence, finally, A is open in Q.

Theorem 8.7.5 has again two corollaries:

8.7.6. *If a set $A \subset Q$ is open in P, it is open in Q.*

8.7.7. *If a set $A \subset Q$ is open in Q and Q is open in P, then A is open in P.*

Generally we will give a fixed metric space P and if we simply say that a point set A is closed (open), we mean closed (open) in P. The sets which are closed or open in $Q \subset P$ are sometimes called *relatively closed* or *relatively open.* Similarly, the closure $\bar{A}$ of a set A is the closure in P, $Q \cap \bar{A}$ is the *relative closure.*

8.8. Let $\{A_n\}_1^\infty$ be a sequence of subsets of a metric space P. We associate with the sequence $\{A_n\}$ two subsets B and C of P as follows: [1] $x \in B$ if and only if there is a sequence $\{a_n\}_{n=m}^\infty$ such that $a_n \in A_n$ for $n \geqq m$ and $a_n \to x$; [2] $x \in C$ if and only if there exist indices $i_1 < i_2 < i_3 < \ldots$ and a sequence $\{a_n\}$ such that $a_n \in A_{i_n}$ and $a_n \to x$. The set B is termed the *lower limit* of $\{A_n\}$, C the *upper limit* of $\{A_n\}$; we denote them

$$B = \underline{\mathrm{Lim}}\, A_n = \underline{\mathrm{Lim}}_{n\to\infty} A_n$$

$$C = \overline{\mathrm{Lim}}\, A_n = \overline{\mathrm{Lim}}_{n\to\infty} A_n\,.$$

Evidently,

$$\underline{\mathrm{Lim}}\, A_n \subset \overline{\mathrm{Lim}}_{n\to\infty} A_n\,. \tag{1}$$

If $B = C$, we write

$$\underline{\mathrm{Lim}}\, A_n = \overline{\mathrm{Lim}}\, A_n = \mathrm{Lim}\, A_n\,. \tag{2}$$

Asserting that $\mathrm{Lim}\, A_n$ exists we indicate the validity of (2).

Exercises

In ex. 8.1—8.8, A and B are point sets embedded into a metric space P.

8.1. $\bar{A} - \bar{B} \subset \overline{A - B}$.

8.2.* $\overline{P - \bar{A}} \subset \overline{P - A}$.

8.3. $\overline{P - \overline{P - \overline{P - \bar{A}}}} = \overline{P - \bar{A}}$.

8.4. If A is closed and B is open, $A - B$ is closed and $B - A$ is open.

In the exercises 8.5.—8.9, 8.11, A_ι denotes the set of all interior points of A. The set A_ι is called the *interior* of the set A.

8.5. $A_\iota = P - \overline{P - A}$.

8.6. A_ι is the largest open set contained in A.

8.7. $A \subset B \Rightarrow A_\iota \subset B_\iota$.

8.8. $(A \cap B)_\iota = A_\iota \cap B_\iota$.

8.9. $A_\iota - B_\iota \supset (A - B)_\iota$.

In ex. 8.10—8.14, P and Q are given metric spaces, $A \subset P$, $B \subset Q$, $A \neq \emptyset \neq B$.

8.10. $\overline{A \times B} = \bar{A} \times \bar{B}$.

8.11. $(A \times B)_\iota = A_\iota \times B_\iota$.

8.12. $A \times B$ is closed in $P \times Q$ if and only if A is closed in P and B is closed in Q.

8.13.* In ex. 8.12 the word closed may be (simultaneously) replaced by the word open.

8.14. To what extent in ex. 8.10—8.13 is the assumption of $A \neq \emptyset \neq B$ substantial?

8.15. If $a \in P$, $r > 0$ then the set $\bar{\Omega}(a, r)$ is closed in P. Thus, $\bar{\Omega}(a, r) \supset \overline{\Omega(a, r)}$. However, it may occur that $\bar{\Omega}(a, r) \neq \overline{\Omega(a, r)}$.

8.16.* $x \in \underline{\mathrm{Lim}}\, A_n$ means that $\varrho(x, A_n) \to 0$; $x \in \overline{\mathrm{Lim}}\, A_n$ means that there is a subsequence $\{A_{i_n}\}$ of $\{A_n\}$ with $\varrho(x, A_{i_n}) \to 0$.

8.17. $\underline{\mathrm{Lim}}\, A_n = \underline{\mathrm{Lim}}\, \bar{A}_n$, $\overline{\mathrm{Lim}}\, A_n = \overline{\mathrm{Lim}}\, \bar{A}_n$.

8.18. * The sets $\underline{\mathrm{Lim}}\, A_n$ and $\overline{\mathrm{Lim}}\, A_n$ are closed.

8.19. Let $A_n = (a_n)$; if $\lim a_n$ exists, then $\underline{\mathrm{Lim}}\, A_n = (\lim a_n)$. Otherwise $\mathrm{Lim}\, A_n = \emptyset$.

8.20.* If $i_1 < i_2 < i_3 < \dots$, we have

$$\underline{\mathrm{Lim}}\, A_n \subset \underline{\mathrm{Lim}}\, A_{i_n} \subset \overline{\mathrm{Lim}}\, A_{i_n} \subset \overline{\mathrm{Lim}}\, A_n;$$

hence,

$$\mathrm{Lim}\, A_n = \mathrm{Lim}\, A_{i_n},$$

if the left-hand side exists.

8.21. One has

$$\bigcap_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i \subset \underline{\mathrm{Lim}}\, A_n \subset \overline{\mathrm{Lim}}\, A_n \subset \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \subset \bigcup_{n=1}^{\infty} A_n.$$

§ 9. Continuous mapping. Homeomorphism

9.1. Let P and Q be given metric spaces. The distance function in both of them will be denoted by ϱ. It will be always evident which one is meant.

Let f be a mapping of P into Q. Let $a \in P$. We say that the mapping f is *continuous* at the point a if

$$x_n \to a \Rightarrow f(x_n) \to f(a) .$$

If a mapping f is continuous at a point a and if $a \in M \subset P$, then the partial mapping f_M is evidently continuous at the point a.

9.1.1. *A mapping f is continuous at a point a if and only if for every $\varepsilon > 0$ there is a $\delta(a, \varepsilon) > 0$ such that $x \in P$, $\varrho(a, x) < \delta(a, \varepsilon)$ imply $\varrho[f(a), f(x)] < \varepsilon$.*

Proof: I. Let the condition be satisfied. Let $x_n \to a$. Choose an $\varepsilon > 0$ and take the $\delta(a, \varepsilon)$. Since $x_n \to a$, $\delta(a, \varepsilon) > 0$, there is an index $p(\varepsilon)$ such that $n > p(\varepsilon)$ implies $\varrho(a, x_n) < \delta(a, \varepsilon)$. Thus, $n > p(\varepsilon)$ implies $\varrho[f(a), f(x_n)] < \varepsilon$ i.e. $f(x_n) \to$ $\to f(a)$. Hence, f is continuous at the point a.

II. Let the mapping f be continuous at a point a. Let us choose an $\varepsilon > 0$ and let us suppose that there exist no suitable $\delta(a, \varepsilon)$. In particular, we cannot put $\delta(a, \varepsilon) = 1/n$ for $n = 1, 2, 3, \ldots$ i.e. there is a point $x_n \in P$ such that $\varrho(a, x_n) < 1/n$, $\varrho[f(a), f(x_n)] \geqq \varepsilon$. Since $\varrho(a, x_n) < 1/n$ we have $x_n \to a$. Since $\varrho[f(a), f(x_n)] \geqq \varepsilon > 0$, it does not hold that $f(x_n) \to f(a)$, which is a contradiction.

A mapping f is said to be *continuous* (without any further determination), if it is continuous at every point of the space P.

If f is a mapping of a metric space P into a metric space Q, $f(P)$ is a point set embedded into the space Q. If Q' is a point set with $f(P) \subset Q' \subset Q$ then Q' is a metric space (see 6.3) and f is a mapping of P into Q'. The definition of continuity of the mapping f obviously does not change, if we take Q' instead of Q.

9.2. *Let f be a mapping of a metric space P onto a metric space Q. A necessary and sufficient condition for f to be continuous is the following: If A is open in Q, $f_{-1}(A)$ is open in P. Another form of the condition: If A is closed in Q, $f_{-1}(A)$ is closed in P.*

Proof: I. Both forms of the condition are equivalent; see ex. 2.13, write P, Q, Q, A instead of A, B, N_1, N_2 respectively.

II. Let $f_{-1}(A)$ be open in P whenever A is open in Q. Choose an $a \in P$, $\varepsilon > 0$. We have to prove that there is a $\delta > 0$ such that $x \in P$, $\varrho(a, x) < \delta$ imply $\varrho[f(a), f(x)] < \varepsilon$. Put $A = \Omega_Q[f(a), \varepsilon]$. A is open in Q and hence $f_{-1}(A)$ is open in P. We have $a \in f_{-1}(A)$ and hence a is an interior point of $f_{-1}(A)$. Thus, there is a $\delta > 0$ such that $\Omega_P(a, \delta) \subset f_{-1}(A)$. If $x \in P$, $\varrho(a, x) < \delta$, we have $x \in \Omega_P(a, \delta)$, hence $x \in f_{-1}(A)$, hence $f(x) \in A$ and hence finally $\varrho[f(a), f(x)] < \varepsilon$.

III. Let the mapping f be continous. Let A be open in Q. Let $a \in f_{-1}(A)$. We have to prove that a is an interior point of $f_{-1}(A)$. Since $a \in f_{-1}(A)$ and since A is open, $f(a)$ is an interior point of A and hence there is an $\varepsilon > 0$ with $\Omega_Q[f(a), \varepsilon] \subset A$.

As the mapping f is continuous at the point a, there is a $\delta = \delta(a, \varepsilon) > 0$ such that the following sequence of implications holds

$$x \in P,\ \varrho(a, x) < \delta \Rightarrow \varrho[f(a), f(x)] < \varepsilon \Rightarrow f(x) \in \Omega_Q[f(a), \varepsilon] \Rightarrow f(x) \in A \Rightarrow x \in f_{-1}(A).$$

Thus, $\Omega_P(a, \delta) \subset f_{-1}(A)$.

9.3. Let f be a *one-to-one* continuous mapping of a metric space P onto a metric space Q. Then the inverse mapping f_{-1} of Q onto P need not be continuous. An example: Let P be the set of all natural numbers $1, 2, 3, \ldots$; let Q be the set of all rational numbers; the distance functions in P and Q are defined by embedding into $\mathbf{E}_1$ (see 6.3). By ex. 3.1 there is a one-to-one mapping f of the set P onto the set Q. It is easy to prove that the mapping f is continuous, while the inverse mapping f_{-1} is continuous at no point of the space Q.

If f is a one-to-one continuous mapping of a metric space P onto a metric space Q and if the inverse mapping f_{-1} is also continuous, we say that f is a *homeomorphic* mapping of the space P onto the space Q. The mapping f_{-1} is then, evidently, a homeomorphic mapping of the space Q onto the space P.

Spaces P and Q are said to be *homeomorphic* if there exists a homeomorphic mapping of P onto Q (or of Q onto P).

A *topological property* of a space P is any property which is preserved on replacing P by an arbitrary homeomorphic space. Every isometry is a homeomorphic mapping; thus, every topological property is a metric property. Of course, the converse is not true.

Let ϱ_1 and ϱ_2 be two equivalent distance functions in a set P. Let us, for clarity, speak about the metric spaces (P, ϱ_1) and (P, ϱ_2), as we did at the beginning of section 7.2. If we assign to every point $x \in P$, considered as a point of (P, ϱ_1) the same point x in (P, ϱ_2), we obtain a mapping f of the space (P, ϱ_1) onto the space (P, ϱ_2). It is easy to see that the mapping f is homeomorphic.

On the other hand, let f be a homeomorphic mapping of a space P (with a distance function ϱ_1) onto a space Q (with a distance function ϱ_2). Let us define a function ϱ_0 on the domain $P \times P$ as follows:

$$\varrho_0(x, y) = \varrho_2[f(x), f(y)].$$

We see easily that ϱ_0 is a distance function equivalent with ϱ_1.

These considerations show that the topological properties are those metric properties which remain preserved after replacing a given distance function by an equivalent one.

Evidently, every property of a metric space P, that may be formulated without explicitly speaking about the distance function, i.e., in terms of convergence in P only, is a topological property. E.g., the closure $\bar{A}$ of a point set A embedded into a metric space P is a topological notion by 8.2.1. (This is not obvious from the definition, however.) Thus, every notion which may be formulated by means of the notion of closure, is topological, e.g. the notion of closed set ($A = \bar{A}$) and the notion of open set ($P - A = \overline{P - A}$) are topological. The notion of continuous mapping of a space P into a space Q is also topological, being explicitly defined only in convergences in P and Q.

9.4. We often meet with spaces in which the definition of convergence is quite natural, while a distance function is defined artificially. E.g., this was the case in the space S in exercise 7.4 with both given distance functions.

A simpler and more important example is the set **R**, consisting of all real numbers and the symbols ∞ and $-\infty$. We define convergence in **R** as follows: If $x_n \in \mathbf{R}$ then: [1] $x_n \to \infty$ means that for every $c \in \mathbf{E}_1$ there is an index $p(c)$ such that $n > p(c)$ implies $x_n > c$; [2] $x_n \to -\infty$ means that for every $c \in \mathbf{E}_1$ there is an index $p(c)$ such that $n > p(c)$ implies $x_n < c$; [3] $x_n \to \alpha$, where $\alpha \in \mathbf{E}_1$ means that there is only a finite number of indices n with $x_n = \infty$ or $x_n = -\infty$ and that, rejecting all terms x_n which are ∞ or $-\infty$, we obtain a subsequence $\{y_n\}$ of $\{x_n\}$ such that $y_n \to \alpha$ in the ordinary sense.

Now, we shall define a distance function ϱ in **R** such that the convergence defined by means of ϱ (see 7.1) coincides with the one just described. This may be done in various ways; it does not matter which one we choose, since we are interested now only in *topological* properties of **R**.

For $x \in \mathbf{R}$, $y \in \mathbf{R}$ set

$$\varrho(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right| .$$

(We put

$$\frac{\infty}{1 + |\infty|} = 1, \qquad \frac{-\infty}{1 + |-\infty|} = -1.)$$

Property [1] (see section 6.1) follows from the fact that for $x \in \mathbf{R}$, $y \in \mathbf{R}$, $x < y$ we have

$$\frac{x}{1 + |x|} < \frac{y}{1 + |y|} .$$

Property [2] is evident, property [3] follows immediately from the inequality $|a| + |b| \geqq |a + b|$ (valid for $a \in \mathbf{E}_1$, $b \in \mathbf{E}_1$) by substituing $a = [x/(1 + |x|)] - [y/(1 + |y|)]$, $b = [y/(1 + |y|)] - [z/(1 + |z|)]$ ($x \in \mathbf{R}$, $y \in \mathbf{R}$, $z \in \mathbf{R}$).

Thus ϱ is a distance function in **R**.

We see easily that the convergence in **R** defined by means of the distance function ϱ coincides with the convergence defined above. Consequently, the partial distance function $\varrho_{\mathbf{E}_1 \times \mathbf{E}_1}$ is equivalent to the ordinary distance function in $\mathbf{E}_1$ (introduced in 6.1.). The distance function $\varrho_{\mathbf{E}_1 \times \mathbf{E}_1}$ is called the *reduced* distance function in $\mathbf{E}_1$. Hence, the space $\mathbf{E}_1$ with the reduced distance function (not with the ordinary one) is a point set embedded into the metric space **R**.

A mapping of a set P into the set $\mathbf{R}$ was termed a function (see 2.3). If P is a metric space, a function f is said to be continuous (at a point $a \in P$), if the mapping f is continuous (at the point a) in the sense of section 9.1. If the function f is finite, we may define continuity by the ordinary or by the reduced distance function in $\mathbf{E}_1$. In the calculus, functions are always finite and continuity is defined by means of the ordinary distance function in $\mathbf{E}_1$.

9.5. 9.5.1. *Let P be a metric space. Let f be a function on P. A necessary and sufficient condition for a function f to be continuous is the following: for every $c \in \mathbf{E}_1$, the sets $\underset{x}{\mathrm{E}}[f(x) > c]$ and $\underset{x}{\mathrm{E}}[f(x) < c]$ are open (in P).*

Proof: I. Let the condition be satisfied. Let $x_n \in P$, $y \in P$, $x_n \to y$. We have to prove that $f(x_n) \to f(y)$. We shall distinguish three cases:

[1] Let $f(y) = \infty$. Choose a $c \in \mathbf{E}_1$. Since the set $M = \underset{x}{\mathrm{E}}[f(x) > c]$ is open and since $y \in M$, there is an $\varepsilon > 0$ such that $\Omega_P(y, \varepsilon) \subset M$. As $x_n \to y$, there is an index p such that the following sequence of implications holds:

$$n > p \Rightarrow \varrho(x_n, y) < \varepsilon \Rightarrow x_n \in \Omega_P(y, \varepsilon) \Rightarrow x_n \in M \Rightarrow f(x_n) > c\,.$$

Thus, for every $c \in \mathbf{E}_1$, there is an index $p(c) = p$ such that $n > p$ implies $f(x_n) > c$, i.e. $f(x_n) \to \infty$ q.e.d.

[2] Let $f(y) = -\infty$. The argument is similar to case [1].

[3] Let $f(y) \in \mathbf{E}_1$. Choose an $\varepsilon > 0$. The set $M_\varepsilon = \underset{x}{\mathrm{E}}[|f(x) - f(y)| < \varepsilon] = \underset{x}{\mathrm{E}}[f(x) > f(y) - \varepsilon] \cap \underset{x}{\mathrm{E}}[f(x) < f(y) + \varepsilon]$ is open and contains the point y; hence there is an $r > 0$ such that $\varrho(x, y) < r$ for $x \in M_\varepsilon$. As $x_n \to y$, there is an index p such that:

$$n > p \Rightarrow \varrho(x_n, y) < r \Rightarrow x_n \in M_\varepsilon \Rightarrow |f(x_n) - f(y)| < \varepsilon.$$

Thus, for every $\varepsilon > 0$ there is an index $p(\varepsilon) = p$ such that $n > p$ implies $|f(x_n) - f(y)| < \varepsilon$, i.e. $f(x_n) \to f(y)$, q.e.d.

II. Let f be a continuous function. It is easy to prove that the set $C = \underset{y}{\mathrm{E}}[y \in \mathbf{R}, y > c]$ is open in $\mathbf{R}$ for every $c \in \mathbf{E}_1$. Hence, by 9.2, the set $f_{-1}(C) = \underset{x}{\mathrm{E}}[f(x) > c]$ is open in P. Similarly we prove that also the set $\underset{x}{\mathrm{E}}[f(x) < c]$ is open in P.

Since

$$\underset{x}{\mathrm{E}}[f(x) \leqq c] = P - \underset{x}{\mathrm{E}}[f(x) > c],$$
$$\underset{x}{\mathrm{E}}[f(x) \geqq c] = P - \underset{x}{\mathrm{E}}[f(x) < c], \tag{1}$$

the condition in the theorem just proved may be expressed as follows: *For every* $c \in \mathbf{E}_1$, *the sets* $\underset{x}{\mathrm{E}}[f(x) \geqq c]$ *and* $\underset{x}{\mathrm{E}}[f(x) \leqq c]$ *are closed in* P.

Since

$$\begin{aligned}
&\underset{x}{\mathrm{E}}[f(x) = c] = \underset{x}{\mathrm{E}}[f(x) \geqq c] \cap \underset{x}{\mathrm{E}}[f(x) \leqq c]\,,\\
&\underset{x}{\mathrm{E}}[f(x) = \infty] = \bigcap_{n=1}^{\infty} \underset{x}{\mathrm{E}}[f(x) \geqq n]\,,\\
&\underset{x}{\mathrm{E}}[f(x) = -\infty] = \bigcap_{n=1}^{\infty} \underset{x}{\mathrm{E}}[f(x) \leqq -n]\,,\\
&\underset{x}{\mathrm{E}}[f(x) < \infty] = P - \underset{x}{\mathrm{E}}[f(x) = \infty]\,,\\
&\underset{x}{\mathrm{E}}[f(x) > -\infty] = P - \underset{x}{\mathrm{E}}[f(x) = -\infty]\,,
\end{aligned} \tag{2}$$

the sets $\underset{x}{\mathrm{E}}[f(x) = c]$ $(c \in \mathbf{R})$ are closed in P and the sets $\underset{x}{\mathrm{E}}[f(x) < \infty]$, $\underset{x}{\mathrm{E}}[f(x) > -\infty]$ (and hence the set $\underset{x}{\mathrm{E}}[f(x) \in \mathbf{E}_1] = \underset{x}{\mathrm{E}}[f(x) < \infty] \cap \underset{x}{\mathrm{E}}[f(x) > -\infty]$) are open in P for every continuous function.

From the calculus, we are acquainted with many continuous functions. We may use the theorems just proved to prove easily that some simple sets in the euclidean spaces are closed or open. E.g., the function f defined on $\mathbf{E}_2$ by $f(x, y) = x^2/a^2 + y^2/b^2$ is continuous; hence, the ellipse $\underset{(x,y)}{\mathrm{E}}[x^2/a^2 + y^2/b^2 = 1]$ is a closed set, its interior $\underset{(x,y)}{\mathrm{E}}[x^2/a^2 + y^2/b^2 < 1]$ and exterior $\underset{(x,y)}{\mathrm{E}}[x^2/a^2 + y^2/b^2 > 1]$ are open sets; the set $\underset{(x,y)}{\mathrm{E}}[x^2/a^2 + y^2/b^2 \leqq 1]$ is closed, etc.

9.6. Let f be a mapping of a metric space P into a metric space Q. The mapping f is said to be *uniformly continuous* if

$$x_n \in P, \qquad y_n \in P, \qquad \varrho(x_n, y_n) \to 0 \quad \text{imply} \quad \varrho[f(x_n), f(y_n)] \to 0\,.$$

Putting $y_n = a \in P$ for every n we see that *every uniformly continuous mapping is continuous.*

Continuity is a *topological* notion. Concerning uniform continuity, however, we can assert that this is a *metric* notion only: it need not be preserved on replacing the distance functions in P or in Q by equivalent ones.

As a rule, the term *uniformly continuous function* is used with *finite* functions only ($Q = \mathbf{E}_1$), assuming the *ordinary distance function* in $\mathbf{E}_1$ (not the *reduced* one).

9.6.1. *A mapping* f (of a space P into a space Q) *is uniformly continuous if and only if for every* $\varepsilon > 0$ *there is a* $\delta(\varepsilon) > 0$ *such that*

$$x \in P, \qquad y \in P, \qquad \varrho(x, y) < \delta(\varepsilon) \quad \textit{imply} \quad \varrho[f(x), f(y)] < \varepsilon\,.$$

Proof: I. Let the condition be satisfied. Let $x_n \in P$, $y_n \in P$, $\varrho(x_n, y_n) \to 0$. Choose an $\varepsilon > 0$ and determine the $\delta(\varepsilon)$. Since $\varrho(x_n, y_n) \to 0$, there is an index p such that $\varrho(x_n, y_n) < \delta(\varepsilon)$ for $n > p$. Thus, $n > p$ implies $\varrho[f(x_n), f(y_n)] < \varepsilon$ and consequently $\varrho[f(x_n), f(y_n)] \to 0$.

II. Let the mapping f be uniformly continuous. Choose an $\varepsilon > 0$ and assume that there is no suitable $\delta(\varepsilon)$. Thus, we may not put $\delta(\varepsilon) = 1/n$ for $n = 1, 2, 3, \ldots$, i.e. there are points $x_n \in P$, $y_n \in P$ such that $\varrho(x_n, y_n) < 1/n$, $\varrho[f(x_n), f(y_n)] \geqq \varepsilon$. Since $\varrho(x_n, y_n) < 1/n$, we have $\varrho(x_n, y_n) \to 0$. Since $\varrho[f(x_n), f(y_n)] \geqq \varepsilon$, $\varrho[f(x_n), f(y_n)] \to 0$ does not hold. This is a contradiction.

Exercises

9.1. Let us define a function f on $\mathbf{E}_1$ as follows: for irrational x put $f(x) = 0$, for rational x put $f(x) = 1$. The function f is continuous at no point $x \in \mathbf{E}_1$.

9.2. Let us define a function f on $\mathbf{E}_1$ as follows: for irrational x put $f(x) = 0$; if m and n are integers without common divisor put $f(m/n) = 1/|n|$. The function f is continuous at a point $x \in \mathbf{E}_1$ if and only if x is irrational.

9.3. Let us define a function f on $\mathbf{E}_2$ as follows: $f(0, y) = 0$, $f(x, y) = (x^2 + y^2)/x$ for $x \neq 0$. The function f is not continuous at the point $(0, 0)$. If A is an arbitrary straight line going through $(0, 0)$, i.e. $A = \underset{(x,y)}{\mathrm{E}}\,[ax + by = 0]$ where $a \in \mathbf{E}_1$, $b \in \mathbf{E}_1$, $|a| + |b| > 0$, the partial function f_A is continuous at the point $(0, 0)$.

In ex. 9.4—9.5, P is a metric space and A and B are point sets embedded into P; f is a mapping of P into Q.

9.4. If $A \cup B = P$, $a \in A \cap B$ and both the partial mappings f_A and f_B are continuous at a, then the mapping f is also continuous at a.

9.5. If $A \cup B = P$, both sets A and B are closed and both the partial mappings f_A and f_B are continuous, then the mapping f is also continuous.

9.6. If a mapping f is continuous, then the set $\underset{(x,y)}{\mathrm{E}}\,[x \in P, y \in Q, y = f(x)]$ is closed in $P \times Q$.

9.7. The characteristic function of a set A is continuous if and only if the set A is both closed and open.

9.8. A mapping f is continuous if and only if $f(\overline{X}) \subset \overline{f(X)}$ for every $X \subset P$.

9.9. A mapping f is continuous if and only if $\overline{f_{-1}(Y)} \subset f_{-1}(\overline{Y})$ for every $Y \subset Q$.

9.10.* Let $A \neq \emptyset$. The function $\varrho(x, A)$ is uniformly continuous.

9.11.* Let $d(A) < \infty$. The function $d(x, A)$ is uniformly continuous.

9.12.* The distance function ϱ of the space P is uniformly continuous in the domain $P \times P$.

In ex. 9.13—9.15, f is a one-to-one mapping of P onto Q.

9.13. A necessary and sufficient condition for f to be homeomorphic is the following: $X \subset P$ is closed in P if and only if $f(X)$ is closed in Q.

9.14. In ex. 9.13, the word closed may be (simultaneously) replaced by the word open.

9.15. A necessary and sufficient condition for f to be homeomorphic is the following: $X \subset P$ implies $f(\overline{X}) = \overline{f(X)}$.

9.16. Let P and Q be metric spaces, $Q \neq \emptyset$. For $x \in P$, $y \in Q$ put $f(x, y) = x$. Then f is a uniformly continuous mapping of the space $P \times Q$ onto the space P.

9.17. For $x \in \mathbf{E}_1$ put $f(x) = x^2$. The function f is continuous, but it is not uniformly continuous. However, if we replace the ordinary distance function in $\mathbf{E}_1$ by the *reduced* one, the mapping f is a uniformly continuous mapping of the space $\mathbf{E}_1$ into the space $\mathbf{E}_1$.

9.18.* The metric space $\mathbf{R}$ is homeomorphic with the interval $\mathrm{E}_t[-1 \leqq t \leqq 1]$. We obtain a homeomorphic mapping e.g. putting $f(\infty) = 1$, $f(-\infty) = -1$ and $f(t) = t/(1 + |t|)$ for $t \in \mathbf{E}_1$.*)

9.19.* Let P be a metric space, let $a \in P$, let f, g be finite functions on P, continuous at the point a. Then the function, the value of which is given in any $x \in P$ by one of the following formulas (by the same one for each $x \in P$):

$$|f(x)|, \quad \max\,[f(x), g(x)], \; \min\,[f(x), g(x)], \; f(x) + g(x), \; f(x) - g(x), \; f(x) \cdot g(x)$$

is continuous at the point a. If, moreover, $g(a) \neq 0$, the function h, defined for $x \in \mathrm{E}_y[y \in P, g(y) \neq 0]$ by the relation $h(x) = f(x)/g(x)$, and defined arbitrarily for $x \in P - \mathrm{E}_y[y \in P, g(y) \neq 0]$ is also continuous at the point a. — The reader should examine how the theorem has to be altered, if we do not assume finiteness of the functions f, g.

9.20.* Let g_i $(i = 1, 2)$ be a mapping of a metric space P_i onto a metric space P_{i+1}; let $a_1 \in P_1$, $a_2 = g_1(a_1)$, hence $a_2 \in P_2$. Let the mappings g_i $(i = 1, 2)$ be continuous at the points a_i. Then the mapping f of P_1 onto P_3 defined by

$$x \in P_1 \Rightarrow f(x) = g_2[g_1(x)]$$

is continuous at the point a_1.

9.21.* Let f be a continuous mapping of a metric space P onto a metric space Q. Let $A_n \subset P$ $(n = 1, 2, 3, \ldots)$. Then

$$f(\overline{\mathrm{Lim}}\, A_n) \subset \overline{\mathrm{Lim}}\, f(A_n)\,, \quad f(\underline{\mathrm{Lim}}\, A_n) \subset \underline{\mathrm{Lim}}\, f(A_n)\,.$$

If the mapping f is homeomorphic, then

$$f(\overline{\mathrm{Lim}}\, A_n) = \overline{\mathrm{Lim}}\, f(A_n)\,, \quad f(\underline{\mathrm{Lim}}\, A_n) = \underline{\mathrm{Lim}}\, f(A_n)$$

and hence $\mathrm{Lim}\, f(A_n)$ exists if and only if $\mathrm{Lim}\, A_n$ exists.

§ 10. Separated point sets; the boundaries of point sets

10.1. 10.1.1. *Let P be a metric space. For arbitrary $A_1 \subset P$, $A_2 \subset P$ there are closed sets F_1, F_2 such that $F_1 \cup F_2 = P$, $A_1 \subset F_1$, $A_2 \subset F_2$, $F_1 \cap F_2 \cap (\bar{A}_1 \cup \bar{A}_2) = \bar{A}_1 \cap \bar{A}_2$.*

Proof: If $A_1 = \emptyset$, put $F_1 = \emptyset$, $F_2 = P$; similarly for $A_2 = \emptyset$. Let $A_1 \neq \emptyset \neq A_2$. If $f(x) = \varrho(x, A_1) - \varrho(x, A_2)$ for $x \in P$, the function f is continuous on P by ex. 9.10. Consequently, by 9.5, the sets $F_1 = \mathrm{E}_x[\varrho(x, A_1) \leqq \varrho(x, A_2)]$ and $F_2 = \mathrm{E}_x[\varrho(x, A_1) \geqq \varrho(x, A_2)]$ are closed. Evidently $F_1 \cup F_2 = P$, $A_1 \subset F_1$, $A_2 \subset F_2$. It remains to prove that $F_1 \cap F_2 \cap (\bar{A}_1 \cup \bar{A}_2) = \bar{A}_1 \cap \bar{A}_2$. *First,* let $x \in F_1 \cap F_2$, i.e. $\varrho(x, A_1) = \varrho(x, A_2)$. If $x \in \bar{A}_1$, we have $\varrho(x, A_1) = 0$, hence $\varrho(x, A_2) = 0$ and hence $x \in \bar{A}_2$.

*) This mapping is an isometry if we take the ordinary distance function in $\mathrm{E}_t[-1 \leqq t \leqq 1]$

Similarly, $x \in \bar{A}_2$ implies $x \in \bar{A}_1$. Hence $F_1 \cap F_2 \cap (\bar{A}_1 \cup \bar{A}_2) \subset \bar{A}_1 \cap \bar{A}_2$. *On the other hand,* let $x \in \bar{A}_1 \cap \bar{A}_2$. Then $\varrho(x, A_1) = 0 = \varrho(x, A_2)$ implies $x \in F_1 \cap F_2$. Hence also $\bar{A}_1 \cap \bar{A}_2 \subset F_1 \cap F_2 \cap (\bar{A}_1 \cup \bar{A}_2)$.

10.1.2. *Let U be a neighborhood of a closed set A. Then there is a neighborhood V of A such that $\bar{V} \subset U$.**)

Proof: In 10.1.1 put $A_1 = A$, $A_2 = P - U$. Then $A_1 = \bar{A}_1$, $A_2 = \bar{A}_2$. Find F_1 and F_2 by the quoted theorem and put $V = P - F_2$. Thus, the set V is open. We have $F_1 \cap F_2 \cap [A \cup (P - U)] = A - U = \emptyset$, and hence $A \cap F_2 = \emptyset$ i.e. $A \subset V$, and $F_1 \subset U$. Since $F_1 \cup F_2 = P$ we have $V = P - F_2 \subset F_1$, and hence finally $\bar{V} \subset \bar{F}_1 = F_1 \subset U$.

10.2. Point sets A and B are said to be *separated* if: [1] $A \cap B = \emptyset$, [2] both A and B are closed in $A \cup B$. Since condition [1] yields that $A = (A \cup B) - B$, $B = (A \cup B) - A$, condition [2] may be replaced by condition [2']: both A and B are open in $A \cup B$. The property of sets A and B being separated is a topological property (see 9.3) depending on the space $A \cup B$ only, not on the whole space P into which $A \cup B$ is embedded.

The following two theorems follow immediately from the definition (see 8.7.3 and 8.7.6).

10.2.1. *Two closed disjoint sets are separated.*

10.2.2. *Two open disjoint sets are separated.*

10.2.3. *Sets A and B are separated if and only if $A \cap \bar{B} = \emptyset = B \cap \bar{A}$; otherwise stated: if and only if* [1] *$x \in B$ implies $\varrho(x, A) > 0$ and* [2] *$x \in A$ implies $\varrho(x, B) > 0$.*

Proof: I. Let the sets A and B be separated. Then $A \cap B = \emptyset$. The set A is closed in $A \cup B$; hence, its relative closure in $A \cup B$, i.e. $(A \cup B) \cap \bar{A}$, is equal to A. Hence $B \cap \bar{A} \subset A \cap B = \emptyset$. Consequently $B \cap \bar{A} = \emptyset$; similarly, $A \cap \bar{B} = \emptyset$.

II. Let $A \cap \bar{B} = \emptyset = B \cap \bar{A}$. As $B \subset \bar{B}$, we have $A \cap B = \emptyset$. As $B \cap \bar{A} = \emptyset$, we have $(A \cup B) \cap \bar{A} = A \cap \bar{A} \cup B \cap \bar{A} = A \cap A = A$; i.e. the set A is equal to its relative closure in $A \cup B$, i.e. A is closed in $A \cup B$. Similarly, B is closed in $A \cup B$.

10.2.4. *Let sets A and B be separated. Let $C \subset A$, $D \subset B$. Then the sets C and D are separated.*

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*) This property of subsets of metric spaces is termed the normality. An analogous statement in more general spaces may be false. (Ed.)

Proof: $\bar{C} \subset \bar{A}$, $\bar{D} \subset \bar{B}$ and hence $A \cap \bar{B} = \emptyset$ implies $C \cap \bar{D} = \emptyset$, $\bar{A} \cap B = \emptyset$ implies $\bar{C} \cap D = \emptyset$.

10.2.5. *Let sets A and B be separated; let also sets A and C be separated. Then the sets A and $B \cup C$ are separated.*

Proof: $\bar{A} \cap B = \emptyset$, $\bar{A} \cap C = \emptyset$ imply $\bar{A} \cap (B \cup C) = \emptyset$. Since $\overline{B \cup C} = \bar{B} \cup \bar{C}$, $A \cap \bar{B} = \emptyset$, $A \cap \bar{C} = \emptyset$ imply $A \cap \overline{B \cup C} = \emptyset$.

10.2.6. *Let sets G and H be open and let $G \cap H = \emptyset$. Then $\bar{G} \cap H = \emptyset$.*

Actually, G and H are separated.

10.2.7. *Sets A and B are separated if and only if there exist open sets U and V such that $U \cap V = \emptyset$, $U \supset A$, $V \supset B$.*

Proof: I. If such sets U and V exist, they are separated. As $A \subset U$, $B \subset V$, also A and B are separated.

II. Let A and B be separated. By 10.1.1 there are closed F_1, F_2 such that

$$F_1 \cup F_2 = P, \qquad F_1 \supset A, \qquad F_2 \supset B, \qquad F_1 \cap F_2 \cap (\bar{A} \cup \bar{B}) = \bar{A} \cap \bar{B}$$

Put $U = P - F_2$, $V = P - F_1$. Then the sets U and V are open and $U \cap V = P - (F_1 \cup F_2) = \emptyset$. If we had $x \in A \cap F_2$, we would obtain $x \in F_1 \cap F_2 \cap (\bar{A} \cup \bar{B})$, hence $x \in \bar{A} \cap \bar{B} \cap A$ and hence $x \in A \cap \bar{B}$. We have however, $A \cap \bar{B} = \emptyset$. Thus, $A \cap F_2 = \emptyset$, so that $A \subset P - F_2 = U$. Similarly, $B \subset V$.

10.3. For every $A \subset P$ denote by $B(A)$, more precisely $B_P(A)$, the set $\bar{A} \cap \overline{(P - A)}$ (i.e. the set $\mathop{\mathrm{E}}_x[\varrho(x, A) = 0,\ \varrho(x, P - A) = 0]$), and call it the *boundary* of the set A (in the space P). The notion of boundary is a topological notion. From the definition it follows:

10.3.1. *The set $B(A)$ is always closed. Evidently we always have*

$$B(P - A) = B(A). \tag{1}$$

It always holds that

$$B(A \cup B) \subset B(A) \cup B(B). \tag{2}$$

Proof: Since $A \subset A \cup B$, we have $P - (A \cup B) \subset P - A$ and hence $\overline{P - (A \cup B)} \subset \overline{P - A}$. Similarly $\overline{P - (A \cup B)} \subset \overline{P - B}$; moreover, $\overline{A \cup B} = \bar{A} \cup \bar{B}$. Thus

$$B(A \cup B) = (\bar{A} \cup \bar{B}) \cap \overline{(P - (A \cup B))} \subset \bar{A} \cap \overline{(P - A)} \cup \bar{B} \cup \overline{(P - B)} = \\ = B(A) \cup B(B).$$

From (2) it follows by induction that

$$B\left(\bigcup_{i=1}^{m} A_i\right) \subset \bigcup_{i=1}^{m} B(A_i). \tag{3}$$

We always have

$$B(A \cap B) \subset B(A) \cup B(B). \tag{4}$$

Actually by (1) and (2)

$$B(A \cap B) = B[P - (A \cap B)] =$$
$$= B[(P - A) \cup (P - B)] \subset B(P - A) \cup B(P - B) = B(A) \cup B(B).$$

From (4) it follows by induction for $m = 1, 2, 3, \ldots$ that

$$B\left(\bigcap_{i=1}^{m} A_i\right) \subset \bigcup_{i=1}^{m} B(A_i). \tag{5}$$

It always holds that

$$B(A - B) \subset B(A) \cup B(B), \tag{6}$$

since, by (4) and (1),

$$B(A - B) = B[A \cap (P - B)] \subset B(A) \cup B(P - B) = B(A) \cup B(B).$$

We always have

$$B(\bar{A}) \subset B(A). \tag{7}$$

Proof: By ex. 8.2, $\overline{P - \bar{A}} \subset \overline{P - A}$; hence

$$B(\bar{A}) = \bar{A} \cap \overline{(P - \bar{A})} \subset \bar{A} \cap \overline{(P - A)} = B(A).$$

Formulas (6) and (7) yield

$$B(A - \bar{B}) \subset B(A) \cup B(B). \tag{8}$$

The notion of boundary is particularly important in the case of open sets.

10.3.2. *If a set A is open, then*

$$B(A) = \bar{A} - A. \tag{9}$$

Proof: The set $P - A$ is closed, hence $\overline{P - A} = P - A$ and hence

$$B(A) = \bar{A} \cap \overline{P - A} = \bar{A} \cap (P - A) = \bar{A} - A.$$

10.4.*) *Let G_n and V_n $(n = 1, 2, 3, \ldots)$ be open sets. Let $S = \bigcap_{n=1}^{\infty} \bar{G}_n$. Let $T \subset \bigcup_{n=1}^{\infty} V_n$.*

*) The so called Menger's addition theorem (Ed.).

Let, for $n = 1, 2, 3, \ldots$, $G_n \supset G_{n+1}$, $V_n \subset G_n$. Then

$$B\Big(\bigcup_{n=1}^{\infty} V_n\Big) \subset \bigcup_{n=1}^{\infty} B(V_n) \cup M,$$

$$M = S \cap B\Big(\bigcup_{n=1}^{\infty} V_n\Big) \subset S - T.$$

Proof: Put $H = B(\bigcup_{n=1}^{\infty} V_n)$, $M = S \cap H$. As $T = \bigcup_{n=1}^{\infty} V_n$ and as the set $\bigcup_{n=1}^{\infty} V_n$ is open, $H \cap T = \emptyset$ and hence $M \subset S - T$. Now, it suffices to show that, for every $a \in H - S$ there is an index k with $a \in B(V_k)$. As $a \in H - S$, $S = \bigcap_{n=1}^{\infty} \overline{G}_n$, there is an index h such that $a \in P - \overline{G}_h$. For every $n > h$ we have $V_n \subset G_n \subset G_h$, hence $\bigcup_{n=h+1}^{\infty} V_n \subset G_h$, hence $\overline{\bigcup_{n=h+1}^{\infty} V_n} \subset \overline{G}_h$ and hence $a \in P - \overline{\bigcup_{n=h+1}^{\infty} V_n}$. On the other hand

$$a \in H \subset \overline{\bigcup_{n=1}^{\infty} V_n} = \overline{V_1 \cup \ldots \cup V_h \cup \bigcup_{n=h+1}^{\infty} V_n} = \overline{V}_1 \cup \ldots \cup \overline{V}_h \cup \overline{\bigcup_{n=h+1}^{\infty} V_n}$$

and hence $a \in \bigcup_{n=1}^{h} \overline{V}_n$. Consequently there is an index k with $a \in \overline{V}_k$. We have

$$a \in H \subset P - \bigcup_{n=1}^{\infty} V_n \subset P - V_k.$$

Thus, $a \in \overline{V}_k - V_k = B(V_k)$.

10.5. 10.5.1. *Let $Q \subset P$, $A \subset P$. Then*

$$B_Q(Q \cap A) \subset Q \cap B_P(A)\,. \tag{10}$$

Proof: By 8.7, $B_Q(Q \cap A) = Q \cap \overline{(Q \cap A)} \cap \overline{(Q - Q \cap A)}$. By formula (3) in 8.1, $\overline{Q \cap A} \subset \overline{A}$, $\overline{Q - Q \cap A} \subset \overline{P - A}$ and hence $(Q \cap \overline{Q \cap A}) \cap \overline{(Q - Q \cap A)} \subset$ $\subset Q \cap \overline{A} \cap \overline{(P - A)} = Q \cap B_P(A)$.

10.5.2. *Let $Q \subset P$. Let a set U_0 be open in Q. Let a set U be open in P; let $U_0 \subset U$. Then there is a set $V \subset U$ open in P and such that*

$$U_0 = Q \cap V, \qquad B_Q(U_0) = Q \cap B_P(V)\,.$$

Proof: The sets U_0 and $Q - \overline{U}_0$ are open in Q, and $U_0 \cap (Q - \overline{U}_0) = \emptyset$. Thus, the sets U_0 and $Q - U_0$ are separated, so that there are T and W open in P such that $T \cap W = \emptyset$, $T \supset U_0$, $W \supset Q - \overline{U}_0$. Since U_0 is open in Q, there is, by 8.7.5, a set G open in P such that $U_0 = Q \cap G$. Put $V = G \cap T \cap U$. Thus, the set $V \subset U$ is open in P. Since $U_0 \subset U$, $U_0 \subset T$, $U_0 = Q \cap G$, we have $U_0 = Q \cap V$. Since $V \subset T$, $Q - \overline{U}_0 \subset W$, $T \cap W = \emptyset$ and since the sets T and W are open in P, the sets V and $Q - \overline{U}_0$ are separated, so that $\overline{V} \cap (Q - \overline{U}_0) = \emptyset$ and hence $Q \cap \overline{V} \subset$ $\subset Q \cap \overline{U}_0$. Thus, as $U_0 = Q \cap V$, we have $Q \cap \overline{V} - Q \cap V \subset Q \cap \overline{U}_0 - U_0$. We have $Q \cap \overline{V} - Q \cap V = Q \cap (\overline{V} - V) = Q \cap \boldsymbol{B}_P(V)$. Since $Q \cap \overline{U}_0$ is the relative closure of the set U_0 in Q, we obtain, by (9), $Q \cap \overline{U}_0 - U_0 = \boldsymbol{B}_Q(U_0)$. Thus, $Q \cap \boldsymbol{B}_P(V) \subset \boldsymbol{B}_Q(U_0)$. By (10), also $Q \cap \boldsymbol{B}_P(V) \supset \boldsymbol{B}_Q(U_0)$.

Exercises

10.1. If sets A and B are closed, the sets $A - B$ and $B - A$ are separated.

10.2. A one-point set (a) and a set A are separated if and only if $\varrho(a, A) > 0$.

10.3.* Let $m = 3, 4, 5, \ldots$. Sets $A_1, A_2, \ldots, A_m$ are said to be separated if, for every i, k with $1 \leqq i < k \leqq m$, A_i and A_k are separated. The sets $A_1, A_2, \ldots, A_m$ are separated if and only if: [1] they are disjoint, [2] they are closed in $\bigcup_{i=1}^{m} A_i$. The word "closed" in [2] may be replaced by the word "open".

10.4. Sets $A_1, A_2, \ldots, A_m$ are separated if and only if for every n, $1 \leqq n \leqq m-1$, the sets $\bigcup_{i=1}^{n} A_i$ and A_{n+1} are separated.

For $A \subset P$, $B \subset P$ put $S(A, B) = A \cap \bar{B} \cup B \cap \bar{A}$. The set $S(A, B)$ is called the *junction* of the sets A and B.

10.5. The junction $S(A, B)$ will not change, if the space P is replaced by a space Q, $A \cup B \subset$ $\subset Q \subset P$.

10.6. $S(A, B) = A \cap B$ if and only if both the sets A and B are closed in $A \cup B$.

10.7. $S[A, S(B, C)] = S[S(A, B), C]$ does not, in general, hold (the junction is not associative).

10.8. $\bar{A} = A \cup \boldsymbol{B}(A)$.

10.9. A set A is closed if and only if $\boldsymbol{B}(A) \subset A$.

10.10. A set A is open if and only if $A \cap \boldsymbol{B}(A) = \emptyset$.

10.11. $\boldsymbol{B}(A)$ is the set of all the points at which the characteristic function of A is not continuous.

10.12. For any closed A, $\boldsymbol{B}[\boldsymbol{B}(A)] = \boldsymbol{B}(A)$.

10.13. $\boldsymbol{B}\{\boldsymbol{B}[\boldsymbol{B}(A)]\} = \boldsymbol{B}[\boldsymbol{B}(A)] \subset \boldsymbol{B}(A)$.

In exercises 10.14 and 10.15 the index ι has the same significance as it had in ex. 8.5 and the following ones.

10.14. $\boldsymbol{B}(A_\iota) \subset \boldsymbol{B}(A)$.

10.15. $\overline{[\boldsymbol{B}(A)]_\iota} = \overline{A \cap [\boldsymbol{B}(A)]_\iota} = \overline{[\boldsymbol{B}(A)]_\iota - A}$.

10.16. Let P and Q be metric spaces, let $A \subset P$, $B \subset Q$. Then $\boldsymbol{B}(A \times B) = \boldsymbol{B}(A) \times \bar{B} \cup \bar{A} \times \boldsymbol{B}(B)$.

For $A \subset B$ put $S(A) = A \cap \boldsymbol{B}(A)$. The set $S(A)$ is called the *frontier* of the set A.

10.17. $\boldsymbol{B}(A) = S(A) \cup S(P - A)$ with disjoint summands.

10.18. $S[S(A)] = S(A)$.

10.19. $\boldsymbol{B}(A) = S(A)$ if and only if A is closed.

§ 11. Dense-in-itself and dispersed spaces

11.1. A point a of a metric space P is said to be an *isolated* point of P if there is a positive ε such that $x \in P$, $\varrho(a, x) < \varepsilon$ imply $x = a$. A non-void space P is said to be *isolated*, if each one of its point is isolated.

A space P is said to be *dense-in-itself* if there are no isolated points in P.

Since a point set Q embedded into a metric space P is also a metric space, we need not define explicitly an isolated point of a (nonvoid) point set, an isolated point set, or a dense-in-itself point set.

Evidently, an isolated point a of a set $A \subset P$ is an isolated point of every the B such that $a \in B \subset A$. Consequently, a union $\bigcup A(z)$ is dense-in-itself whenever every $A(z)$ is dense-in-itself.

11.1.1. *A set $A \subset P$ is dense-in-itself if and only if the set $\bar{A}$ is dense-in-itself.*

Proof: I. Let A be dense-in-itself. Then $A \neq \emptyset$ and hence $\bar{A} \neq \emptyset$. If $\bar{A}$ is not dense-in-itself, there is an isolated point $a \in \bar{A}$. There is an $\varepsilon > 0$ such that $x \in \bar{A}$, $\varrho(a, x) < \varepsilon$ imply $x = a$. Since $a \in \bar{A}$, we have $\varrho(a, A) = 0$ and hence there is a point $b \in A$ with $\varrho(a, b) < \varepsilon$. As $A \subset \bar{A}$, we have $b = a$ and hence $a \in A$. Since a is an isolated point of the set $\bar{A} \supset A$ and since $a \in A$, a is an isolated point of A. This is a contradiction.

II. Let the set $\bar{A}$ be dense-in-itself. Then $\bar{A} \neq \emptyset$ and hence $A \neq \emptyset$. If the set A is not dense-in-itself, it has an isolated point a. There is an $\varepsilon > 0$ such that $x \in A$, $\varrho(a, x) < \varepsilon$ imply $x = a$. As the set $\bar{A}$ has no isolated points and as $a \in A \subset \bar{A}$, there is a point $b \in \bar{A}$ such that $a \neq b$, $\varrho(a, b) < \frac{1}{2}\varepsilon$. Since $b \in \bar{A}$, we have $\varrho(b, A) = 0$. Hence, there is a point $c \in A$ with $\varrho(b, c) < \varrho(a, b)$. We have $\varrho(a, c) \leqq \varrho(a, b) + \varrho(b, c) < 2\varrho(a, b) < \varepsilon$, $c \in A$ and hence $c = a$. This is a contradiction, since $\varrho(b, c) < \varrho(a, b)$.

Let P be an arbitrary metric space. If there is no dense-in-itself subset $A \subset P$, we put $K = \emptyset$. Otherwise, K is the union of all dense-in-itself $A \subset P$. By the remark above, the set K is dense-in-itself. Thus, it is the *largest dense-in-itself set embedded into* P. The set $\bar{K}$ is also dense-in-itself, hence $\bar{K} \subset K$ so that $\bar{K} = K$; this holds, of course, with $K = \emptyset$, too. The set K is called the *kernel* of the space P. Again, we need not define explicitly the kernel of a point set.

A set $A \subset P$ is termed by many authors *perfect* in P, if: [1] it is dense-in-itself, [2] it is closed in P. Notice, that the property [1] depends on the set A only, while [2] depends on the space P.

Many authors consider $\emptyset$ as a dense-in-itself set.

11.2. A space P is said to be *dispersed*, if its kernel is void, i.e. if P does not contain a non-void dense-in-itself set, i.e., if for every $A \neq \emptyset$, $A \subset P$, A has an isolated point. We need not define a dispersed point set $Q \subset P$.

If a space P is dispersed, then evidently every $A \subset P$ is dispersed. Evidently $\emptyset$ and every isolated set is dispersed. Obviously no (nonvoid) set is simultaneously both dense-in-itself and dispersed.

11.2.1. *If sets $A \subset P$ and $B \subset P$ are dispersed, then the set $A \cup B$ is also dispersed.*

Proof: On the contrary, let there be a dense-in-itself set $S \subset A \cup B$. As the set B is dispersed, it is not the case that $S \subset B$ and hence $A \cap S \neq \emptyset$; similarly $B \cap S \neq \emptyset$. Since $\emptyset \neq A \cap S \subset A$ and since A is dispersed, there exists an isolated point a of the set $A \cap S$. There is an $\varepsilon > 0$ such that $x \in A \cap S$, $\varrho(a, x) < \varepsilon$ imply $x = a$. As $a \in S$ and as S is dense-in-itself, there is a point $b \in S$ such that $b \neq a$, $\varrho(a, b) < \varepsilon$. Thus, b is not contained in $A \cap S$ and hence $b \in B \cap S$.

Thus, the set $B \cap S \cap \Omega(a, \varepsilon)$ is non-void; since $B \cap S$ is dispersed the set $B \cap S \cap \Omega(a, \varepsilon)$ has an isolated point c. Hence, there exists an $\eta > 0$ such that $x \in B \cap S \cap \Omega(a, \varepsilon)$, $\varrho(c, x) < \eta$ imply $x = c$. As $c \in S$ and as S is dense-in-itself, there is a point $d \in S$ such that $d \neq c$, $\varrho(c, d) < \eta$, $\varrho(c, d) < \varepsilon - \varrho(a, c)$, $d \neq a$ (since $c \in \Omega(a, \varepsilon)$, we have $\varepsilon - \varrho(a, c) > 0$). As $d \in S = (A \cap S) \cup (B \cap S)$, we have either $d \in A \cap S$ or $d \in B \cap S$. But $d \in A \cap S$ does not hold, as $\varrho(a, d) \leqq \varrho(a, c) + \varrho(c, d) < \varepsilon$, $d \neq a$; also $d \in B \cap S$ does not hold, for $\varrho(c, d) < \eta$, $d \neq c$.

Exercises

11.1. A point $a \in P$ is isolated if and only if the set (a) is open.

Let A_j designate the set of all isolated points of a point set A; by A_h we denote the set $A - A_j$.

11.2. The set A_h is closed in A.

11.3. $A \subset B$ implies $A_h \subset B_h$.

The set $(\bar{A})_h$ is denoted by A' and called the *derived set* of the set A. The points of the set A' are termed, as a rule, the *accumulation points* (or *cluster points*, or *limit points*) of A. While A_h depends only on the space A, A' depends also on the space $P \supset A$.

11.4. $\bar{A} = A \cup A'$.

11.5. $A_h = A \cap A'$.

11.6. $A' = \bar{A} - A_j$.

11.7. $(A \cup B)' = A' \cup B'$.

11.8. The set A' is always closed.

11.9. A' is the set of all limits of convergent one-to-one sequences $\{x_n\}$ such that $x_n \in A$.

11.10. The set A_m consisting of zero and of all the numbers of the form $\sum_{i=1}^{m} 1/n_i$ $(m, n_1, \ldots, n_m = 1, 2, 3, \ldots)$ is dispersed and closed in the space $\mathbf{E}_1$. The set $A_1 - (0)$ is isolated. We have $A'_{m+1} = (A_{m+1})_h = A_m$. The set $\bigcup_{m=1}^{\infty} A_m$ is dense-in-itself and is not closed in $\mathbf{E}_1$.

11.11. Let $A \subset B \subset \bar{A}$. The set B is dense-in-itself if and only if the set A is dense-in-itself.

11.12. If a set A is dense-in-itself, the set A' is also dense-in-itself.

11.13. If a space is dense-in-itself, then each of its open nonvoid subsets is dense-it-itself.

11.14. A space P is isolated if and only if every function on P is continuous.

In the following exercises, P and Q are two given metric spaces.

11.15. Let $a \in P$, $b \in Q$. The point (a, b) is isolated in $P \times Q$ if and only if the point a is isolated in P and the point b is isolated in Q.
11.16. If P is dense-in-itself, then $P \times Q$ is dense-in-itself.
11.17. If $P \times Q$ is dense-in-itself, then either P or Q is dense-in-itself.
11.18. If P and Q are dispersed, then $P \times Q$ is dispersed.
11.19. If $P \neq \emptyset \neq Q$ and if $P \times Q$ is dispersed, then P and Q are dispersed.

§ 12. Dense and nowhere dense sets. Sets of the first category

12.1. Let P be a metric space. A point set $A \subset P$ is said to be *dense*, more precisely, dense in P, if $\bar{A} = P$, i.e. if $\varrho(x, A) = 0$ for every point $x \in P$. The density of a set A is a topological property (similar to the property of being closed or open) depending on the "position" of A in P. This contrasts with the property of being dense-in-itself which depends on the "form" of A only.

The following theorem is obvious by the definition:

12.1.1. *If $A \subset B \subset P$ and if A is dense, then B is also dense.*

12.1.2. *A set $A \subset P$ is dense if and only if $A \cap G \neq \emptyset$ for every open $G \neq \emptyset$.*

Proof: I. Let $\bar{A} = P$. Let G be open and let $A \cap G = \emptyset$. Then $A \subset P - G$, hence $P = \bar{A} \subset P - G = \overline{P - G}$ and hence $G = \emptyset$.

II. Let $\bar{A} \neq P$. The set $G = P - \bar{A}$ is non-void and open, and we have $G \neq \emptyset = A \cap G$.

12.1.3. *Let A be a dense set, let G be an open dense set. Then the set $A \cap G$ is dense.*

Proof: Let $\Gamma \neq \emptyset$ be open. The set $G \cap \Gamma$ is open and nonvoid, since G is dense. Hence, as A is dense, we have $A \cap G \cap \Gamma \neq \emptyset$. Thus, $A \cap G \cap \Gamma \neq \emptyset$ for every open $\Gamma \neq \emptyset$ and hence $A \cap G$ is dense.

12.2. A set $A \subset P$ is said to be *nowhere dense*, more precisely, nowhere dense in P, if the set $P - \bar{A}$ is dense. It is again a topological property depending on the position of the set A in the space P, in contrast with dispersedness which depends on the form of A only.

12.2.1. *If $A \subset B \subset P$ and if B is nowhere dense, then A is nowhere dense.*

Proof: As $\overline{P - \bar{B}} = P$ and $\bar{A} \subset \bar{B}$, we have $P - \bar{A} \supset P - \bar{B}$ and hence $\overline{P - \bar{A}} \supset \overline{P - \bar{B}} = P$. Hence $\overline{P - \bar{A}} = P$.

From the definition follows immediately:

12.2.2. *If $\bar{A} = \bar{B}$ (e.g. if $A \subset B \subset \bar{A}$), and if the set A is nowhere dense, then the set B is also nowhere dense.*

12.2.3. *A set $A \subset P$ is nowhere dense if and only if every open $G \neq \emptyset$ contains an open $\Gamma \neq \emptyset$ with $A \cap \Gamma = \emptyset$.*

Proof: I. Let A be nowhere dense; then, $P - \bar{A}$ is dense. If G is a non-void open set, the set $\Gamma = G \cap (P - \bar{A})$ is non-void. The set Γ is open and $A \cap \Gamma = \emptyset$.

II. Let A not be nowhere dense; hence, $P - \bar{A}$ is not dense. Then there is an open non-void set G with $G \cap (P - \bar{A}) = \emptyset$, i.e. $G \subset \bar{A}$. Let Γ be a non-void open subset of G. We have to prove that $A \cap \Gamma \neq \emptyset$. If $A \cap \Gamma = \emptyset$, then $A \subset P - \Gamma$ and hence $\Gamma \subset G \subset \bar{A} \subset \overline{P - \Gamma} = P - \Gamma$; consequently $\Gamma = \emptyset$. This is a contradiction.

12.2.4. *Let A_i $(1 \leqq i \leqq m;\ m = 1, 2, 3, \ldots)$ be nowhere dense sets. Then the set $\bigcup_{i=1}^{m} A_i$ is nowhere dense.*

Proof: This is evident for $m = 1$. If the statement holds for some m and if sets A_i $(1 \leqq i \leqq m + 1)$ are nowhere dense, then the sets $\bigcup_{i=1}^{m} A_i$ and A_{m+1} are nowhere dense and hence the sets $P - \overline{\bigcup_{i=1}^{m} A_i} = P - \bigcup_{i=1}^{m} \bar{A}_i$ and $P - \bar{A}_{m+1}$ are dense. As the set $P - \bar{A}_{m+1}$ is open, the set $(P - \bigcup_{i=1}^{m} \bar{A}_i) \cap (P - \bar{A}_{m+1}) = P - \bigcup_{i=1}^{m+1} \bar{A}_i =$
$= P - \overline{\bigcup_{i=1}^{m+1} A_i}$ is dense (by 12.1.3) and hence the set $\bigcup_{i=1}^{m+1} A_i$ is nowhere dense.

12.3. A set $A \subset P$ is called a set of the *first category*, more precisely, of the first category in P, if there is a sequence $\{A_n\}$ of nowhere dense sets such that $A =$
$= \bigcup_{n=1}^{\infty} A_n$. This is again a topological property of the position of A in P. A set which is not of the first category is termed by many authors a *set of the second category.* A set A such that $P - A$ is of the first category is said to be *residual.*

12.3.1. *If $A \subset B \subset P$ and if B is a set of the first category, then A is a set of the first category, too.*

Proof: $B = \bigcup_{n=1}^{\infty} B_n$ with nowhere dense sets B_n. Hence, $A = \bigcup_{n=1}^{\infty} A \cap B_n$ and the sets $A \cap B_n \subset B_n$ are nowhere dense.

The definition yields immediately:

12.3.2. *Every nowhere dense set is a set of the first category.*

12.3.3. *If A_n ($n = 1, 2, 3, \dots$) are sets of the first category, $\bigcup_{n=1}^{\infty} A_n$ is also a set of the first category.*

Proof: We have $A_n = \bigcup_{i=1}^{\infty} A_{ni}$ with nowhere dense A_{ni}. By 3.5 there is a one-to-one sequence $\{(n_k, i_k)\}_{k=1}^{\infty}$ consisting of all the pairs (n, i). We have $A = \bigcup_{k=1}^{\infty} A_{n_k i_k}$.

12.4. Let a point set Q be embedded into a space P. Then (see 6.3) Q is also a metric space. A point set A embedded into Q is also embedded into P. The set A may be dense in Q, dense in P, nowhere dense in Q, nowhere dense in P, of the first category in Q, of the first category in P.

12.4.1. *The set $A \subset Q$ is dense in Q if and only if $\bar{A} \supset Q$ and if and only if $\bar{A} = \bar{Q}$.*

Proof: By 8.7.1, A is dense in Q if and only if $Q \cap \bar{A} = Q$. $Q \cap \bar{A} = Q$ implies $Q \subset \bar{A} \subset \bar{Q} \subset \bar{A}$ and hence $\bar{A} = \bar{Q}$ and $Q \cap \bar{A} = Q \cap \bar{Q} = Q$.

12.4.2. *If a set $A \subset Q$ is dense in P, then A is dense in Q and Q is dense in P.*

Proof: As $\bar{A} = P$, we have $\bar{A} \supset Q$, i.e. A is dense in Q. Q is dense in P by 12.1.1.

12.4.3. *If a set $A \subset Q$ is dense in Q and if Q is dense in P, then A is dense in P.*

Proof: $\bar{A} = \bar{Q}$, $\bar{Q} = P$ and consequently $\bar{A} = P$.

12.4.4. *If a set $A \subset Q$ is nowhere dense in Q, then A is nowhere dense in P.*

Proof: Let G be a non-void set open in P. We have to prove that there is a non-void open $\Gamma \subset G$ with $\Gamma \neq \emptyset = A \cap \Gamma$. Since $A \subset Q$, in the case $Q \cap G = \emptyset$ we may choose $\Gamma = G$. Thus, let $Q \cap G \neq \emptyset$. The set $Q \cap G$ is open in Q and non-void. Since A is nowhere dense in Q, there is a non-void $\Delta \subset Q \cap G$ open in Q such that $\Delta \neq \emptyset = A \cap \Delta$. As Δ is open in Q, there is a set H open in P such that $\Delta = Q \cap H$. Put $\Gamma = G \cap H$. The set Γ is open in P and we have $\Gamma \subset G$. Since $\Delta \subset Q \cap G$, $\Delta = Q \cap H$, we have $\Delta = Q \cap \Gamma$, and hence $\Gamma \neq \emptyset$, since $\Delta \neq \emptyset$. Since $A \subset Q$, $\Delta = Q \cap \Gamma$, we have $A \cap \Gamma = A \cap \Delta = \emptyset$.

12.4.5. *If $A \subset Q$ is a set of the first category in Q, then it is a set of the first category in P.*

Proof: $A = \bigcup_{n=1}^{\infty} A_n$ with A_n nowhere dense in Q and hence nowhere dense in P.

Exercises

12.1. A dense subset A of P contains every isolated point of P.

12.2.* A set A is dense in P if and only if for every $x \in P$ there exists a sequence $\{x_n\}$ with $x_n \in A$ and $x_n \to x$.

12.3.* If A is dense in P and if G is open in P, then $A \cap G$ is dense in G.

12.4. If A is dense in a space P, then A is dense-in-itself if and only if P is dense-in-itself.

12.5.* A finite set A is nowhere dense in a space P if and only if there is no isolated point of P in A.

12.6. If A is nowhere dense in $\bar{Q}$, then $A \cap Q$ is nowhere dense in Q.

12.7.* If a set A is either closed or open or nowhere dense, then the set $B(A)$ is nowhere dense.

12.8. If the sets $B(A)$ and $B(B)$ are nowhere dense, then the sets $B(A \cup B)$, $B(A \cap B)$, $B(A - B)$ are nowhere dense.

12.9. If G is an open set and if A is a nowhere dense set, then $A \cap G$ is nowhere dense in G and $A \cap \bar{G}$ is nowhere dense in $\bar{G}$.

12.10. Let $A \subset P$ be a dispersed set. Let P be dense in itself. Then A is dispersed in P.

12.11. If A and B are separated sets, then the set $\bar{A} \cap \bar{B}$ is nowhere dense.

12.12. No set of the first category in P contains an isolated point of P.

12.13.* A countable set A is a set of the first category in P if and only if it contains no isolated point of P.

12.14. If A is a set of the first category in $\bar{Q}$, then $A \cap Q$ is a set of the first category in Q.

12.15. If G is an open set and if A is a set of the first category, then the set $A \cap G$ is of the first category in G and the set $A \cap \bar{G}$ is of the first category in $\bar{G}$.

In the following exercises, P and Q are two metric spaces, $A \subset P$, $B \subset Q$.

12.16. If A is dense in P and if B is dense in Q, then $A \times B$ is dense in $P \times Q$.

12.17. If $P \neq \emptyset \neq Q$ and if $A \times B$ is dense in $P \times Q$, then A is dense in P and B dense in Q.

12.18. If A is nowhere dense in P, then $A \times B$ is nowhere dense in $P \times Q$.

12.19. If $A \times B$ is nowhere dense in $P \times Q$, then either A is nowhere dense in P or B is nowhere dense in Q.

§ 13. $\mathbf{G}_\delta$-sets and $\mathbf{F}_\sigma$-sets

13.1. A point set A embedded into a metric space P is said to be $\mathbf{G}_\delta$ (or a $\mathbf{G}_\delta$-set), more precisely $\mathbf{G}_\delta(P)$, if there exist *open* sets $A_n \subset P$ such that $A = \bigcap_{n=1}^{\infty} A_n$. Moreover, in such a case *there are open sets $G_n \subset P$ such that $A = \bigcap_{n=1}^{\infty} G_n$ and $G_n \supset G_{n+1}$*; it suffices to put $G_n = \bigcap_{i=1}^{n} A_i$. The notion of $\mathbf{G}_\delta$-set is a topological notion.

Obviously:

13.1.1. *Every open set A is* $\mathbf{G}_\delta$ (it suffices to put $A_n = A$).

13.1.2. *If A_n $(n = 1, 2, 3, \ldots)$ are $\mathbf{G}_\delta$-sets, the set $\bigcap_{n=1}^{\infty} A_n$ is also $\mathbf{G}_\delta$.*

Proof: There are open A_{ni} such that $A_n = \bigcap_{i=1}^{\infty} A_{ni}$. By 3.5 there exists a one-to-one sequence $\{(n_k, i_k)\}_{k=1}^{\infty}$ of all the pairs (n, i). We have $\bigcap_{n=1}^{\infty} A_n = \bigcap_{k=1}^{\infty} A_{n_k i_k}$.

13.1.3. *If A and B are $\mathbf{G}_\delta$-sets, the set $A \cup B$ is also $\mathbf{G}_\delta$.*

Proof: There exist open sets A_n and B_i such that $A = \bigcap_{n=1}^{\infty} A_n$, $B = \bigcap_{i=1}^{\infty} B_i$. If $\{(n_k, i_k)\}_{k=1}^{\infty}$ is again a one-to-one sequence of all the pairs (n, i), then, as $A_n \cup B_i$ are open, it suffices to prove that

$$A \cup B = \bigcap_{k=1}^{\infty} (A_{n_k} \cup B_{i_k}).$$

The left-hand side is evidently a subset of the right hand side. Let $x \in \bigcap_{k=1}^{\infty} (A_{n_k} \cup \cup B_{i_k})$. If x does not belong to $A \cup B$, we have neither $x \in A = \bigcap_{n=1}^{\infty} A_n$ nor $x \in \bigcap_{i=1}^{\infty} B_i$; thus, there are indices n and i such that neither $x \in A_n$ nor $x \in B_i$ and hence x does not belong to $A_n \cup B_i$. This is a contradiction, since there is an index k with $n = n_k$ and $i = i_k$ and $x \in A_{n_k} \cup B_{i_k}$.

13.2. *Every closed set A is $\mathbf{G}_\delta$. Moreover, for every closed set A there are open sets G_n such that*

$$A = \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \bar{G}_n, \qquad \bar{G}_{n+1} \subset G_n.$$

Proof: Put $G_n = \Omega(A, 1/n) = \underset{x}{\mathrm{E}}[\varrho(x, A) < 1/n]$. By 9.5 and by ex. 9.10, the sets G_n are open. (This also holds for $A = \emptyset$ since then $G_n = \emptyset$.) Moreover, the sets $\underset{x}{\mathrm{E}}[\varrho(x, A) \leqq \leqq 1/n]$ are closed and hence, by 8.4, $\bar{G}_n \subset \underset{x}{\mathrm{E}}[\varrho(x, A) \leqq 1/n]$ and hence $A \subset G_n \subset \subset \bar{G}_n \subset G_{n-1}$ so that $A \subset \bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \bar{G}_n$. It remains to be proved that $A \supset \bigcap_{n=1}^{\infty} G_n$. If $x \in \bigcap_{n=1}^{\infty} G_n$, then $\varrho(x, A) < 1/n$ for every n, hence $\varrho(x, A) = 0$ and hence finally $x \in \bar{A} = A$.

13.3. A point set A embedded into a metric space P is said to be $\mathbf{F}_\sigma$ (or a $\mathbf{F}_\sigma$-set), more precisely $\mathbf{F}_\sigma(P)$, if there exist *closed* sets $A_n \subset P$ such that $A = \bigcup_{n=1}^{\infty} A_n$. Moreover, in such a case *there are closed sets $F_n \subset P$ such that $A = \bigcup_{n=1}^{\infty} F_n$, $F_n \subset F_{n+1}$*; it suffices to put $F_n = \bigcup_{i=1}^{n} A_i$. The notion of $\mathbf{F}_\sigma$-set is a *topological notion*.

13.3.1. *A set A is $\mathbf{F}_\sigma$ if and only if $P - A$ is $\mathbf{G}_\sigma$.* In fact,

$$A = \bigcup_{n=1}^{\infty} A_n \Rightarrow P - A = \bigcup_{n=1}^{\infty} (P - A_n),$$

$$P - A = \bigcap_{n=1}^{\infty} B_n \Rightarrow A = \bigcup_{n=1}^{\infty} (P - B_n).$$

Thus, the preceding theorems yield:

13.3.2. *Every closed set is* $\mathbf{F}_\sigma$.

13.3.3. *If* A_n $(n = 1, 2, 3, \ldots)$ *are* $\mathbf{F}_\sigma$*-sets, then the set* $\bigcup_{n=1}^{\infty} A_n$ *is also* $\mathbf{F}_\sigma$.

13.3.4. *If* A *and* B *are* $\mathbf{F}_\sigma$*-sets, then the set* $A \cap B$ *is also* $\mathbf{F}_\sigma$.

13.3.5. *Every open set is* $\mathbf{F}_\sigma$.

13.4. *Let* f *be a mapping of a metric space* P *into a metric space* Q. *Let* C *be the set of all the* $x \in P$ *at which the mapping* f *is continuous; put* $D = P - C$. *Then* C *is* $\mathbf{G}_\delta(P)$ *and consequently* D *is* $\mathbf{F}_\sigma(P)$.

Proof: It is easy to see that C is the set of all the $x \in P$ having for every n $(= 1, 2, 3, \ldots)$ a neighbourhood G (in the space P) such that

$$y \in G, \ z \in G \Rightarrow \varrho[f(y), f(z)] < \frac{1}{n}. \tag{1}$$

Denote (for $n = 1, 2, 3, \ldots$) by $\mathfrak{A}_n$ the system of all sets G open in P having the property (1). Then, $\Gamma_n = \bigcup_{X \in \mathfrak{A}_n} X$ are open sets in P and $C = \bigcap_{n=1}^{\infty} \Gamma_n$ so that C is $\mathbf{G}_\delta(P)$.

13.5.*) *Let* $A \subset B \subset P$. *Let* A *be* $\mathbf{G}_\delta$ *and let* B *be* $\mathbf{F}_\sigma$. *Then there exists a set* $C \subset P$ *such that:* [1] C *is* $\mathbf{G}_\delta$, [2] C *is* $\mathbf{F}_\sigma$, [3] $A \subset C \subset B$.

Proof: We have $A = \bigcap_{n=1}^{\infty} G_n$, $B = \bigcup_{n=1}^{\infty} F_n$ where the sets G_n are open and the sets F_n closed. Let us define recursively the sets H_n and K_n as follows:

$$H_1 = G_1, \quad K_1 = G_1 \cap F_1, \quad H_{n+1} = K_n \cup \bigcap_{i=1}^{n+1} G_i,$$
$$K_{n+1} = H_{n+1} \cap \bigcup_{i=1}^{n+1} F_i. \tag{1}$$

Put

$$H = \bigcap_{n=1}^{\infty} H_n, \qquad K = \bigcup_{n=1}^{\infty} K_n. \tag{2}$$

*) From the statement of the theorem we obtain: if A_1, A_2 are two disjoint subsets of P, both of them $\mathbf{G}_\delta$, $\mathbf{F}_\sigma$, or both of them there are C_1, C_2 which are both $\mathbf{G}_\delta$ and $\mathbf{F}_\sigma$ such that $C_1 \cap C_2 = \emptyset$, $A_i \subset C_i$ $(i = 1, 2)$.

We may prove by induction that the sets H_n and K_n are both $\mathbf{G}_\delta$ and $\mathbf{F}_\sigma$ so that the set H is $\mathbf{G}_\delta$ and the set K is $\mathbf{F}_\sigma$. Thus, it suffices to prove that

$$A \subset H = K \subset B.$$

By (1) we have

$$K_n \subset H_n. \tag{3}$$

Moreover, $\bigcap_{i=1}^{n+1} G_i \subset \bigcap_{i=1}^{n} G_i \subset H_n$ and hence

$$H_{n+1} \subset H_n. \tag{4}$$

Finally, $K_n \subset \bigcup_{i=1}^{n} F_i \subset \bigcup_{i=1}^{n+1} F_i$, $K_n \subset H_{n+1}$ and hence

$$K_{n+1} \supset K_n. \tag{5}$$

I. Let $x \in A$. Then we have $x \in G_i$ for every i, hence, by (1), $x \in H_n$ for every n, hence $x \in H$. Consequently $A \subset H$.

II. Let $x \in K$. Then there is an index m such that $x \in K_m$. By (1), $K_m \subset \bigcup_{i=1}^{m} F_i$, hence $x \in \bigcup_{i=1}^{\infty} F_i = B$. Hence, $K \subset B$.

III. Let $x \in K$. Then there is an index m such that $x \in K_m$. Thus, by (5), $n \geqq m$ implies $x \in K_n$. Thus, by (3) $n \geqq m$ implies $x \in H_n$, so that, by (4), $x \in \bigcap_{i=1}^{\infty} H_n = H$. Hence, $K \subset H$.

IV. Let $x \in H - K$. Then, for every n, $x \in H_{n+1} - K_n \subset \bigcap_{i=1}^{n+1} G_i$. Hence, $x \in \bigcap_{n=1}^{\infty} G_n = A \subset B = \bigcup_{n=1}^{\infty} F_n$, so that there exists an index m such that $x \in \bigcup_{n=1}^{m} F_n$. Since also $x \in H_m$, we have $x \in H_m \cap \bigcup_{n=1}^{m} F_n = K_m$, which is a contradition. Thus, $H - K = \emptyset$, i.e. $H = K$.

13.6. Let Q by a point set embedded into a metric space P, so that Q is also a metric space.

13.6.1. *A set $A \subset Q$ is $\mathbf{G}_\delta(Q)$ if and only if there is a set B such that* [1] $A = Q \cap B$, [2] *B is $\mathbf{G}_\delta(P)$.*

Proof: Let A be a $\mathbf{G}_\delta(Q)$-set. Then $A = \bigcap_{n=1}^{\infty} A_n$ with A_n open in Q for every n. By 8.7.5 there exist sets B_n open in P such that $A_n = Q \cap B_n$. It suffices to put $B = \bigcap_{n=1}^{\infty} B_n$.

II. Let B be $\mathbf{G}_\delta(P)$. Then $B = \bigcap_{n=1}^{\infty} B_n$ with B_n open in P for every n. The sets $Q \cap B_n$ are open in Q, so that $Q \cap B = \bigcap_{n=1}^{\infty} Q \cap B_n$ is $\mathbf{G}_\delta(Q)$.

Similarly we may prove the following:

13.6.2. *A set $A \subset Q$ is $\mathbf{F}_\sigma(Q)$ if and only if there is a set B such that* [1] $A = Q \cap B$, [2] B is $\mathbf{F}_\sigma(P)$.

Exercises

If A is a point set embedded into a metric space P, we use the following terms: [1] A is $\mathbf{G}_{\delta\sigma}$, more precisely $\mathbf{G}_{\delta\sigma}(P)$, if there exist sets $A_n \subset P$ such that $A = \bigcup_{n=1}^{\infty} A_n$ and every A_n is $\mathbf{G}_\delta$; [2] A is $\mathbf{F}_{\sigma\delta}$, more precisely $\mathbf{F}_{\sigma\delta}(P)$, if there exist sets $A_n \subset P$ such that $A = \bigcap_{n=1}^{\infty} A_n$ and every A_n is $\mathbf{F}_\sigma$.

13.1. A set A is $\mathbf{F}_{\sigma\delta}$ if and only if $P - A$ is $\mathbf{G}_{\delta\sigma}$.

13.2. If A is $\mathbf{G}_\delta$ or if A is $\mathbf{F}_\sigma$, then A is both $\mathbf{G}_{\delta\sigma}$ and $\mathbf{F}_{\sigma\delta}$.

13.3. If A_n are $\mathbf{G}_{\delta\sigma}$-sets, then the set $\bigcup_{n=1}^{\infty} A_n$ is $\mathbf{G}_{\delta\sigma}$.

13.4. If A_n are $\mathbf{F}_{\sigma\delta}$ sets, then the set $\bigcap_{n=1}^{\infty} A_n$ is $\mathbf{F}_{\sigma\delta}$.

13.5. If A and B are $\mathbf{G}_{\delta\sigma}$, then the set $A \cap B$ is $\mathbf{G}_{\delta\sigma}$.

13.6. If A and B are $\mathbf{F}_{\sigma\delta}$, then the set $A \cup B$ is $\mathbf{F}_{\sigma\delta}$.

13.7.* Let f be a continuous mapping of a metric space P into a metric space Q. Let $A \subset Q$. If A is $\mathbf{G}_\delta(Q)$, then $f_{-1}(A)$ is $\mathbf{G}_\delta(P)$; if A is $\mathbf{F}_\sigma(Q)$, then $f_{-1}(A)$ is $\mathbf{F}_\sigma(P)$; if A is $\mathbf{G}_{\delta\sigma}(Q)$, then $f_{-1}(A)$ is $\mathbf{G}_{\delta\sigma}(P)$; if A is $\mathbf{F}_{\sigma\delta}(Q)$, then $f_{-1}(A)$ is $\mathbf{F}_{\sigma\delta}(P)$.

13.8. Let $A \subset B \subset P$. Let A be $\mathbf{F}_{\sigma\delta}$ and let B be $\mathbf{G}_{\delta\sigma}$. Then there exists a set $C \subset P$ such that: [1] C is $\mathbf{G}_{\delta\sigma}$, [2] C is $\mathbf{F}_{\sigma\delta}$, [3] $A \subset C \subset B$.

13.9. Let $A \subset Q \subset P$. The set A is $\mathbf{G}_{\delta\sigma}(Q)$ if and only if there exists a set $B \subset P$ such that [1] $A = Q \cap B$, [2] B is $\mathbf{G}_{\delta\sigma}(P)$. The set A is $\mathbf{F}_{\sigma\delta}(Q)$ if and only if there exists a set $B \subset P$ such that [1] $A = Q \cap B$, [2] B is $\mathbf{F}_{\sigma\delta}(P)$.

13.10.* Let $A \subset Q \subset P$. If Q is $\mathbf{G}_\delta(P)$, then A is $\mathbf{G}_\delta(P)$ if and only if it is $\mathbf{G}_\delta(Q)$. It is permitted to write simultaneously $\mathbf{F}_\sigma$ or $\mathbf{G}_{\delta\sigma}$ or $\mathbf{F}_{\sigma\delta}$ instead of $\mathbf{G}_\delta$.

13.11.* Every countable set $A \subset P$ is $\mathbf{F}_\sigma(P)$.

13.12. If $A \subset P$ is a set of the first category, then there is a set $B \subset P$ such that [1] $A \subset B$, [2] B is $\mathbf{F}_\sigma$, [3] B is a set of the first category

13.13. Let a set $A \subset P$ be $\mathbf{F}_\sigma$. Let $P - A$ be a dense set. Then A is a set of the first category.

13.14. Let P and Q be metric spaces. Let $C \subset P \times Q$. For every $x \in P$ put $\sigma''_x(C) = \underset{y}{\mathrm{E}}[(x, y) \in C]$.

If a set C is open (closed, $\mathbf{F}_\sigma$, $\mathbf{G}_\delta$, $S_{\sigma\delta}$, $\mathbf{G}_{\delta\sigma}$) in $P \times Q$, also the set $\sigma''_x(C)$ is, for every $x \in P$ open (closed, $\mathbf{F}_\sigma$, $\mathbf{G}_\delta$, $\mathbf{F}_{\sigma\delta}$, $\mathbf{G}_{\delta\sigma}$) in Q.

Similarly for $\sigma'_y(C) = \underset{x}{\mathrm{E}}[(x, y) \in C]$ $(y \in Q)$.

13.15. Let P and Q be two metric spaces. Let $\emptyset \neq A \subset P$, $\emptyset \neq B \subset Q$. The set $A \times B$ is $\mathbf{G}_\delta(P \times Q)$ if and only if A is $\mathbf{G}_\delta(P)$ and B is $\mathbf{G}_\delta(Q)$. It is permitted to write (simultaneously) $\mathbf{F}_\sigma$, or $\mathbf{G}_{\delta\sigma}$ or $\mathbf{F}_{\sigma\delta}$ instead of $\mathbf{G}_\delta$.

§ 14. Functions of the first class

14.1. Let P be a metric space. Let f be a function on P. We say that f is a *function of the first class*, if there is a sequence $\{f_n\}$ of *continuous* functions on P such that $f_n(x) \to f(x)$ for every point $x \in P$. We may always attain this by means of *bounded* functions f_n; if f_n are not bounded, it suffices to replace them by functions g_n defined as follows:

$$\begin{aligned} g_n(x) &= f_n(x) \quad \text{if} \quad |f_n(x)| \leqq n\,, \\ g_n(x) &= n \quad \text{if} \quad f_n(x) > n, \\ g_n(x) &= -n \quad \text{if} \quad f_n(x) < -n\,. \end{aligned}$$

The following theorems are evident:

14.1.1. *Every continuous function is of the first class.*

14.1.2. *If c_i are real numbers and if f_i are finite functions of the first class, then $\sum_{i=1}^{m} c_i f_i$ is a finite function of the first class.*

14.1.3. *If Q is a point set embedded into a metric space P and if f is a function of the first class on P, then the partial function f_Q is a function of the first class on Q.*

14.2. Let f and f_n be finite functions on P. We say that f is the *uniform limit* of the sequence $\{f_n\}$, if for every $\varepsilon > 0$ there is an index p such that for every $x \in P$

$$n \geqq p \Rightarrow |f_n(x) - f(x)| < \varepsilon\,.$$

The index p depends on ε only, not on x (otherwise every limit would be uniform).

14.2.1. *Let f be a finite function. Let $\{f_n\}$ be a sequence of finite functions of the first class. Let f be the uniform limit of the sequence $\{f_n\}$. Then f is a function of the first class.*

Proof: For $i = 1, 2, 3, \ldots$ there is an index n_i such that

$$n \geqq n_i \quad \text{implies} \quad |f_n(x) - f(x)| < \frac{1}{2^{i+1}}\,,$$

hence

$$m \geqq n_i, \qquad n \geqq n_i \quad \text{imply} \quad |f_m(x) - f_n(x)| < \frac{1}{2^i}\,.$$

Evidently, we may assume that $n_1 < n_2 < \ldots$. Put

$$F(x) = f(x) - f_{n_1}(x), \quad F_i(x) = f_{n_{i+1}}(x) - f_{n_i}(x) \quad (i = 1, 2, 3, \ldots)\,.$$

Then $|F_i(x)| < \dfrac{1}{2^i}$ and

$$\sum_{i=1}^{\infty} F_i(x) = \lim_{m\to\infty} \sum_{i=1}^{m} F_i(x) = \lim_{m\to\infty} [\![f_{n_{m+1}}(x) - f_{n_1}(x)]\!] = f(x) - f_{n_1}(x) = F(x).$$

Since $f(x) = F(x) + f_{n_1}(x)$, if suffices to prove that F is a function of the first class. The functions F_i are of the first class and hence there are finite continuous functions ψ_{in} such that $\lim\limits_{n\to\infty} \psi_{in}(x) = F_i(x)$ for $i = 1, 2, 3, \ldots$ and for every $x \in P$. Define functions φ_{in} as follows:

$$\varphi_{in}(x) = \psi_{in}(x) \quad \text{for} \quad |\psi_{in}(x)| < \frac{1}{2^i},$$

$$\varphi_{in}(x) = \frac{1}{2^i} \quad \text{for} \quad \psi_{in}(x) \geqq \frac{1}{2^i},$$

$$\varphi_{in}(x) = -\frac{1}{2^i} \quad \text{for} \quad \psi_{in}(x) \leqq -\frac{1}{2^i}.$$

Then, φ_{in} are continuous functions; since $|F_i(x)| < 1/2^i$, we have

$$|\varphi_{in}(x) - F_i(x)| \leqq |\psi_{in}(x) - F_i(x)|,$$

and hence $\lim\limits_{n\to\infty} \varphi_{in}(x) = F_i(x)$. Put

$$\Phi_n(x) = \varphi_{1n}(x) + \ldots + \varphi_{nn}(x).$$

The functions Φ_n are continuous; hence it suffices to deduce that $\lim\limits_{n\to\infty} \Phi_n(x) = F(x)$.

Choose a point $x \in P$ and a number $\varepsilon > 0$. Choose an index k such that $1/2^k < \varepsilon/3$ and such that $|F(x) - \sum\limits_{i=1}^{k} F_i(x)| < \varepsilon/3$. Since $\lim\limits_{n\to\infty} \varphi_{in}(x) = F_i(x)$, there is an index $m > k$ such that for $i = 1, 2, \ldots, k$ and for every $n > m$ we have $|\varphi_{in}(x) - F_i(x)| < \varepsilon/3k$. Let $n > m$. Then

$$|\Phi_n(x) - F(x)| \leqq \left|\sum_{i=1}^{k} F_i(x) - F(x)\right| + \sum_{i=1}^{k} |\varphi_{in}(x) - F_i(x)| +$$
$$+ \sum_{i=k+1}^{\infty} |\varphi_{in}(x)| < \frac{\varepsilon}{3} + k \cdot \frac{\varepsilon}{3k} + \sum_{i=k+1}^{\infty} \frac{1}{2^i} = \frac{2\varepsilon}{3} + \frac{1}{2^k} < \varepsilon.$$

Hence, $n > m$ implies $|\Phi_n(x) - F(x)| < \varepsilon$ and therefore $\lim\limits_{n\to\infty} \Phi_n(x) = F(x)$.

14.3. 14.3.1. *Let P be a metric space. Let f be a function on P. A necessary and sufficient condition for f to be of the first class is the following: for every $c \in \mathbf{E}_1$ the sets $\underset{x}{\mathrm{E}}[f(x) > c]$ and $\underset{x}{\mathrm{E}}[f(x) < c]$ are $\mathbf{F}_\sigma(P)$.*

Another form of the condition: for every $c \in \mathbf{E}_1$, the sets $\underset{x}{\mathrm{E}}[f(x) \leqq c]$ *and* $\underset{x}{\mathrm{E}}[f(x) \geqq c]$ *are* $\mathbf{G}_\delta(P)$.

Proof: I. Let f be a function of the first class and let $\{f_n\}$ be a sequence of continuous functions such that $f_n(x) \to f(x)$ for every $x \in P$. Let $c \in \mathbf{E}_1$. If $f(x) > c$, there is an index m such that

$$n \geqq m \quad \text{implies} \quad f_n(x) \geqq c + \frac{1}{m}. \tag{1}$$

On the other hand, if there is an index m with (1), we have $f(x) > c$. Hence

$$\underset{x}{\mathrm{E}}[f(x) > c] = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} A_{nm}, \qquad A_{nm} = \underset{x}{\mathrm{E}}\left[f_n(x) \geqq c + \frac{1}{m}\right].$$

Since the functions f_n are continuous, the sets A_{nm} are closed by 9.5, hence the sets $\bigcap_{n=m}^{\infty} A_{nm}$ are closed and consequently, the set $\underset{x}{\mathrm{E}}[f(x) > c]$ is $\mathbf{F}_\sigma$. Similarly we may prove that also the set $\underset{x}{\mathrm{E}}[f(x) < c]$ is $\mathbf{F}_\sigma$. Hence, the sets $\underset{x}{\mathrm{E}}[f(x) \leqq c]$ and $\underset{x}{\mathrm{E}}[f(x) \geqq c]$ are $\mathbf{G}_\delta(P)$ by 13.3.1.

II. Let f be the characteristic function of a set A, which is simultaneously both $\mathbf{G}_\delta$ and $\mathbf{F}_\sigma$. Let us prove that f is a function of the first class. Since A is both $\mathbf{G}_\delta$ and $\mathbf{F}_\sigma$, there are closed sets F_n and open sets G_n such that

$$F_n \subset F_{n+1}, \quad G_{n+1} \subset G_n, \quad A = \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} G_n.$$

If $F_n \neq \emptyset$ and $G_n \neq P$ put

$$f_n(x) = \frac{\varrho(x, P - G_n)}{\varrho(x, F_n) + \varrho(x, P - G_n)}.$$

Since F_n and $P - G_n$ are closed sets, $\varrho(x, F_n) = 0$ holds for $x \in F_n$ and $\varrho(x, P - G_n) = 0$ for $x \in P - G_n$ only; as $F_n \subset G_n$, we have $\varrho(x, F_n) + \varrho(x, P - G_n) > 0$ for every point x. The function f_n is, by ex. 9.10, continuous and obviously has the following properties

$$x \in F_n \quad \text{implies} \quad f_n(x) = 1, \quad x \in P - G_n \quad \text{implies} \quad f_n(x) = 0 \tag{2}$$

If $F_n = \emptyset$ and $G_n \neq P$ put $f_n(x) = 0$ for every point x; if $G_n = P$, put $f_n(x) = 1$ for every point x. In both cases, f_n is again a continuous function with the properties (2). It suffices to show that $f_n(x) \to f(x)$ for very point x. *First*, if $x \in A = \bigcup_{n=1}^{\infty} F_n$, there is an index m such that $x \in F_m$. Since $F_n \subset F_{n+1}$ for $n \geqq m$, $x \in F_n$ and hence $f_n(x) = 1$; thus, $\lim_{n \to \infty} f_n(x) = 1 = f(x)$. *Secondly*, if $x \in P - A =$

$= P - \bigcap_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} (P - G_n)$, there is an index m such that $x \in P - G_m$. As $G_{n+1} \subset G_n$, we have, for $n \geqq m$, $x \in P - G_n$, hence $f_n(x) = 0$, and hence $\lim f_n(x) = 0 = f(x)$.

III. Let f be finite and let it gain only a finite number of values $c_1, c_2, \ldots, c_m$. Let every set $C_i = \underset{x}{\mathrm{E}}[f(x) = c_i]$ $(1 \leqq i \leqq m)$ be simultaneously both $\mathbf{G}_\delta$ and $\mathbf{F}_\sigma$. If f_i is the characteristic function of the set C_i, we have evidently $f(x) = \sum_{i=1}^{\infty} c_i f_i(x)$ for every point x. By II, the functions f_i are of the first class, so that f is of the first class.

IV. Let f be such that: [1] $-1 \leqq f(x) \leqq 1$ for every point x, [2] the sets $\underset{x}{\mathrm{E}}[fx) > c]$ and $\underset{x}{\mathrm{E}}[f(x) < c]$ are $\mathbf{F}_\sigma$ for every real number c. For $n = 1, 2, 3, \ldots$ and for an integer i with $-n \leqq i \leqq n - 1$ put

$$A_{in} = \underset{x}{\mathrm{E}}\left[\frac{i}{n} \leqq f(x) \leqq \frac{i+1}{n}\right] = P - \left(\underset{x}{\mathrm{E}}\left[f(x) < \frac{i}{n}\right] \cup \underset{x}{\mathrm{E}}\left[f(x) > \frac{i+1}{n}\right]\right),$$

$$B_{in} = \underset{x}{\mathrm{E}}\left[\frac{i-1}{n} < f(x) < \frac{i+2}{n}\right] = \underset{x}{\mathrm{E}}\left[f(x) > \frac{i-1}{n}\right] \cap \underset{x}{\mathrm{E}}\left[f(x) < \frac{i+2}{n}\right].$$

Then

$$P = \bigcup_{i=-n}^{i-1} A_{in}, \qquad A_{in} \subset B_{in}; \qquad x \in B_{in} \Rightarrow \left|f(x) - \frac{i}{n}\right| < \frac{2}{n},$$

the sets B_{in} are $\mathbf{F}_\sigma$ and the sets A_{in} are $\mathbf{G}_\delta$. By 13.5 there are sets C_{in} such that: [1] $A_{in} \subset C_{in} \subset B_{in}$ and hence $\bigcup_{i=-n}^{n-1} C_{in} = P$ and $|f(x) - (i/n)| < 2/n$ for $x \in C_{in}$ [2] C_{in} are $\mathbf{F}_\sigma$, [3] C_{in} are $\mathbf{G}_\delta$. Put $D_{-n,n} = C_{-n,n}$, $D_{i+1,n} = C_{i+1,n} - \bigcup_{j=-n} C_{jn}$ $(-n \leqq i \leqq n-2)$.

Then: [1] $P = \bigcup_{i=-n}^{n-1} D_{in}$ with disjoint summands, [2] $x \in D_{in}$ implies $|f(x) - i/n| < 2/n$, [3] D_{in} are $\mathbf{F}_\sigma$, [4] D_{in} are $\mathbf{G}_\delta$. By the property [1] of the sets D_{in} there are functions f_n on P such that $f_n(x) = i/n$ for $x \in D_{in}$. By properties [3] and [4] of the sets D_{in} and by III, the functions f_n are of the first class. By property [2] of the sets D_{in}, the function f is a uniform limit of the sequence $\{f_n\}$; thus, by 14.2.1, f is a function of the first class.

V. Let f be a function such that the sets $\underset{x}{\mathrm{E}}[f(x) > c]$ and $\underset{x}{\mathrm{E}}[f(x) < c]$ are $\mathbf{F}_\sigma$ for every $c \in \mathbf{E}_1$, or, which is by 13.3.1 the same, such that $\underset{x}{\mathrm{E}}[f(x) \leqq c]$ and $\underset{x}{\mathrm{E}}[f(x) \geqq c]$ are $\mathbf{G}_\delta$. By ex. 9.18 there exists a homeomorphic mapping φ of the set **R** onto the interval $\underset{t}{\mathrm{E}}[-1 \leqq t \leqq 1]$. Set $F(x) = \varphi[f(x)]$. Then the function F has (on P) the following property

$$-1 \leqq F(x) \leqq 1 \quad \text{for} \quad x \in P .$$

Let $c \in \mathbf{E}_1$. If, *first*, $c \geqq 1$, then $\underset{x}{\mathrm{E}}[F(x) > c] = \emptyset$; *secondly*, if $c < -1$ then $\underset{x}{\mathrm{E}}[f(x) > c] = P$. *Thirdly*, if $-1 < c < 1$, then $\underset{x}{\mathrm{E}}[F(x) > c] = \underset{x}{\mathrm{E}}[f(x) > \varphi_{-1}(c)]$; *fourthly*, $\underset{x}{\mathrm{E}}[F(x) > -1] = \bigcup_{n=1}^{\infty} \underset{x}{\mathrm{E}}[F(x) > -1 + (1/n)]$. Hence, for every $c \in \mathbf{E}_1$, the set $\underset{x}{\mathrm{E}}[F(x) > c]$ is $\mathbf{F}_\sigma$ and this may be similarly proved for the set $\underset{x}{\mathrm{E}}[F(x) < c]$. Thus, by IV, F is a function of the first class and hence there is a sequence $\{F_n\}$ of continuous functions such that $F_n(x) \to F(x)$ for every point x. Put

$$\begin{aligned} G_n(x) &= F_n(x) \quad \text{for} \quad |F_n(x)| \leqq 1 , \\ G_n(x) &= 1 \quad \text{for} \quad F_n(x) > 1 , \\ G_n(x) &= -1 \quad \text{for} \quad F_n(x) < -1 . \end{aligned}$$

Then G_n are continuous functions such that $G_n(P) \subset \underset{t}{\mathrm{E}}[-1 \leqq t \leqq 1]$ and $G_n(x) \to$ $\to F(x)$. Put

$$f_n(x) = \varphi_{-1}[G_n(x)] .$$

Then, f_n are continuous functions, and, for every point x,

$$f_n(x) \to \varphi_{-1}[F(x)] = f(x) .$$

Hence, f is a function of the first class.

Thus, the proof of theorem 14.3.1 is finished. By formulas (2) in 9.5, for every function f of the first class the sets $\underset{x}{\mathrm{E}}[f(x) = c]$ $(c \in \mathbf{R})$ are $\mathbf{G}_\delta$ and the sets $\underset{x}{\mathrm{E}}[f(x) < \infty]$, $\underset{x}{\mathrm{E}}[f(x) > -\infty]$ are $\mathbf{F}_\sigma$.

14.4. *Let f and g be finite functions of the first class. Then $f \, . \, g$ is a function of the first class. If $g(x) \neq 0$ for every point $x \in P$, f/g is also a function of the first class.*

Proof: I. Let $c \in \mathbf{E}_1$. If $c < 0$, then $\underset{x}{\mathrm{E}}[f^2(x) > c] = P$. If $c \geqq 0$, then $\underset{x}{\mathrm{E}}[f^2(x) > c] =$ $= \underset{x}{\mathrm{E}}[f(x) > \sqrt{c}] \cup \underset{x}{\mathrm{E}}[f(x) < -\sqrt{c}]$. If $c \leqq 0$, then $\underset{x}{\mathrm{E}}[f^2(x) < c] = \emptyset$. If $c > 0$, then $\underset{x}{\mathrm{E}}[f^2(x) < c] = \underset{x}{\mathrm{E}}[f(x) < \sqrt{c}] \cap \underset{x}{\mathrm{E}}[f(x) > -\sqrt{c}]$. Hence, by 14.3, the sets $\underset{x}{\mathrm{E}}[f^2(x) > c]$ and $\underset{x}{\mathrm{E}}[f^2(x) < c]$ are $\mathbf{F}_\sigma$ and f^2 is a function of the first class.

II. Since

$$f \, . \, g = \frac{1}{4}(f + g)^2 - \frac{1}{4}(f - g)^2,$$

$f \, . \, g$ is also a function of the first class.

III. If $c > 0$, then (assuming $g(x) \neq 0$)

$$\mathop{\mathrm{E}}_x\left[\frac{1}{g(x)} > c\right] = \mathop{\mathrm{E}}_x\left[g(x) < \frac{1}{c}\right] \cap \mathop{\mathrm{E}}_x[g(x) > 0],$$

$$\mathop{\mathrm{E}}_x\left[\frac{1}{g(x)} < c\right] = \mathop{\mathrm{E}}_x\left[g(x) > \frac{1}{c}\right] \cup \mathop{\mathrm{E}}_x[g(x) < 0].$$

If $c < 0$, then

$$\mathop{\mathrm{E}}_x\left[\frac{1}{g(x)} > c\right] = \mathop{\mathrm{E}}_x\left[g(x) < \frac{1}{c}\right] \cup \mathop{\mathrm{E}}_x[g(x) > 0],$$

$$\mathop{\mathrm{E}}_x\left[\frac{1}{g(x)} < c\right] = \mathop{\mathrm{E}}_x\left[g(x) > \frac{1}{c}\right] \cap \mathop{\mathrm{E}}_x[g(x) < 0].$$

Finally,

$$\mathop{\mathrm{E}}_x\left[\frac{1}{g(x)} > 0\right] = \mathop{\mathrm{E}}_x[g(x) > 0],$$

$$\mathop{\mathrm{E}}_x\left[\frac{1}{g(x)} < 0\right] = \mathop{\mathrm{E}}_x[g(x) < 0].$$

Hence, by 14.3, $1/g$ is a function of the first class and consequently, by II, $f/g = f \,.\, 1/g$ is a function of the first class.

14.5. 14.5.1. *Let f be a finite function on P. Let $\{f_n\}$ be a sequence of finite functions on P. Let $f(x_n) \to f(x)$ for every $x \in P$. Let $a \in P$. Let all the functions f_n be continuous at the point a. A necessary and sufficient condition for f to be continuous at a is: for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ and an index $m(\varepsilon)$ such that*

$$\varrho(a, x) < \delta(\varepsilon) \quad \textit{implies} \quad |f_{m(\varepsilon)}(x) - f(x)| < \varepsilon .$$

Proof: I. Let the condition be satisfied. Choose an $\varepsilon > 0$ and determine the $\delta(\varepsilon)$ and $m(\varepsilon) = m$. As f_m is continuous at the point a, there is an $\eta(\varepsilon)$ with $0 < \eta(\varepsilon) < \delta(\varepsilon)$ such that $\varrho(a, x) < \eta(\varepsilon)$ implies $|f_m(x) - f_m(a)| < \varepsilon$. Then

$$\varrho(a, x) < \eta(\varepsilon) \Rightarrow |f(x) - f(a)| \leqq f(x) - f_m(x)| + |f_m(x) - f_m(a)| + |f_m(a) - f(a)| < 3\varepsilon .$$

Hence, for every $\varepsilon > 0$, there is an $\eta(\varepsilon) > 0$ such that $\varrho(a, x) < \eta(\varepsilon)$ implies $|f(x) - f(a)| < 3\varepsilon$. Thus, the function f is continuous at the point a.

II. Let the function f be continuous at the point a. Choose an $\varepsilon > 0$. There exists a $\delta_1 > 0$ such that $\varrho(a, x) < \delta_1$ implies $|f(x) - f(a)| < \varepsilon/3$. Since $f_n(a) \to f(a)$, there is an index m such that $|f_m(a) - f(a)| < \varepsilon/3$. Since f_m is continuous at the point a, there is a $\delta_2 > 0$ such that $\varrho(a, x) < \delta_2$ implies $|f_m(x) - f_m(a)| < \varepsilon/3$. Put $\delta = \min(\delta_1, \delta_2)$. Then

$$\varrho(a, x) < \delta \Rightarrow |f_m(x) - f(x)| \leqq |f_m(x) - f_m(a)| + |f_m(a) - f(a)| + |f(a) - f(x)| < \varepsilon .$$

14.5.2. *Let f be a function of the first class on P. Let D be the set of all $x \in P$ at which the function f is not continuous. Then D is a set of the first category in P.**)

Proof: I. Let the function f be finite. There is a sequence $\{f_n\}$ of finite continuous functions on P such that $f_n(x) \to f(x)$ for every point x. For every $\varepsilon > 0$ and for $m = 1, 2, 3, \ldots$ put

$$A_{m,\varepsilon} = \mathop{\mathrm{E}}_x[\mu > m, \nu > m \Rightarrow |f_\mu(x) - f_\nu(x)| \leqq \varepsilon]. \tag{1}$$

As the functions $f_\mu(x) - f_\nu(x)$ are continuous, the sets $A_{m,\varepsilon}$ are closed by 9.5. As the sequence $\{f_n(x)\}$ is convergent in the ordinary sense for every $x \in P$, we have

$$\bigcup_{m=1}^{\infty} A_{m,\varepsilon} = P \tag{2}$$

for every $\varepsilon > 0$. Put

$$B_{m,\varepsilon} = B(A_{m,\varepsilon}). \tag{3}$$

By ex. 12.7, the sets $B_{m,\varepsilon}$ are nowhere dense. Thus, it suffices to prove that

$$D \subset \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} B_{m,1/n}.$$

Let

$$a \in P - \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} B_{m,1/n}. \tag{4}$$

We have to prove that the function f is continuous at the point a.

Choose an $\varepsilon > 0$ and an index $p > 1/\varepsilon$. By (2) and (4) we have

$$a \in \bigcup_{m=1}^{\infty} A_{m,1/p} - \bigcup_{m=1}^{\infty} B_{m,1/p},$$

so that there is an index q such that $a \in A_{q,1/p} - B_{q,1/p}$. By (3), we have $a \in P - \overline{P - A_{q,1/p}}$, so that the number $\varrho(a, P - A_{q,1/p})$ is positive. Let $0 < \delta < \varrho(a, P - A_{q,1/p})$. Then

$$\Omega(a, \delta) \subset A_{q,1/p},$$

so that, by (1)

$$\varrho(a, x) < \delta, \quad \mu > q, \quad \nu > q \quad \Rightarrow \quad |f_\mu(x) - f_\nu(x)| \leqq \frac{1}{p}.$$

*) Theorem 14.5.2 (called sometimes Baire's theorem) gives a necessary condition for f to be a function of the first class in P. By 14.1.3 the following theorem also holds:

14.5.3. *Let f be a function of the first class on P. Let $Q \subset P$. Let D_Q be the set of such $x \in Q$ at which the partial function f_Q is not cotinuous. The set D_Q is of the first category in Q.*

We shall prove later (see, in particular, theorem 16.6.3 and the associated footnote) that the *necessary* condition, stated in theorem 14.5.3 is, in some special spaces, also a *sufficient* condition. The steps, by which we reach theorem 16.6.3, are, in particular, theorems 14.5.2, 15.8.3, 16.6.2.

As $f_n(x) \to f(x)$, we have

$$\varrho(a, x) < \delta, \quad n > q \quad \Rightarrow \quad |f_n(x) - f(x)| \leqq \frac{1}{p} < \varepsilon .$$

Thus, by 14.5.1, the function f is continuous at the point a.

II. Let f be an arbitrary function of the first class. Let $\{f_n\}$ be a sequence of continuous functions such that $f_n(x) \to f(x)$. By ex. 9.18 there is a homeomorphic mapping φ of the set $\mathbf{R}$ onto the interval $\mathop{\mathrm{E}}_t[-1 \leqq t \leqq 1]$. Put $\varphi[f_n(x)] = F_n(x)$, $\varphi[f(x)] = F(x)$. Then the functions F_n are continuous and $F_n(x) \to F(x)$. Thus, F is a finite function of the first class. Evidently, the set D of all points at which the function f is not continuous coincides with the set of all points at which the function F is not continuous. Thus, D is a set of the first category by I.

14.6. *Let f be a function on P. Let D, the set of all $x \in P$ at which f is not continuous, be countable. Then f is a function of the first class.*

Proof: Choose a $c \in \mathbf{E}_1$, and put $A = \mathop{\mathrm{E}}_x[f(x) > c]$. Put $C = P - D$.

If $x \in A \cap C$, we have $f(x) > c$ and the function f is continuous at x; consequently, for every $x \in A \cap C$ there is a number $\delta(x) > 0$ such that $\Omega[x, \delta(x)] \subset A$. Put

$$B = \bigcup_{x \in A \cap C} \Omega[x, \delta(x)] .$$

Then $A \cap C \subset B \subset A$ and the set B is open. We have $A - B \subset D$ so that the set $A - B$ is countable. By 13.3.5, B is $\mathbf{F}_\sigma$, by ex. 13.11 $A - B$ is $\mathbf{F}_\sigma$. Thus the set $\mathop{\mathrm{E}}_x[f(x) > c] = A = B \cup (A - B)$ is $\mathbf{F}_\sigma$ by 13.3.3. Similarly, we may prove that the set $\mathop{\mathrm{E}}_x[f(x) < c]$ is also $\mathbf{F}_\sigma$. Thus, f is a function of the first class by 14.3.1.

14.7. Let f be a function on P. Let $a \in P$. We say that the function f is *upper semicontinuous* at the point a if, for every $\alpha \in \mathbf{E}_1$ with $f(a) < \alpha$, there is a $\delta > 0$ such that

$$x \in P, \ \varrho(a, x) < \delta \quad \text{imply} \quad f(x) < \alpha .$$

Similarly, we say that the function f is *lower semicontinuous* at the point a, if, for every $\alpha \in \mathbf{E}_1$ with $f(a) > \alpha$, there is a $\delta > 0$ such that

$$x \in P, \ \varrho(a, x) < \delta \quad \text{imply} \quad f(x) > \alpha .$$

We say that a function f is *semicontinuous* at the point a if it is either upper semicontinuous or lower semicontinuous at the point a. We say that a function f is *upper* (*lower*) *semicontinuous* if f is upper (lower) semicontinuous at every point $a \in P$.

Finally, we say that f is *semicontinuous*, if it is either upper semicontinuous or lower semicontinuous.

The following two theorems are evident:

14.7.1. *A function f is lower semicontinuous at a point a if and only if the function $-f$ is upper semicontinuous at the point a.*

14.7.2. *A function f is continuous at a point a if and only if it is both upper and lower semicontinuous at the point a.*

14.7.3. *Let f be a function on P. The function f is upper semicontinuous if and only if for every $c \in \mathbf{E}_1$ the set $\underset{x}{\mathrm{E}}[f(x) < c]$ is open.*) A function f is lower semicontinuous if and only if for every $c \in \mathbf{E}_1$ the set $\underset{x}{\mathrm{E}}[f(x) > c]$ is open.**)*

Proof: I. Let the set $\underset{x}{\mathrm{E}}[f(x) < c]$ be open for every $c \in \mathbf{E}_1$. Let $a \in P$, $\alpha \in \mathbf{E}_1$, $f(a) < \alpha$. We have $a \in \underset{x}{\mathrm{E}}[f(x) < \alpha]$; as the set on the right-hand side is open, there is a $\delta > 0$ such that $\Omega(a, \delta) \subset \underset{x}{\mathrm{E}}[f(x) < \alpha]$ i.e. such that $\varrho(a, x) < \delta$ implies $f(x) < \alpha$. Hence, the function f is upper semicontinuous at the point a.

II. Let the function f be upper semicontinuous. Choose a $c \in \mathbf{E}_1$. Put $C = \underset{x}{\mathrm{E}}[f(x) < c]$. If $a \in C$, $f(a) < c$, hence there is a $\delta > 0$ such that $\varrho(a, x) < \delta$ implies $f(x) < c$, i.e. $\Omega(a, \delta) \subset C$. Hence, the set C is open.

III. We have finished the proof for the case of the upper semicontinuous function. The case of the lower semicontinuous function may be reduced to the first one by 14.7.1.

14.7.4. *Let f be a function on P. The function f is upper semicontinuous if and only if and only if there exists a sequence $\{f_n\}$ of continuous functions on P such that for every $x \in P$: [1] $f_n(x) \geqq f_{n+1}(x)$ for $n = 1, 2, 3, \ldots$; [2] $\lim_{n \to \infty} f_n(x) = f(x)$. The function f is lower semicontinuous if and only if there is a sequence $\{f_n\}$ of continuous functions on P such that for every $x \in P$: [1] $f_n(x) \leqq f_{n+1}(x)$ for $n = 1, 2, 3, \ldots$; [2] $\lim_{n \to \infty} f_n(x) = f(x)$.*

Proof: By 14.7.1, we may do the proof for the case of lower semicontinuous functions only.

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*) By (1) in 9.5, this condition may be stated as follows: *for every $c \in \mathbf{E}_1$ the set $\underset{x}{\mathrm{E}}[f(x) \geq c]$ is closed.*

**) This condition may be stated: *for every $c \in \mathbf{E}_1$ the set $\underset{x}{\mathrm{E}}[f(x) \leqq c]$ is closed.*

I. Let $\{f_n\}$ be a sequence of continuous functions on P such that for every $x \in P$: [1] $f_n(x) \leqq f_{n+1}(x)$; [2] $f_n(x) \to f(x)$. Let $a \in P$, $\alpha \in \mathbf{E}_1$, $f(a) > \alpha$. As $f(a) = \lim f_n(a)$, there is an index p with $f_p(a) > \alpha$. As the function f_p is continuous at the point a, there is a $\delta > 0$ such that $\varrho(a, x) < \delta$ implies $f_p(x) > \alpha$. Since $f_n(x) \leqq f_{n+1}(x)$ and $f_n(x) \to f(x)$, $f_p(x) > \alpha$ implies $f(x) > \alpha$. Hence, $\varrho(a, x) < \delta$ implies $f(x) > \alpha$. Thus, the function f is lower semicontinuous.

II. Let the function f be lower semicontinuous and let $-1 \leqq f(x) \leqq 1$ for every point $x \in P$. For $n = 1, 2, 3, \ldots$ and for $x \in P$ put

$$f_n(x) = \inf_{z \in P} [f(z) + n \,.\, \varrho(x, z)] \,.$$

Hence for every $z \in P$, $f_n(x) \leqq f(z) + n \,.\, \varrho(x, z)$. If we put $z = x$, we obtain $f_n(x) \leqq f(x)$, so that $f_n(x) \leqq 1$ and $f_n(x) = 1$ implies $f(x) = 1$. As $f(z) \geqq -1$, $\varrho(x, z) \geqq 0$, we have $f_n(x) \geqq -1$. Evidently, $f(x) = -1$ implies $f_n(x) = -1$. Further, if $f(x) = a > -1$, there exists a $\delta > 0$ such that $z \in P$, $\varrho(x, z) < \delta$ imply $f(z) > \frac{1}{2}(-1 + a)$. Hence

$$z \in P \quad \text{implies} \quad f(z) + n \,.\, \varrho(x, z) \geqq \min\,[-1 + n\delta, \tfrac{1}{2}(-1 + a)] \,,$$

and hence

$$f_n(x) \geqq \min\,[-1 + n\delta, \tfrac{1}{2}(-1 + a)] > -1 \,.$$

Thus, $f_n(x) = -1$ if and only if $f(x) = -1$. Since $f(z) + n \,.\, \varrho(x, z) \leqq f(z) + n \,.\, \varrho(y, z) + n \,.\, \varrho(x, y)$, we have

$$\inf_{z \in P} [f(z) + n \,.\, \varrho(x, z)] \leqq \inf_{z \in P} [f(z) + n \,.\, \varrho(y, z)] + n \,.\, \varrho(x, y)$$

i.e. $f_n(x) \leqq f_n(y) + n \,.\, \varrho(x, y)$, and, of course, $f_n(y) \leqq f_n(x) + n\varrho(x, y)$, hence $|f_n(x) - f_n(y)| \leqq n \,.\, \varrho(x, y)$. Thus, for every n, f_n is a continuous (moreover: uniformly continuous) function. Obviously, $f_n(x) \leqq f_{n+1}(x) \leqq 1$ so that there exists $g(x) = \lim f_n(x)$. As $f_n(x) \leqq f(x)$, we have $g(x) \leqq f(x)$. Let $\varepsilon > 0$. Since the function f is lower semicontinuous at the point x, there exists a $\delta > 0$ such that $\varrho(x, y) < \delta$ implies $f(y) > f(x) - \varepsilon$. Since $f_n(x) = \inf_{x \in P} [f(z) + n \,.\, \varrho(x, z)]$, there exists (for the given x) a point $z_n \in P$ such that $f(z_n) + n \,.\, \varrho(x, z_n) < f_n(x) + 1/n \leqq f(x) + 1/n$, hence $1 \geqq f(x) > n \,.\, \varrho(x, z_n) - 1/n + f(z_n) \geqq n \,.\, \varrho(x, z_n) - 1/n - 1$, and hence $\varrho(x, z_n) \leqq 2/n + 1/n^2$. There exists an index q such that for $n > q$ we have $2/n + 1/n^2 < \delta$, hence $\varrho(x, z_n) < \delta$ and hence $f(z_n) > f(x) - \varepsilon$. On the other hand, $f(z_n) \leqq f(z_n) + n \,.\, \varrho(x, z_n) < f_n(x) + 1/n$ and hence $f(x) < f_n(x) + 1/n + \varepsilon$ for every $n > q$, so that $f(x) \leqq \lim_{n\to\infty} (f_n(x) + 1/n) + \varepsilon = g(x) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $f(x) \leqq g(x)$; since also $g(x) \leqq f(x)$, we have $f(x) = g(x) = \lim_{n\to\infty} f_n(x)$.

III. Let the function f be lower semicontinuous. By ex. 9.18 there is a homeomorphic mapping φ of the set $\mathbf{R}$ onto the interval $\underset{t}{\mathrm{E}}[-1 \leqq t \leqq 1]$. Put $\varphi[f(x)] = F(x)$. Evidently, for $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$ we have $\alpha < \beta$ if and only if $\varphi(\alpha) < \varphi(\beta)$. Since f is lower semicontinuous, we decide easily that the function F is also lower semicontinuous. We have $-1 \leqq F(x) \leqq 1$, $F(x) = -1$ if and only if $f(x) = -\infty$, $F(x) = 1$ of and only if $f(x) = \infty$.

By II, there is a sequence $\{F_n\}$ of continuous functions such that [1] $F_n(x) \leqq F_{n+1}(x)$, [2] $\lim_{n\to\infty} F_n(x) = F(x)$, [3] $-1 \leqq F_n(x) \leqq 1$, [4] $F_n(x) = -1 \Leftrightarrow F(x) = -1$, [5] $F_n(x) = 1 \Rightarrow F(x) = 1$.

Put $\varphi_{-1}[F_n(x)] = f_n(x)$. Then f_n are continuous functions such that $f_n(x) \leqq f_{n+1}(x)$ and $\lim_{n\to\infty} f_n(x) = f(x)$. The proof is finished. Moreover, from the proof we obtain that $f_n(x) = -\infty$ if and only if $f(x) = -\infty$ and that $f_n(x) = \infty$ implies $f(x) = \infty$. Thus, we may formulate the following theorem:

14.7.5. *Let f be a finite function on P. The function f is upper semicontinuous if and only if there exists a sequence $\{f_n\}$ of finite continuous functions on P such that for every $x \in P$: [1] $f_n(x) \geqq f_{n+1}(x)$ for $n = 1, 2, 3, \ldots$, [2] $\lim_{n\to\infty} f_n(x) = f(x)$. The function f is lower semicontinuous if and only if there exists a sequence $\{f_n\}$ of finite continuous functions on P such that for every $x \in P$: [1] $f_n(x) \leqq f_{n+1}(x)$ for $n = 1, 2, 3, \ldots$; [2] $\lim_{n\to\infty} f_n(x) = f(x)$.*

Theorem 14.7.4 yields:

14.7.6. *Every semicontinuous function is a function of the first class.*

14.8. 14.8.1. *Let g and h be functions on P. Let g be upper semicontinuous; let h be lower semicontinuous. Let $g(x) \leqq h(x)$ for every $x \in P$. Then there exists a continuous function f on P such that $g(x) \leqq f(x) \leqq h(x)$ for every $x \in P$.*

Proof: I. Let the functions g and h be finite. For $t \in \mathbf{E}_1$ put: [1] $\lambda(t) = t$ for $t \geqq 0$, [2] $\lambda(t) = 0$ for $t < 0$. By 14.7.5 there exist sequences $\{g_n\}$ and $\{h_n\}$ of finite continuous functions on P such that $g_n(x) \geqq g_{n+1}(x)$, $h_n(x) \leqq h_{n+1}(x)$, $g_n(x) \to g(x)$, $h_n(x) \to h(x)$. We have

$$g_n(x) - h_n(x) \geqq g_n(x) - h_{n+1}(x) \geqq g_{n+1}(x) - h_{n+1}(x)\,, \tag{1}$$

and hence the absolute values of the terms (with the exception of the first one) of the series

$$\begin{aligned} &h_1(x) + \lambda[g_1(x) - h_1(x)] - \lambda[g_1(x) - h_2(x)] + \\ &+ \lambda[g_2(x) - h_2(x)] - \lambda[g_2(x) - h_3(x)] + \ldots \end{aligned} \tag{2}$$

converge monotonically to zero. As the terms of the series (2) are (with the exception of the first one) alternately $\geqq 0$ and $\leqq 0$, the series (2) is, by the well-known Leibniz criterion, convergent. Denote its sum by $f(x)$. If $f_n(x)$ designates the sum of the sum of the first n terms of the series (2), then f_n are continuous functions and $f_{2n}(x) \geqq f_{2n+2}(x)$, $f_{2n-1}(x) \leqq f_{2n+1}(x)$, $f_n(x) \to f(x)$, so that, by 14.7.2 and 14.7.5, the function f is continuous. It remains to prove that, for every $x \in P$, $g(x) \leqq f(x) \leqq h(x)$. Let us distinguish two cases.

First, let $g(x) = h(x)$. Then $g_n(x) \geqq h_n(x)$, so that the series (2) goes as follows

$$h_1(x) + [g_1(x) - h_1(x)] + [h_2(x) - g_1(x)] + [g_2(x) - h_2(x)] + \ldots$$

and its sum $f(x)$ is equal to $\lim g_n(x) = \lim h_n(x) = g(x) = h(x)$.

Secondly, let $g(x) < h(x)$. Then, beginning with a certain n, $g_n(x) < h_n(x)$, $g_n(x) < h_{n+1}(x)$. Let m be the least index such that in the n-th term of the series (2) we have a negative number after the sign λ. By (1), for $2 \leqq n \leqq m - 1$, in the n-th term of the series (2) we have a non-negative number after λ, so that $f(x)$ is equal to the sum of the first $m - 1$ terms of the series

$$h_1(x) + [g_1(x) - h_1(x)] + [h_2(x) - g_1(x)] + \\ + [g_2(x) - h_2(x)] + [h_3(x) - g_2(x)] + \ldots.$$

Hence, if $m = 2i$ is even, we have $f(x) = h_i(x)$ and $g_i(x) < h_i(x)$. If $m = 2i + 1$, we have $f(x) = g_i(x)$ and $g_i(x) < h_{i+1}(x)$. Since $g_n(x) \geqq g_{n+1}(x)$, $h_n(x) \leqq h_{n+1}(x)$, $g_n(x) \to g(x)$, $h_n(x) \to h(x)$ we have obviously $g(x) \leqq f(x) \leqq h(x)$.

II. In the general case we proceed as follows: By ex. 9.18 there is a homeomorphic mapping φ of the set $\mathbf{R}$ onto the interval $\underset{t}{\mathrm{E}}[-1 \leqq t \leqq 1]$ such that $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$, $\alpha < \beta$ imply $\varphi(\alpha) < \varphi(\beta)$. Putting $G(x) = \varphi[g(x)]$, $H(x) = \varphi[h(x)]$, we obtain that $-1 \leqq G(x) \leqq 1$, $-1 \leqq H(x) \leqq 1$, $G(x) \leqq H(x)$, G is upper semicontinuous and H lower semicontinuous. By I there exists a continuous function F such that $G(x) \leqq F(x) \leqq H(x)$, hence $-1 \leqq F(x) \leqq 1$, so that we may put $f(x) = \varphi_{-1}[F(x_y)]$. Obviously f is a continuous function and $g(x) \leqq f(x) \leqq h(x)$.

14.8.2. *Let $A \subset P$ be a closed set. Let k be a continuous function on A. Then there exists a continuous function f on P such that the partial function f_A is identical with k.*

Proof. Define functions g and h on P as follows:

$$g(x) = h(x) = k(x) \quad \text{for} \quad x \in A,$$
$$g(x) = -\infty \quad \text{and} \quad h(x) = \infty \quad \text{for} \quad x \in P - A.$$

Let $c \in \mathbf{E}_1$. We have

$$\underset{x}{\mathrm{E}}[x \in P, g(x) \geqq c] = \underset{x}{\mathrm{E}}[x \in A, k(x) \geqq c].$$

Hence, by 9.5.1 and 8.7.4, the set $\underset{x}{\mathrm{E}}[x \in P, g(x) \geqq c]$ is closed, so that by 14.7.3 g is an upper semicontinuous function on P. Similarly we prove that h is a lower semicontinuous function on P. As $g(x) \leqq h(x)$ for $x \in P$, there is by 14.8.1 a continuous function f on P such that $x \in P$ implies $g(x) \leqq f(x) \leqq h(x)$. For $x \in A$ we have $g(x) = k(x) = h(x)$ and hence $f(x) = k(x)$.

14.8.3. *Let $A \subset P$ be a closed set. Let k be a finite continuous function on A. Then there exists a finite continuous function f on P such that the partial function f_A is identical with k.*

Proof: The case $A = \emptyset$ is trivial; thus, let $A \neq \emptyset$. By ex. 9.18 there is a homeomorphic mapping φ of the set $\mathbf{R}$ onto the interval $\underset{t}{\mathrm{E}}[-1 \leqq t \leqq 1]$ such that $\varphi(-\infty) = -1$, $\varphi(\infty) = 1$ and such that $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$, $\alpha < \beta$ imply $\varphi(\alpha) < \varphi(\beta)$. By 14.8.2 there is a continuous function l on P such that $x \in A$ implies $l(x) = k(x)$. Put $B = \underset{x}{\mathrm{E}}[l(x) = \infty] \cup \underset{x}{\mathrm{E}}[l(x) = -\infty]$. By 9.5 B is closed; evidently $A \cap B = \emptyset$. If $B = \emptyset$ we may put $f = l$; thus, let $B \neq \emptyset$. For $x \in P$ put $L(x) = \varphi[l(x)]$. Then L is a continuous function on P such that $x \in P$ implies $|L(x)| \leqq 1$ and such that $\underset{x}{\mathrm{E}}[|L(x)| = 1] = B$. For $x \in P$ put

$$r(x) = \frac{\varrho(x, A)}{\varrho(x, A) + \varrho(x, B)}.$$

We have

$$\varrho(x, A) = 0 \Leftrightarrow x \in \bar{A} \Leftrightarrow x \in A, \quad \varrho(x, B) = 0 \Leftrightarrow x \in B.$$

Thus, r is a finite function on P such that $r(x) \geqq 0$ for $x \in P$, $r(x) = 0$ for $x \in A$, $r(x) = 1$ for $x \in B$. By ex. 9.10, the function r is continuous. For $x \in P$ put

$$F(x) = \frac{L(x)}{1 + r(x)}.$$

Then, F is a continuous function on P such that: [1] $|F(x)| < 1$ for $x \in P$, [2] $F(x) = \varphi[k(x)]$ for $x \in A$. For $x \in P$ put $f(x) = \varphi_{-1}[F(x)]$. Then f is a finite continuous function on P such that $x \in A$ implies $f(x) = k(x)$.

Exercises

In exercises 14.1—14.3, χ is a characteristic function of a point set $A \subset P$.

14.1. The function χ is upper semicontinuous if and only if the set A is closed.

14.2. The function χ is lower semicontinuous if and only if the set A is open.

14.3. The function χ is of the first class if and only if the set A is simultaneously both $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$.

14.4. Deduce theorem 14.1.2 from theorem 14.3.1.

14.5. Deduce theorem 14.2.1 from theorem 14.3.1.

14.6. If finite functions f and g are both upper semicontinuous, then also the function $f+g$ is upper semicontinuous.
Deduce: [1] directly from the definition, [2] from theorem 14.7.3, [3] from theorem 14.7.5.

14.7. If P is a countable space, then every function on P is of the first class.

14.8. Let f be a function on P. For $a \in P$ there exist the limits:

$$\varphi_1(a) = \lim_{n\to\infty} \sup_{\varrho(a,x)<1/n} f(x), \qquad \varphi_2(a) = \lim_{n\to\infty} \sup_{0<\varrho(a,x)<1/n} f(x),$$

$$\psi_1(a) = \lim_{n\to\infty} \inf_{\varrho(a,x)<1/n} f(x), \qquad \psi_2(a) = \lim_{n\to\infty} \inf_{0<\varrho(a,x)<1/n} f(x).$$

The functions φ_1 and φ_2 are upper semicontinuous, the functions ψ_1 and ψ_2 are lower semicontinuous.

14.9. Let $\mathbf{C} \neq \emptyset$ be an arbitrary set. For every $z \in \mathbf{C}$ let f_z be an upper semicontinuous function on P. For $x \in P$ put $f(x) = \inf_{z\in\mathbf{C}} f_z(x)$. Then f is an upper semicontinuous function. Deduce: [1] directly from the definition, [2] from theorem 14.7.3.

14.10. Let f be a function on $\mathbf{E}_1$ such that $x < y$ implies $f(x) \leqq f(y)$. Then f is a function of the first class.
Deduce: [1] directly from the definition, [2] from theorem 14.3.1, [3] from theorem 14.6.

14.11. Let $\{f_n\}$ be a sequence of continuous mappings of a metric space P into a metric space Q. For every $x \in P$ let there exist $\lim_{n\to\infty} f_n(x) = f(x) \in Q$. Then for every set $A \subset f(P)$ open in $f(P)$ the set $\mathop{\mathrm{E}}_x[f(x) \in A]$ is $\mathbf{F}_\sigma$ in P.

14.12. Let f be a mapping of a metric space P into the euclidean space $\mathbf{E}_m$ such that, for every set $A \subset f(P)$ open in $f(P)$, $\mathop{\mathrm{E}}_x[f(x) \in A]$ is an $\mathbf{F}_\sigma$-set in P. Then there is a sequence $\{f_n\}$ of continuous mappings of the space P into $\mathbf{E}_m$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in P$.

A function f on P is said to be of the *second class*, if there exists a sequence $\{f_n\}$ of functions of the first class on P such that $f_n(x) \to f(x)$ for every $x \in P$.

14.13. If f is a finite function of the second class on P, there exists a sequence $\{f_n\}$ of finite functions of the first class on P such that $f_n(x) \to f(x)$ for every $x \in P$.

14.14. Let f be a finite function. Let $\{f_n\}$ be a sequence of finite functions of the second class. Let f be the uniform limit of the sequence $\{f_n\}$. Then f is a function of the second class.

14.15. Let f be a function on P. A necessary and sufficient condition for f to be a function of the second class is: for every $c \in \mathbf{E}_1$ the sets $\mathop{\mathrm{E}}_x[f(x) > c]$ and $\mathop{\mathrm{E}}_x[f(x) < c]$ are $\mathbf{G}_{\delta\sigma}(P)$.