

Forgotten mathematician Henry Lowig (1904–1995)

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Löwig's works in functional analysis

In: Martina Bečvářová (author); Jindřich Bečvář (author); Vlastimil Dlab (author); Antonín Slavík (author): Forgotten mathematician Henry Lowig (1904–1995). (English). Praha: MATFYZPRESS, Vydavatelství Matematicko-fyzikální fakulty v Praze, 2012. pp. 87–122.

Persistent URL: <http://dml.cz/dmlcz/402299>

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LÖWIG'S WORKS IN FUNCTIONAL ANALYSIS

The basic structures investigated in functional analysis are infinite-dimensional real and complex linear (vector) spaces endowed with a topology that is compatible with the algebraic structure. The topology can be induced by a metric, the metric can be derived from a norm, and the norm from an inner product. Consequently, functional analysis studies topological linear spaces, metric linear spaces, normed linear spaces, and inner product spaces. Every inner product space is also a normed linear space, every normed linear space is also a metric linear space, and every metric linear space is a topological linear space.

A topology compatible with the algebraic structure is sufficient to define concepts such as the limit of a sequence, the sum of an infinite series, continuity of a linear mapping (an operator or a functional). It allows us to speak about Cauchy (fundamental) sequences, complete spaces, to construct the completion of a given space, etc.

In an inner product space, it is also possible to introduce orthogonality (in a real linear space, one can define the angle between two nonzero vectors), orthogonal and orthonormal sets, etc.

* * *

The notion of a linear space was already investigated by Hermann Günther Grassmann (1809–1877) in his obscure book *Die lineale Ausdehnungslehre* from 1844. Even the somewhat more intelligible version, which appeared in 1862 under the title *Die Ausdehnungslehre. Vollständig und in strenger Form bearbeitet*, met with a mild reception. Giuseppe Peano (1858–1932) decided to follow Grassmann's ideas in his book *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazioni della logica deduttiva*; among other things, he gave the axiomatic definition of a linear space (*sistema lineare*), introduced linear mappings (*operazione distributiva, trasformazione lineare*), and clearly formulated certain basic facts which are now classified as belonging to linear algebra. Again, his book received almost no response. Peano's axiomatic definition of linear spaces was later taken over by Salvatore Pincherle (1853–1936). In his extensive monograph *Le operazioni distributive e le loro applicazioni all'analisi* published in 1901, he presented a systematic theory of linear spaces (*sistema lineare, spazio lineare*), especially function spaces, inner product spaces, etc. The book's title was chosen to emphasise its main goal, namely the study of linear operators on function spaces. Pincherle's book is sometimes considered to be the first monograph on functional analysis (see e.g. [Me2]).

In a famous book by Hermann Weyl (1885–1955) entitled *Raum. Zeit. Materie. Vorlesungen über allgemeine Relativitätstheorie* and published in 1918, the axioms of a linear space reappeared in the context of affine spaces and

their axiomatic definition. The book was very successful and had its fifth edition published in 1923, while the English and French translations appeared in 1922.

Nevertheless, the notion of a linear space was still not a part of common knowledge. In 1922, the Polish mathematician Stefan Banach (1892–1945) found it necessary to provide a detailed explanation of this concept in his work *Sur les opérations dans les ensembles abstraits et leurs application aux équations intégrales* [Ba1]. A completely modern definition of a linear space can be found in the chapter *Espaces vectoriels généraux* of his book *Théorie des opérations linéaires*, which was published ten years later (see [Ba2], p. 26).

The concept of a linear space became generally known only in the 1930s, partially in connection with the emergence of functional analysis. At the same time, linear algebra started to constitute itself as an independent discipline, and later became one of the cornerstones of higher mathematics (see e.g. [B2]).

A basis is nowadays defined as a linearly independent spanning set. A simple construction proves the existence of a basis for finitely generated spaces; in the general case, it is sufficient to apply Zorn's lemma.

The well-known fact that all bases have the same number of elements follows either from the Steinitz exchange lemma (for finitely generated spaces), or from the basic properties of infinite cardinals (see e.g. [B1], pp. 86–87). The dimension of a linear space is then defined as the cardinality of an arbitrary basis.

The concepts of dimension and n -dimensional space were already known to Grassmann in 1862 (*Stufenzahl, Gebiet n -ter Stufe*). Moreover, he stated the theorem concerning the dimensions of the intersection and the sum of two subspaces. In 1888, Peano was also familiar with the notions of basis and dimension (*enti di riferimento, numero delle dimensioni*). Both of them restricted themselves to finite-dimensional spaces, but Peano mentioned the space of polynomials as an example of an infinite-dimensional space. Similarly in 1901, Pincherle initially studied these concepts in finite-dimensional spaces (*sistema fondamentale, insieme lineare ad n dimensioni*); his ideas on infinite-dimensional spaces were not yet completely rigorous. This should come as no surprise when we consider that set theory (including the basic facts concerning cardinalities) was first developed by Georg Cantor (1845–1918) in the period from 1873 to 1884, and that his extensive unifying work entitled *Beiträge zur Begründung der transfiniten Mengenlehre*¹ was published in the years 1895 and 1897. Only in the beginning of the 20th century did set theory begin to influence other branches of mathematics.²

¹ *Mathematische Annalen* 46 (1895), 481–512, 49 (1897), 207–246. See also *Gesammelte Abhandlungen*, 282–351 (Anmerkungen 351–356). English translation (P. Jourdain): *Contributions to the Founding of the Theory of Transfinite Numbers*, Open Court, Chicago, 1915 (reprint 1952), pp. 85–136, 137–201.

² Let us recall the first study texts, textbooks, and monographs: A. Schoenflies: *Die Entwicklung der Lehre von den Punktmannigfaltigkeiten* (1st volume: Jahresbericht

In 1905, the German mathematician Georg Hamel (1877–1954) published a short paper entitled *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x+y) = f(x) + f(y)$* [Ha] in the journal *Mathematische Annalen*, where he proved the existence of a basis of the linear space of all reals numbers over the field of rational numbers.³ His original result reads as follows:

Es existiert eine Basis aller Zahlen, d. h. es gibt eine Menge von Zahlen a, b, c, \dots derart, daß sich jede Zahl x in einer und auch nur einer Weise in der Form

$$x = \alpha a + \beta b + \gamma c + \dots$$

darstellen läßt, wo die Zahlen $\alpha, \beta, \gamma, \dots$ rational sind, aber in jedem einzelnen Falle nur eine endliche Anzahl von ihnen von Null verschieden ist. ([Ha], p. 459)

His proof made use of the so-called Zermelo's theorem, which states that every set can be well-ordered.⁴ Using the last result, it was easy for him to conclude that an arbitrary mapping of a given basis into the set of real numbers can be uniquely extended into a linear mapping f (in general, the mapping is not continuous). In this way, he obtained a solution of the functional equation $f(x+y) = f(x) + f(y)$.

We remark that the famous Zorn's lemma, which plays a crucial role in the proof of the existence of a basis of a linear space, was published by Max Zorn (1906–1993) in his work *A remark on method in transfinite algebra* [Zo] from 1935 in the following form:

DEFINITION 2. A set \mathfrak{A} of sets A is said to be closed (right-closed), if it contains the union $\Sigma_{\mathfrak{B} \ni B} B$ of every chain \mathfrak{B} contained in \mathfrak{A} .

Then our maximum principle is expressible in the following form.

der Deutschen Mathematiker-Vereinigung 8(1900), 1–250, 2nd volume: Jahresbericht der Deutschen Mathematiker-Vereinigung, Ergänzungsband 2, Teubner, Leipzig, 1908, x+331 pp.), A. Schoenflies: *Die Entwicklung der Mengenlehre und ihrer Anwendungen. Erster Hälfte: Allgemeine Theorie der unendlichen Mengen und Theorie der Punktmengen*, Teubner, Leipzig, 1913, G. Hessenberg: *Grundbegriffe der Mengenlehre* (Abhandlungen der Friesschen Schule, Neue Folge 1, Heft IV, 1906, 220 pp., also Vandenhoeck und Ruprecht, Göttingen, 1906), W. H. Young, G. Ch. Young: *The theory of sets of points* (Cambridge University Press, 1906, xii+316 pp.), W. Sierpiński: *Zarys teorii mnogości* (Warszawa, 1912, 158 pp.), F. Hausdorff: *Grundzüge der Mengenlehre* (Veit & Comp., 1914, viii+476 pp., 2nd edition: *Mengenlehre*, 1927, 285 pp.), A. Fraenkel: *Einleitung in die Mengenlehre. Eine elementare Einführung in das Reich des Unendlichgrossen* (Springer-Verlag, Berlin, 1919, iv+155 pp., 2nd edition: 1923, ix+251 pp., 3rd edition: 1928, 424 pp.).

³ It is well known today that this space is of uncountable dimension.

⁴ This result of Ernst Zermelo (1871–1953) was published in the paper *Beweis, daß jede Menge wohlgeordnet werden kann* [Z1] in 1904. It included a part of Zermelo's letter to David Hilbert dating from 24th September 1904. Zermelo proved the well-ordering theorem using a new axiom, which became known as the *axiom of choice*. See e.g. [M] and [E]. This topic is also discussed in Zermelo's later works [Z2], [Z3] from 1908.

(MP). In a closed set \mathfrak{A} of sets A there exists at least one, A^* , not contained as a proper subset in any other $A \in \mathfrak{A}$. ([Zo], p. 667)⁵

Both Zermelo's theorem and Zorn's lemma (as well as the Hausdorff maximal principle) are equivalent to the axiom of choice.⁶

* * *

The birth of functional analysis is usually dated to the 1920s and 1930s. However, the basic ideas have their origin in the works of several important mathematicians from the turn of the century, namely Salvatore Pincherle, Vito Volterra (1860–1940), David Hilbert (1862–1943), Jacques Hadamard (1865–1963), and Maurice Fréchet (1878–1973). In their later works, these mathematicians were analysing the ideas which led to the birth and development of functional analysis (see e.g. [Me1]), such as the study of differential and integral equations (often motivated by real-world phenomena), the investigation of function spaces, particular operators, functionals, etc. Their ideas were later picked up by other mathematicians – Ivar Fredholm (1866–1927), Felix Hausdorff (1868–1942), Ernst Fischer (1875–1954), Hans Hahn (1879–1934), Frigyes (Frédéric, Friedrich) Riesz (1880–1956), Paul Lévy (1886–1971), Stefan Banach, John von Neumann (1903–1957), and others. Nowadays, functional analysis is understood as the axiomatic theory of topological linear spaces and linear as well as nonlinear operators.

A decisive step was the birth of the theory of normed linear spaces in the 1920s. Particular examples of various different spaces were already known at the time, and their common properties created an impetus for the development of a new theory. Mathematicians had new tools at their disposal, such as the axiom of choice, Zermelo's theorem, or transfinite induction. Axiomatisation and abstraction witnessed a significant progress – it was no longer a problem to discard the particular nature of elements of specific spaces (functions, sequences, etc.) and consider general spaces, whose elements, instead of being given explicitly, were now characterised by a certain collection of axioms.

The notion of a Banach space, a complete normed linear space, appeared almost simultaneously in several places: in the doctoral dissertation of Banach from 1920, published in 1922 as a journal article under the title *Sur les opérations dans les ensembles abstraits et leurs application aux équations intégrales* [Ba1], in the work *Über Folgen linearer Operationen*⁷ written by Hans Hahn, in the paper *Limit in terms of continuous transformations*,⁸ whose author was Norbert Wiener (1894–1964), and also in the work *Über lineare*

⁵ See P. J. Campbell: *The origin of "Zorn's lemma"*, *Historia mathematica* 5 (1978), pp. 77–89.

⁶ See e.g. [Ku], pp. 23–25, and [Ru1].

⁷ *Monatshefte für Mathematik und Physik* 32 (1922), 1–88.

⁸ *Bulletin de la Société Mathématique de France* 50 (1922), 119–134.

Funktionaloperationen,⁹ published by Eduard Helly (1884–1943). The term *Banach space* was coined by Fréchet in his 1928 monograph *Les espaces abstraits et leurs théorie considérée comme introduction à l'Analyse générale* [F] (*Les espaces de M. Banach*, p. 141). In 1932, Banach used the term *B-space* in his book *Théorie des opérations linéaires* [Ba2] (*Espaces du type (B)*, p. 53).

The theory of normed linear spaces evolved successfully in the 1920s and in the first half of the 1930s. The main results were summarised in Banach's monograph *Théorie des opérations linéaires* [Ba2] from 1932. The recent developments in functional analysis were also discussed at the International Congress of Mathematicians held in Bologna in 1928: Hadamard delivered the lecture *Le développement et le rôle scientifique du calcul fonctionnel*, and Fréchet chose the topic *L'analyse générale et les espaces abstraits*.¹⁰ There was a clear distinction between functional analysis and general analysis at the time.¹¹

According to the present definition, a Hilbert space is a real or complex inner product space that is complete.¹² The original definition of a Hilbert space included the requirement of separability (i.e. the existence of a countable dense subset). The abstract definition of a Hilbert space was formulated in the late 1920s and early 1930s.¹³ For example, in 1930, von Neumann gave a thorough definition of a Hilbert space in his work *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren* [N1]¹⁴:

A. \mathfrak{H} ist ein linearer Raum. . . .

B. Es gibt in \mathfrak{H} ein, zu dem der Vektorrechnung analoges, inneres Produkt, das eine Metrik erzeugt. . . .

C. In der Metrik $|f - g|$ ist \mathfrak{H} separabel. D. h.: eine gewisse abzählbare Menge ist in \mathfrak{H} überall dicht.

D. \mathfrak{H} besitzt beliebig (endlich!) viele lin. unabh. Elemente.

E. \mathfrak{H} ist vollständig. D. h.: wenn eine Folge f_1, f_2, \dots in \mathfrak{H} der Cauchyschen

⁹ Sitzungsberichte der mathematisch-naturwissenschaftliche Klasse der kaiserlichen Akademie der Wissenschaften (Wien) 121 (1912), 265–297.

¹⁰ Atti del Congresso internazionale dei matematici, Bologna 3–10 settembre 1928, T. I, Bologna, 1929, 143–161, 267–274.

¹¹ General analysis grew out of the ideas of Eliakim Hastings Moore (1862–1932), namely from his works *On a form of general analysis with application to linear differential and integral equations* (Atti del IV Congresso Internazionale dei matematici, Roma 6–11 Aprile 1908, Vol. II, Roma, 1909, pp. 98–114), *Introduction to a form of general analysis* (Yale University Press, 1910, 150 pp.), and *On the foundation of the theory of linear integral equations* (Bulletin of the American Mathematical Society 18 (1912), pp. 334–362). See e.g. [Bro].

¹² Halmos gave a clear and succinct definition in his book *Introduction to Hilbert space* from 1951: *A Hilbert space is an inner product space which, as a metric space, is complete*. His definition of a Banach space is similarly brief: . . . *a Banach space is a normed vector space which, as a metric space, is complete*. ([H], p. 17)

¹³ The term *Hilbert space* was already coined by Schoenflies in the second part of his 1908 book *Die Entwicklung der Lehre von den Punktmannigfaltigkeiten*, see p. 266, 298.

¹⁴ See also Neumann's work [N0] from 1927, pp. 15–17.

Konvergenzbedingung genügt (zu jedem $\varepsilon > 0$ gibt es ein $N = N(\varepsilon)$, so daß aus $m, n \geq N$ $|f_m - f_n| \leq \varepsilon$ folgt), so ist sie konvergent (es existiert ein f aus \mathfrak{H} , so daß es zu jedem $\varepsilon > 0$ ein $N = N(\varepsilon)$ gibt, so daß aus $n \geq N$ $|f_n - f| \leq \varepsilon$ folgt). ([N1], pp. 63–66)

In a similar way, Riesz introduced the notion of a Hilbert space in his work *Über die linearen Transformationen des komplexen Hilbertschen Raumes* [R2] from 1930; its first paragraph, entitled *Der komplexe Hilbertsche Raum*, reads as follows:

Erstens ist \mathfrak{H} linear, d. h. ...

Zweitens ist jedem Paare f, g eine bestimmte reelle oder komplexe Zahl, das innere Produkt (f, g) , zugeordnet ...

Drittens ist \mathfrak{H} vollständig, d. h. ...

Viertens ist \mathfrak{H} separabel, d. h. er enthält eine abzählbare, überall dichte Teilmenge. ...

Fünftens enthält \mathfrak{H} beliebig viele linear unabhängige Elemente, d. h. ... ([R2], pp. 26–27)

This type of space was intensively studied in the 1930s, and became gradually known as the Hilbert space.¹⁵ This is evidenced e.g. by the works *Linear Transformations in Hilbert Space* [St1] and *Linear Transformations in Hilbert Space and their applications to analysis* [St2] written by Marshall Harvey Stone (1903–1989). It was in this period when it became clear that the assumption of separability is not always necessary (Löwig [L6] and Rellich [Re]) – and that’s why this requirement was dropped from the definition of a Hilbert space.

For a more detailed look at the history of functional analysis, see e.g. the following books:

A. F. Monna: *Functional Analysis in Historical Perspective* [Mo],

J. Dieudonné: *History of Functional Analysis* [Di],

A. Pietsch: *History of Banach Spaces and Linear Operators* [P].

A wealth of information is contained in a number of journal articles, e.g. [BK], [NB1], [NB2], [Bor], [Br1], [Br2], [Bo2], [Bro], [Du], [G], [Hor] [Hu1], [Hu2], [Ka], [K2], [K3], [K4], [Me1], [Me2], [Si1], [Si2], [Sm], and in the monographs [DS], [Kli]. Very interesting and inspiring is the publication

J.-L. Dorier (ed.): *On the Teaching of Linear Algebra* [Do].

¹⁵ In the epilog of his publication [HS], Albrecht Pietsch (born 1934) mentions the following lovely story, but expresses doubt about its authenticity: *Nachdem Weyl (1909) einen Seminarvortrag über seinen Beweis des Riesz-Fischer-Theorems gehalten hatte, soll ihn Hilbert gefragt haben: „Weyl, sagen Sie mir bitte, was ist ein Hilbertscher Raum? Das habe ich nicht verstanden!“* ([HS], p. 283) See also L. Young: *Mathematicians and their times*, Mathematics Studies 48, North Holland, Amsterdam, New York, Oxford, 1981, p. 312, and A. E. Taylor: *A study of Maurice Fréchet: I. His early work on point set theory and the theory of functionals*, Archive for History of Exact Sciences 27 (1982), 233–295, p. 283.

Another book which is worth recommending is

K. Saxe: *Beginning Functional Analysis* [Sa],

which contains many historical notes, an extensive bibliography, and biographical sketches (Banach, Enflo, Fréchet, Fourier, Hilbert, Lebesgue, von Neumann, Riesz, Stone). The textbook

H. Schröder: *Funktionalanalysis* [Sch],

also provides numerous historical and bibliographic comments. Each chapter in the monograph

F. Deutsch: *Best Approximation in Inner Product Spaces* [De]

is also followed by historical notes. For the relationship between functional analysis and its applications, see the monograph

A. W. Naylor, G. R. Sell: *Linear Operator Theory in Engineering and Science* [NS],

or the books

E. Zeidler: *Applied Functional Analysis. Applications to Mathematical Physics* [Ze1],

E. Zeidler: *Applied Functional Analysis. Main Principles and their Applications* [Ze2],

H. Heuser: *Funktionalanalysis. Theorie und Anwendung* [Hu2].¹⁶

* * *

Felix Hausdorff started his 1932 paper *Zur Theorie der linearen metrischen Räume* [H1]¹⁷ by providing definitions of certain basic linear algebraic notions: linear space (*linearer Raum*), linearly independent set (*linear unabhängige Menge*), linear subspace (*lineare Menge, lineare Teilmenge*), etc.

He denoted the linear span (*lineare Hülle*) of a set A (the intersection of all subspaces containing the set A , or equivalently, the set of all linear combinations of the elements of A) by the symbol A_λ ; he referred to a linearly independent set A as the basis (*Basis*) of the subspace A_λ . He mentioned Zermelo's theorem, and remarked that by discarding unnecessary elements, a set A can always be reduced to a basis of A_λ . After a few preparatory paragraphs, he stated the following definition:

Ein linearer metrischer Raum E entsteht aus einem linearen Raum, wenn jedem Punkt x eine reelle Zahl $|x|$, der Betrag von x , gemäß den Vorschriften

$$\begin{aligned} |0| &= 0, & |x| &> 0 \quad \text{für } x \neq 0, \\ |\alpha x| &= |\alpha||x|, & |x + y| &\leq |x| + |y| \end{aligned}$$

¹⁶ The monograph of Harro Heuser (1927–2011) includes an extensive historical treatise entitled *Ein Blick auf die werdende Funktionalanalysis* (pp. 599–663).

¹⁷ See also the commentary [Ch] written by Srishti D. Chatterji (born 1935).

zugeordnet und $|x - y|$ als Entfernung der Punkte x, y erklärt wird. ([H1], p. 295)

Thus Hausdorff introduced a *normed linear space* and its corresponding metric linear space. This definition allowed him to discuss the convergence of sequences, sums of infinite series, and the closure of a set A (*Konvergenz einer Punktfolge, Summe einer unendlichen Reihe, abgeschlossene Hülle \bar{A} von A*). He stated that the closure of a subspace is a subspace again, and emphasized that the operations of closure and linear span are not interchangeable; in general, the closure $\overline{A_\lambda}$ of the linear span of a set A and the linear span $(\bar{A})_\lambda$ of the closure of A satisfy

$$(\bar{A})_\lambda \subseteq \overline{A_\lambda}.$$

He designated a linearly independent set A as the fundamental set (*Grundmenge*) of the closed linear span $\overline{A_\lambda}$. We quote the corresponding excerpt:

Der Durchschnitt beliebig vieler abgeschlossener Mengen ist wieder abgeschlossen; demnach gibt es zu einer Menge A die kleinste abgeschlossene Menge $\underline{\geq} A$, die abgeschlossene Hülle \bar{A} von A . Die abgeschlossene Hülle \bar{L} einer linearen Menge L ist wieder linear ...

Der Durchschnitt beliebig vieler abgeschlossener linearer Mengen ist wieder abgeschlossen und linear; demnach gibt es zu einer Menge A die kleinste abgeschlossene lineare Menge $\underline{\geq} A$, die abgeschlossene lineare Hülle von A . Sie ist die abgeschlossene Hülle $\bar{L} = \overline{A_\lambda}$ der linearen Hülle $L = A_\lambda$ von A ; denn einerseits ist jede abgeschlossene lineare Menge $\underline{\geq} A$ auch $\underline{\geq} L$ und $\underline{\geq} \bar{L}$, andererseits ist \bar{L} selbst abgeschlossen linear. (Wenn man in umgekehrter Reihenfolge erst die abgeschlossene Hülle $F = \bar{A}$ und dann deren lineare Hülle F_λ bildet, so ist nur $F_\lambda \leq \bar{L}$, da die lineare Hülle einer abgeschlossenen Menge nicht notwendig abgeschlossen ist.) Wenn A linear unabhängig ist, so heie sie eine Grundmenge ihrer abgeschlossenen linearen Hülle \bar{L} ; man kann wie oben aus jeder Menge A eine linear unabhngige Teilmenge B ohne nderung der linearen Hülle L , also eine Basis von L oder Grundmenge von \bar{L} aussondern. ([H1], p. 295)

According to Hausdorff, a space is called separable if it has a countable dense subset. He emphasised that a separable space has a finite or countable fundamental set, but its basis can be finite, countable as well as uncountable.

* * *

The following text is devoted to Lwig's works [L6], [L7], and [L8]. Their character is illustrated by several excerpts. However, we remark that the terminology has been subject to many changes over the years. This is evident when we compare works from the first decades of the 20th century with the latest monographs and textbooks (e.g. [Sa], [De], [Sch]), or even with books which already belong to the classics (e.g. [Ru2]).

* * *

In 1934, the journal *Acta Litterarum ac Scientiarum* (Szeged) published Löwig's work *Komplexe euklidische Räume von beliebiger endlicher oder transfiniter Dimensionszahl* [L6] consisting of thirty-three pages.¹⁸ In February 1934, he submitted the same work as a habilitation thesis to the Faculty of Science of the German University in Prague, and successfully became a Privatdozent in 1935.

As he explained in the introduction, his motivation was to show that many statements concerning Hilbert spaces (as they were understood at the time – see the above mentioned definitions by von Neumann and Riesz) are still valid for arbitrary complete inner product spaces. In other words, the assumption of separability, which was then a usual part of the definition of a Hilbert space, is not really substantial:

In dieser Abhandlung soll gezeigt werden, daß viele Sätze, welche man bisher nur für den Hilbertschen Raum bewiesen hat, auch für beliebige euklidische Räume, d. h. lineare metrische Räume, in denen ein inneres Produkt definiert ist, gelten, oder – mit andern Worten – daß die Voraussetzung der Separabilität des Raumes, die man beim Beweise dieser Sätze bisher zu machen pflegte, unwesentlich ist. ([L6], p. 1)

The first paragraph entitled *Vorbemerkungen über allgemeine komplexe lineare metrische Räume* ([L6], pp. 1–9) served to introduce the reader to the topic. Löwig first recalled the notions of a complex linear space¹⁹ (*komplexer linearer Raum* \mathfrak{R}) and norm (*absoluter Betrag des Elements* \mathfrak{r}), which enabled him to define a complex normed linear space (*komplexer linearer metrischer Raum* \mathfrak{R}). As a next step, he proposed to call two subsets \mathfrak{M} , \mathfrak{N} of the spaces \mathfrak{R} , \mathfrak{S} isomorphic if there is a bijection between \mathfrak{M} and \mathfrak{N} , and for the corresponding elements \mathfrak{r}_k , \mathfrak{n}_k and arbitrary complex numbers a_k , we have

$$\left| \sum_{k=1}^n a_k \mathfrak{r}_k \right| = \left| \sum_{k=1}^n a_k \mathfrak{n}_k \right|.$$

With the norm at his disposal, he was able to define the neighbourhood of an element \mathfrak{r}_0 (*starke Umgebung der Stelle* \mathfrak{r}_0), the concepts of a limit point (*starke Häufungsstelle einer Teilmenge*), closed subset (*starkabgeschlossene Menge*), and convergent sequence (*stark konvergente Folge*). The closed linear span of a set \mathfrak{M} is the set of all limit points of the span of \mathfrak{M} .

In the following pair of definitions, he introduced fundamental sequences (*starke Fundamentalfolge*) and complete sets (*starkvollständige Teilmenge*). He noted that complete sets are necessarily closed.

In the case when the closed linear span of a set \mathfrak{M} is complete, he referred to it as the complete linear span of the set \mathfrak{M} . He proved that if two subsets of normed linear spaces are isomorphic, then their complete linear spans are also

¹⁸ It was received by the journal on 24th February 1934.

¹⁹ He made a reference to the laws of affine vector algebra.

isomorphic (provided they exist). In the usual way, he showed the construction of the completion of a linear metric space:

Jeden komplexen linearen metrischen Raum \mathfrak{R} , welcher nicht starkvollständig ist, kann man zu einem starkvollständigen komplexen linearen metrischen Raum erweitern ... Man betrachte als die Elemente von \mathfrak{R}^ die Gesamtheiten äquivalenter starker Fundamentalfolgen von \mathfrak{R} wir wollen diese Erweiterung die kleinste starkvollständige Erweiterung von \mathfrak{R} nennen. ([L6], p. 4)*

Löwig also recalled the concepts of a linear mapping between linear spaces (*lineare Abbildung*), bounded linear mapping between normed linear spaces²⁰ (*beschränkte lineare Abbildung*), he mentioned that the set of all bounded linear operators between two spaces \mathfrak{R} and \mathfrak{S} forms a normed linear space, and defined the notions of a linear operator (*linearer Operator*) and linear functional (*lineares Funktional*), which led him to the concept of a weak neighbourhood.

Es seien L_k ($k = 1, 2, \dots, n$) endlich viele beschränkte lineare Funktionale in \mathfrak{R} und ε eine beliebige positive reelle Zahl. Ferner sei τ_0 ein beliebiges Element von \mathfrak{R} . Dann heiÙe die Gesamtheit der Stellen τ von \mathfrak{R} , welche den Ungleichungen

$$|L_k(\tau - \tau_0)| < \varepsilon \quad (k = 1, 2, \dots, n)$$

genügen, eine schwache Umgebung der Stelle τ_0 . ([L6], p. 5)

He noted that this definition generalises the definition proposed by von Neumann²¹ and that the four axioms of neighbourhood formulated by Hausdorff in his book *Grundzüge der Mengenlehre*²² are satisfied.

The definition of a weak neighbourhood was followed by a number of related concepts: weak limit point of a set \mathfrak{M} (*schwache Häufungsstelle einer Menge \mathfrak{M}*), weakly closed set (*schwachabgeschlossene Menge*), weakly convergent sequence (*schwach konvergente Folge*); he introduced the notation

$$\lim_{n \rightarrow \infty} \tau_n = \tau \quad \text{and} \quad \text{Lim}_{n \rightarrow \infty} \tau_n = \tau$$

for the strong and weak convergence, respectively.

²⁰ A linear mapping A is bounded if the set of all numbers $|A\tau|$, where $|\tau| \leq 1$, is bounded. The supremum of this set is then referred to as the *norm* of A .

²¹ *Schwache Topologie in \mathfrak{H} . Sei $\mathcal{U}_2(f_0; \varphi_1, \dots, \varphi_s, \varepsilon)$ die Menge aller f mit $|(f - f_0, \varphi_1)| < \varepsilon, \dots, |(f - f_0, \varphi_s)| < \varepsilon$; alle $\mathcal{U}_2(f_0; \varphi_1, \dots, \varphi_s, \varepsilon)$ mit beliebigen $\varphi_1, \dots, \varphi_s$ (aus \mathfrak{H} , s beliebig) und $\varepsilon > 0$ sind die Umgebungen von f_0 . ([N2], p. 379)*

²² *Umgebungsaxiome:*

(A) *Jedem Punkt x entspricht mindestens eine Umgebung U_x ; jede Umgebung U_x enthält den Punkt x .*

(B) *Sind U_x, V_x zwei Umgebungen desselben Punktes x , so gibt es eine Umgebung W_x , die Teilmenge von beiden ist ($W_x \subseteq \mathfrak{D}(U_x, V_x)$).*

(C) *Liegt der Punkt y in U_x , so gibt es eine Umgebung U_y , die Teilmenge von U_x ist ($U_y \subseteq U_x$).*

(D) *Für zwei verschiedene Punkte x, y gibt es zwei Umgebungen U_x, U_y ohne gemeinsamen Punkt ($\mathfrak{D}(U_x, U_y) = \emptyset$). ([H2], p. 213)*

He mentioned the fact that a sequence τ_n is weakly convergent to an element τ if and only if the sequence of numbers $L\tau_n$ converges to $L\tau$ for every bounded linear functional L (*Satz 2*). He pointed out that a weak limit point of a set \mathfrak{M} need not be the weak limit of any sequence of elements from \mathfrak{M} (and gave a reference to the work of von Neumann [N2], pp. 380–381).

Then he recalled the following result (*Satz 3*): If \mathfrak{M} is a subset of a normed complex linear space and the element τ_0 lies outside the closed linear span of \mathfrak{M} , then there exists a bounded linear functional L which is nonzero at τ_0 and vanishes on \mathfrak{M} .²³

As he remarked, this theorem is usually stated and proved for real linear spaces; however, he showed that a real-linear functional can be simply extended to a complex-linear functional by setting

$$L\tau = R\tau - iR(i\tau),$$

where R is the real-linear functional. The previous result leads to the following theorem:

Satz 4. Jede schwache Häufungsstelle einer Menge \mathfrak{M} gehört der starkabgeschlossenen linearen Hülle von \mathfrak{M} an. ([L6], p. 6)

As a consequence, he obtained the following pair of corollaries:

Satz 5. Ist

$$\lim_{n \rightarrow \infty} \tau_n = \tau,$$

*dann gehört τ der starkabgeschlossenen linearen Hülle der Menge der Elemente τ_n ($n = 1, 2, 3, \dots$) an.*²⁴

Satz 6. Jede starkabgeschlossene lineare Mannigfaltigkeit ist auch schwachabgeschlossen. ([L6], p. 7)²⁵

Thus, given a linear subspace (or a linear span), it is enough to consider its closedness. Löwig then stated the following definition:

Eine Teilmenge \mathfrak{M} von \mathfrak{X} heie schwachvollstndig, wenn sie in der kleinsten starkvollstndigen Erweiterung von \mathfrak{X} schwachabgeschlossen ist. ([L6], p. 7)²⁶

Every weakly complete set is also weakly closed, but the converse is not necessarily true.

Eine Folge τ_n ($n = 1, 2, 3, \dots$) von Elementen von \mathfrak{X} heie eine schwache Fundamentalfolge, wenn fr jedes beschrnkte lineare Funktional L in \mathfrak{X} $\lim_{n \rightarrow \infty} L\tau_n$ existiert. ([L6], p. 8)

²³ Löwig cited Hausdorff's work [H1], p. 306.

²⁴ This theorem was also formulated by Banach in [Ba1], p. 134.

²⁵ The theorem generalises a result due to Schmidt [S]. See also von Neumann [N2], p. 396.

²⁶ The reader should be aware that in the existing literature, the notions of weak closedness and weak completeness usually have a different meaning.

In the first paragraph, Löwig cited Hausdorff's work [H1], Banach's monograph [Ba2], von Neumann's work [N2], Schmidt's article [S], and Mazur's paper [Ma]; these works were published (except the article [S]) in the period from 1930 to 1932.

Löwig divided the core of his work in two parts, depending on whether or not he made use of Zermelo's theorem.

In the second paragraph entitled *Sätze über komplexe euklidische Räume, die ohne Benützung des Wohlordnungssatzes bewiesen werden können* (pp. 9–25), he first gave the definition of a general inner product (*hermiteische bilineare Funktion*). In the next theorem, he showed that if the inner product is also positive definite, then it induces a norm, which satisfies the Cauchy-Schwarz inequality and the triangle inequality. Thus a linear space with a positive definite inner product (*innere Produkt*) is also a normed linear space; the n -dimensional complex Euclidean space (*komplexer euklidischer Raum*) and complex Hilbert space (*komplexe Hilbertsche Raum*) are included as special cases.

The following series of theorems is concerned with various properties of Hilbert spaces. We quote some of them in Löwig's original wording.

Satz 11. Zu jedem beschränkten linearen Funktional L eines vollständigen komplexen euklidischen Raumes \mathfrak{R} gibt es ein (und daher auch nur ein) erzeugendes Element, d. h. ein Element u von \mathfrak{R} von der Beschaffenheit, daß für jedes Element τ von \mathfrak{R}

$$L\tau = (\tau, u)$$

ist. ([L6], p. 11)

Löwig noted that Theorem 11 is well known to be true for complex Euclidean finite-dimensional spaces as well as for complex Hilbert spaces (he cited Riesz's work [R2]), and that Theorem 12 is a corollary of these statements. He then proved Theorem 13, generalised the statement of Theorem 12 to arbitrary complex spaces, and finally proved Theorem 11.

Satz 12. Ist L ein beschränktes lineares Funktional in einem endlichdimensionalen komplexen euklidischen Raume oder im komplexen Hilbertschen Raume, dann gibt es stets ein Element u des betreffenden Raumes mit $|u| = |L|$ und $Lu = |u|^2$. ([L6], p. 11)

Satz 13. Gibt es zu einem beschränkten linearen Funktional L in einem komplexen euklidischen Raume \mathfrak{R} ein Element u von \mathfrak{R} mit $|u| = |L|$ und $Lu = |u|^2$, dann ist identisch

$$L\tau = (\tau, u).$$

Löwig introduced the notion of a „regular subspace“ – a subspace whose orthogonal complement is also its algebraic complement – and in the following pair of theorems, he explained the relation between regularity, completeness and closedness.

Definition 10. Eine lineare Mannigfaltigkeit \mathfrak{M} eines komplexen euklidischen Raumes heie regulr, wenn man jedes Element \mathfrak{r} von \mathfrak{R} in Bezug auf \mathfrak{M} in eine Tangentialkomponente \mathfrak{t} und eine Normalkomponente \mathfrak{n} zerlegen kann, d. h. wenn man zu jedem Element \mathfrak{r} von \mathfrak{R} zwei ebensolche Elemente \mathfrak{t} und \mathfrak{n} angeben kann, so da

$$\mathfrak{r} = \mathfrak{t} + \mathfrak{n}$$

ist, \mathfrak{t} \mathfrak{M} angehrt und \mathfrak{n} zu allen Elementen von \mathfrak{M} orthogonal ist. ([L6], p. 12)

Satz 14. Fr die Regularitt einer linearen Mannigfaltigkeit \mathfrak{M} von \mathfrak{R} ist notwendig, aber nicht hinreichend, da \mathfrak{M} in \mathfrak{R} abgeschlossen sei.

Satz 15. Fr die Regularitt einer linearen Mannigfaltigkeit \mathfrak{M} von \mathfrak{R} ist hinreichend, aber nicht notwendig, da \mathfrak{M} vollstndig sei. ([L6], pp. 11–14)

He summarised these results in the following lucid form:

Vollstndigkeit \longrightarrow Regularitt \longrightarrow Abgeschlossenheit.

Satz 16. Ist \mathfrak{M} eine Teilmenge eines vollstndigen komplexen euklidischen Raumes \mathfrak{R} , deren abgeschlossene lineare Hlle nicht mit \mathfrak{R} zusammenfllt, dann gibt es stets mindestens ein vom Nullelement verschiedenes Element von \mathfrak{R} , welches zu allen Elementen von \mathfrak{M} orthogonal ist. ([L6], p. 15)

Lwig also studied adjoint and self-adjoint operators and derived some of their basic properties.²⁷

The presentation of his definitions and theorems is completely modern.

Definition 11. Ein linearer Operator B in einem komplexen euklidischen Raume \mathfrak{R} heie zu dem linearen Operator A in \mathfrak{R} adjungiert, wenn fr beliebige Elemente \mathfrak{r} und \mathfrak{n} von \mathfrak{R}

$$(A\mathfrak{r}, \mathfrak{n}) = (\mathfrak{r}, B\mathfrak{n})$$

ist.

Given a linear operator A , he noted there is at most one adjoint operator, which he denoted by A^* .

Definition 12. Ein linearer Operator A in \mathfrak{R} heie selbstadjungiert, wenn A^* existiert und mit A identisch ist.

Satz 23. Ist \mathfrak{R} vollstndig, dann existiert zu jedem beschrnkten linearen Operator A in \mathfrak{R} der adjungierte lineare Operator. ([L6], p. 20)

He also studied operators E satisfying $E^2 = E$ (so-called *Einzelooperator*).

In the second part of his treatise, Lwig cited the works of Hilbert [Hi], Mazur [Ma], von Neumann [N1], and Riesz [R1], [R2].

²⁷ This topic originated in the works of Schmidt. Self-adjoint operators (*hypermaximalen Operatoren*) were studied by von Neumann in his 1930 work [N1], the term (*self-adjoint*) was coined by Stone in 1932. von Neumann and Stone gave a fairly complete presentation of the theory of self-adjoint operators in their monographs [N3] and [St2] published in 1932.

The third paragraph is called *Beweis weiterer Sätze über komplexe euklidische Räume unter Anwendung des Wohlordnungssatzes* (pp. 25–32). Löwig starts by introducing the space \mathfrak{R} of complex functions φ defined on a certain set \mathfrak{N} , such that φ vanishes with the exception of at most countably values (i.e., the function has a countable support), and such that the series

$$\sum_{n=1}^{\infty} |\varphi(\mathbf{u}_n)|^2$$

is convergent for every sequence \mathbf{u}_n with the property that $\varphi(\mathbf{u}) = 0$ for every $\mathbf{u} \neq \mathbf{u}_n$ (i.e., the sequence “covers” the support). Given a pair of functions $\mathbf{r} = \varphi(\mathbf{u})$, $\mathbf{n} = \psi(\mathbf{u})$ from \mathfrak{R} , he defined their inner product by the formula

$$(\mathbf{r}, \mathbf{n}) = \sum_{n=1}^{\infty} \varphi(\mathbf{u}_n) \overline{\psi(\mathbf{u}_n)},$$

where $\varphi(\mathbf{u}) = 0$ and $\psi(\mathbf{u}) = 0$ for every $\mathbf{u} \neq \mathbf{u}_n$. In this way, he obtained a complete complex Euclidean space (i.e., a Hilbert space). Moreover, the collection of all functions satisfying $\varphi(\mathbf{u}_0) = 1$ for a single value \mathbf{u}_0 and $\varphi(\mathbf{u}) = 0$ for all remaining \mathbf{u} forms a maximal orthonormal set (see below). We remark that von Neumann [N3] (pp. 37–38) investigated the case when \mathfrak{R} is the set of all real numbers.

For an arbitrary cardinal number \aleph , Löwig used the symbol \mathfrak{R}_{\aleph} to denote the above mentioned space of complex functions defined on a set \mathfrak{N} whose cardinality is \aleph . The space \mathfrak{R}_{\aleph} has a finite dimension if \aleph is finite, and \mathfrak{R}_{\aleph} is a Hilbert space if \aleph is countable.

Löwig defined a maximal orthonormal set (*vollständiges normiertes Orthogonalsystem*) in a complete complex Euclidean space, and proved its existence using Zermelo’s theorem:

Satz 28. In jedem vollständigen komplexen euklidischen Raume gibt es mindestens ein vollständiges normiertes Orthogonalsystem. ([L6], p. 27)²⁸

He developed the theory further and obtained the following results (theorems 29, 30, 31, 32 on pages 27–29):

- The complete linear span of every maximal orthonormal set in a complete complex Euclidean space coincides with the whole space.
- Two complete complex Euclidean spaces possessing maximal orthonormal sets of the same cardinality are necessarily isomorphic.²⁹

²⁸ Although it is possible to define a maximal orthonormal set and prove its existence in a general inner product space (without the assumption of completeness), Löwig was interested only in the theory of complete spaces.

²⁹ *Satz 30. Besitzen zwei vollständige komplexe euklidische Räume vollständige normierte Orthogonalsysteme von gleicher Mächtigkeit, dann sind sie isomorph.* ([L6], p. 27)

- A complete complex Euclidean space with a maximal orthonormal set of cardinality \aleph is necessarily isomorphic to \mathfrak{R}_\aleph .³⁰
- If \mathfrak{M} is a maximal orthonormal set in a complete complex Euclidean space, then for every element \mathfrak{r} , only countably many of the numbers $(\mathfrak{r}, \mathfrak{e})$, $\mathfrak{e} \in \mathfrak{M}$, might be nonzero. If $\mathfrak{e}_n \in \mathfrak{M}$ and $(\mathfrak{r}, \mathfrak{e}) = 0$ for every $\mathfrak{e} \neq \mathfrak{e}_n$, then $\mathfrak{r} = \sum_{n=1}^{\infty} (\mathfrak{r}, \mathfrak{e}_n) \mathfrak{e}_n$.³¹

Löwig then showed the important fact that all maximal orthonormal sets in a complete complex Euclidean space have the same cardinality:

Satz 33. Zwei verschiedene vollständige normierte Orthogonalsysteme eines vollständigen komplexen euklidischen Raumes \mathfrak{R} haben stets die gleiche Mächtigkeit. ([L6], p. 31)

In the proof of this statement, he had to use the properties of infinite cardinal numbers; he gave a reference to both editions of Hausdorff's monograph on set theory.³² Theorem 33 then enabled him to give the following definition of a dimension:

Definition 20. Ist \mathfrak{R} ein vollständiger komplexer euklidischer Raum und \mathfrak{M} ein vollständiges normiertes Orthogonalsystem von \mathfrak{R} , dann heie die Mächtigkeit von \mathfrak{M} die Dimensionszahl von \mathfrak{R} . ([L6], p. 32)

As Löwig noted, it follows that the dimension of the complete complex Euclidean space \mathfrak{R}_\aleph is \aleph . He then defined the dimension of an arbitrary complex Euclidean space as the dimension of its smallest completion. He concluded by saying that the method of proof of Theorem 33 can be used to prove the fact that all bases of a complex linear space have the same number of elements. He recalled that the definition was given by Hausdorff in [H1], p. 395.

In the third part of his work, Löwig cited Hausdorff's publications [H1], [H2], [H3], and von Neumann's work [N3].

* * *

Let us remark that Löwig's work [L6] published in the journal *Acta Litterarum ac Scientiarum* (Szeged) was immediately followed by the five-page article *Zur Theorie des Hilbertschen Raumes* [R3]³³ written by Riesz, who cited

³⁰ *Satz 31. Hat ein vollständiges normiertes Orthogonalsystem eines vollständigen komplexen euklidischen Raumes die Mächtigkeit \aleph , dann ist dieser Raum \mathfrak{R}_\aleph isomorph.* ([L6], p. 27)

³¹ *Satz 32. Ist \mathfrak{M} ein vollständiges normiertes Orthogonalsystem eines vollständigen komplexen euklidischen Raumes \mathfrak{R} und \mathfrak{r} ein beliebiges Element von \mathfrak{R} , dann sind von den Zahlen $(\mathfrak{r}, \mathfrak{e})$ mit $\mathfrak{e} \in \mathfrak{M}$ höchstens abzählbar viele von Null verschieden. Ist weiter $\mathfrak{e}_n \in \mathfrak{M}$ ($n = 1, 2, 3, \dots$) und $(\mathfrak{r}, \mathfrak{e}) = 0$ für $\mathfrak{e} \in \mathfrak{M}$, $\mathfrak{e} \neq \mathfrak{e}_n$ ($n = 1, 2, 3, \dots$), dann ist $\mathfrak{r} = \sum_{n=1}^{\infty} (\mathfrak{r}, \mathfrak{e}_n) \mathfrak{e}_n$, wobei das Summenzeichen im Sinne der starken Konvergenz zu verstehen ist.* ([L6], pp. 28–29)

³² [H2] from 1914, p. 127, [H3] from 1927, p. 71.

³³ It was received by the journal on 30th April 1934.

Löwig's work [L6].³⁴ The paper represented a continuation of Riesz's previous work *Über die linearen Transformationen des komplexen Hilbertschen Raumes* [R2] published in the same journal; it contained a very detailed definition of a Hilbert space.

In [R3], Riesz proved the following two well-known theorems (he still found it necessary to recall the definitions of the basic notions) and referred the reader to [R2], which also included these two theorems (see [R2], p. 28):

Die folgenden anspruchslosen Bemerkungen betreffen die beiden wohlbekannten, für die Theorie des reellen oder komplexen Hilbertschen Raumes grundlegenden Sätze, die ich auch in einer früheren Arbeit der Behandlung an die Spitze stellte.

*Satz A.*³⁵ *Ist eine lineare Mannigfaltigkeit \mathfrak{L} des Hilbertschen Raumes \mathfrak{H} nicht überall dicht in \mathfrak{H} , so gibt es ein Element g aus \mathfrak{H} mit $|g| = 1$, das zu allen Elementen aus \mathfrak{L} orthogonal ist.*

*Satz B.*³⁶ *Für jede lineare Funktion $l(f)$ gibt es ein eindeutig bestimmtes „erzeugendes“ Element g , so daß*

$$l(f) = (f, g).$$

Dabei ist unter linearen Mannigfaltigkeit von \mathfrak{H} eine Teilmenge zu verstehen, die mit f für jede komplexe Zahl c auch cf und mit f und g auch $f + g$ enthält. Eine in \mathfrak{H} definierte (skalare) Funktion $l(f)$ heißt linear, wenn sie den Gleichungen $l(cf) = cl(f)$, $l(f + g) = l(f) + l(g)$ genügt und auf der Einheitskugel $|f| = 1$ beschränkt ist.

Gewöhnlich stützt man den Beweis dieser beiden Sätze auf die Separabilität ... ([R3], p. 34)

* * *

Löwig's work [L6] was closely related to certain essential results of functional analysis, which were discovered in the previous three decades and had their origin in the works of Riesz, Fischer, and Fréchet from the beginning of the century. These results were concerned with the isometry between the spaces L_2 and l_2 , its construction based on maximal orthonormal sets, completeness of these spaces, the general form of a continuous (bounded) linear functional on L_2 and l_2 , etc.³⁷ Another important topic was the Hahn-Banach theorem

³⁴ *Die unmittelbare Anregung zu diesen Zeilen verdanke ich der voranstehenden Arbeit (i.e. the work [L6]), in welcher unter andern gezeigt wird, daß auch Satz B ohne Dimensionsabgrenzung nach oben richtig ist. ([R3], p. 35)*

³⁵ Cf. Löwig's Satz 16.

³⁶ Cf. Löwig's Satz 11.

³⁷ Let us mention the related pioneering works: F. Riesz: *Sur les systèmes orthogonaux de fonctions*, Comptes Rendus de l'Académie des Sciences, Paris 144 (1907), 615–619, *Sur les systèmes orthogonaux de fonctions et l'équation de Fredholm*, Comptes Rendus de l'Académie des Sciences, Paris 144 (1907), 734–736, *Sur un espèce de géométrie analytique des systèmes*

discovered in the late 1920s.³⁸ Löwig's main contribution was his consideration of complex linear spaces, the study of Hilbert spaces without the assumption of separability, the investigation of orthonormal sets, etc.

The work [L6] received a wide response. It was reviewed in the journal *Zentralblatt für Mathematik* by Stone, who also mentioned the paper [L6] in his subsequent review of Riesz's work [R3].³⁹

Georg Aumann (1906–1980) wrote a report on Löwig's work [L6] for the journal *Jahrbuch über die Fortschritte der Mathematik*. In the same journal, Riesz's work [R3] was reviewed by Gustav Heinrich Adolf Doetsch (1892–1977), who also recalled the results obtained by Löwig in [L6].⁴⁰

On 12th September 1934, Franz Rellich (1906–1955) delivered the lecture *Abstrakte Spektraltheorie und fastperiodische Funktionen* at the meeting of the Deutsche Mathematiker-Vereinigung; Löwig's work [L6] was also presented on that occasion.⁴¹

Pasqual Jordan (1902–1980) and John von Neumann cited the work [L6] in their paper *On inner products in linear, metric spaces* [JN] published in 1935:

The hyper-Hilbert spaces (without E) have been first discussed by H. Löwig, Acta Szeged, vol. 7(1934), pp. 1–33. ([JN], p. 719)

In 1936, Francis Joseph Murray (1911–1996) and John von Neumann cited Löwig's article [L6] in their extensive work *On rings of operators* [MN] consisting of more than 110 pages; the article [L6] is one of the 22 items in their list of references. In the same year, Oswald Teichmüller (1913–1943) cited the paper [L6] in his work *Operatoren im Wachsschen Raum* [Te].

Béla Adalbert Lengyel (1910–2002) and Marshall Harvey Stone cited the work [L6] in their 1936 paper *Elementary Proof of the Spectral Theorem* [LS]. They wrote:

de fonctions sommables, Comptes Rendus de l'Académie des Sciences, Paris 144 (1907), 1409–1411, Über orthogonale Funktionensysteme, Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse, 1907, 116–122 (also Oeuvres I, 378–381, 382–385, 386–388, 389–395); E. Fischer: Sur la convergence en moyenne, Comptes Rendus de l'Académie des Sciences, Paris 144 (1907), 1022–1024, M. Fréchet: Sur les opérations linéaires, III., Transactions of the American Mathematical Society 8 (1907), 433–446, Sur les ensembles de fonctions et les opérations linéaires, Comptes Rendus de l'Académie des Sciences, Paris 144 (1907), 1414–1416.

³⁸ See H. Hahn: *Über lineare Gleichungssysteme in linearen Räumen*, Journal für die reine und angewandte Mathematik 157 (1927), 214–229, S. Banach: *Sur les fonctionnelles linéaires I, II*, Studia Mathematica 1 (1929), 211–216, 223–239.

³⁹ Zbl 0009.25901, Zbl 0009.25902.

⁴⁰ See JFM 60.0324.01 and JFM 60.0325.01.

⁴¹ *Auszüge aus den auf der Tagung in Bad Pyrmont am 11. und 12. September 1934 gehaltenen Vorträgen*, Jahresbericht der Deutschen Mathematiker-Vereinigung 45 (1935), 82–83.

Since the dimensionality postulates occurring in the definition of Hilbert space are actually so little used in the theory, it is convenient to suppress them. Thus we shall consider those spaces which are called following Löwig [12], complex Euclidean spaces. ([LS], pp. 853–854)

Their article was reviewed⁴² by Edgar Raymond Lorch (1907–1990), who also mentioned Löwig's work [L6]:

Der Beweis wird für den allgemeineren Fall eines komplexen euklidischen Raumes \mathfrak{L} gegeben (vgl. Löwig ...), was bekanntlich keine wesentlichen neuen Schwierigkeiten macht.

Von Neumann included Löwig's paper [L6] among the 15 references in his extensive work *On infinite direct products* [N4] from 1939.

In the same year, the paper [L6] was cited by Tosio Kitagawa in the work *The Parseval theorem of the Cauchy series and the inner products of certain Hilbert spaces* [Ki]. He appreciated Löwig's precise formulation of the basic concepts.

Also in the same year, Franz Wecken (1912 – before 1945) cited Löwig's work [L6] in the article *Unitärinvarianten selbstadjungierter Operatoren* [W]:

Während nun der Spektralsatz für selbstadjungierte Operatoren im Hilbertschen Raum nahezu unverändert auch im nichtseparablen Raum gilt (Rellich [11], Löwig [6]), treten für die Theorie der unitären Invarianten beim Übergang von abzählbarer zu nichtabzählbarer Dimension die im folgenden angedeuteten neuen Verhältnisse ein. ([W], p. 422)⁴³

In 1940, Ralph Saul Phillips (1913–1998) cited Löwig's work in the paper *A characterization of euclidean spaces* [Ph]; his reference [4] corresponds to the article [L6]:

A Banach space is a linear, normed, complete space [3, chap. 5]. A euclidean space of dimension α , where α is any cardinal number, is defined to be the Banach space of sequences x_ν of real numbers where ν ranges over a class of cardinal number α , and $\sum x_\nu^2$ is finite and equal to the square of the norm [4]. ([Ph], p. 930)

In 1941, the name of Löwig appeared in the cyclostyled lectures *Invariant measures* [N5] by John von Neumann with comments due to Paul Richard Halmos (1916–2006).⁴⁴

In the same year, the article [L6] was cited by J. W. Calkin in his extensive work *Two-sides ideals and congruences in the ring of bounded operators in Hilbert space* [Ca] consisting of 35 pages.

⁴² See JFM 62.0450.02.

⁴³ Rellich's reference [11] corresponds to the work [Re].

⁴⁴ Page 124 or page 74 in the printed version from 1999, respectively.

In 1941, Abram Iezekiilovič Plesner (1900–1961) cited the work [L6] in the extensive article *Spektralnaja teorija linejnych operatorov* [P1].

In 1947, Gottfried Maria Hugo Köthe (1905–1989) made use of Löwig's results from [L6] in his paper *Eine axiomatische Kennzeichnung der linearen Räume* [Kö1].

In the same year, Robert C. James (1918–2003) cited Löwig's Theorem 32 from [L6] in his article *Inner products in normed linear spaces* [J1]:

Any complete normed linear space T which has an inner product is characterized by its (finite or transfinite) cardinal "dimension-number" n . It is equivalent to the space of all sets $x = (x_1, x_2, \dots)$ of n real numbers satisfying $\sum_i x_i^2 < +\infty$, where $\|x\| = (\sum_i x_i^2)^{1/2}$ [7, Theorem 32]. ([J1], p. 559)

Also in the same year, he cited the work [L6] in his paper *Orthogonality and linear functionals in normed linear spaces* [J2]; he made a reference to Löwig's Theorems 11 and 16 (see the aboved mentioned *Satz 11* and *Satz 16*).

In 1948, Helmut Schiek (1915–1981) cited the work [L6] in the paper *Mengen mit affiner Anordnung* [Sc].

Naum Il'jič Achieser (also Achieser, 1901–1980) and Izrail' Markovič Glazman (also Glasmann, 1916–1968) included Löwig's work [L6] among the references in their monograph *Teorija linejnych operatorov v Gil'bertovom prostanstve* [AG] published in 1950; the second edition of the book appeared in 1966. The monograph was also translated to German and English.

Frédéric Riesz and Béla Szökefalvi-Nagy (1913–1998) cited Löwig's work [L6] in their monograph *Leçons d'analyse fonctionnelle* [RS] from 1952. Another French edition of this book appeared in the next year, followed by several English, German, and Russian editions. The authors also mentioned Riesz's work *Zur Theorie des Hilbertschen Raumes* [R3], the work of Rellich entitled *Spektraltheorie in nichtseparablen Räumen* [Re], and von Neumann's work *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren* [N1]. The chapter *Espace de Hilbert abstrait* contains the following remark related to the definition of a Hilbert space:

Cf. J. v. Neumann [1]; cet auteur énonce encore deux axiomes exigeant que l'espace soit séparable et de dimension infinie (donc de dimension dénombrablement infinie). Nous préférons de ne pas exclure dès commencement les espaces de dimension finie et les espaces non séparables. Voir, pour les espaces non séparables, Löwig [1], Riesz [15], Rellich [1]. ([RS], 1952, p. 195)

The work [L6] is also cited in the 1960 monograph *Topologische lineare Räume I*. [Kö2] by Köthe (the English version appeared in 1969), and in the book *Introductory Functional Analysis with Applications* [K1] written by Erwin O. Kreyszig (1922–2008) and published in 1978 (reprinted in 1989).

It was the work [L6] which Bourbaki had in mind when referring to Löwig

in the treatise on the history of topological vector spaces:

Dès ce moment les points essentiels de la théorie des espaces hilbertiens peuvent être considérés comme acquis; parmi les progrès plus récents, il faut notamment mentionner la présentation axiomatique de la théorie donnée vers 1930 par M. H. Stone et J. von Neumann, ainsi que l'abandon des restrictions de «séparabilité», qui s'effectue aux environs de 1934, dans les travaux de Rellich, Löwig, et F. Riesz (IXe). ([Bo1], 1955, p. 168)⁴⁵

In 1958, Nelson Dunford (1906–1986) and Jacob T. Schwartz (1930–2009) cited the work [L6] in the extensive monograph *Linear Operators*, which was later translated into Russian. They mentioned Löwig's proof of the facts that all orthonormal bases of a Hilbert space have the same cardinality (see [L6], *Satz 33*, p. 31 – see above; see also Rellich's work [Re], p. 355), and that two Hilbert spaces are isometrically isomorphic if and only if they have the same dimension (see [L6], *Satz 30*, *Satz 31*, p. 27 – see above). They also acknowledged Löwig's contribution in suppressing the requirement of separability in the definition of a Hilbert space.⁴⁶

In 1987, Vasile I. Istratescu cited Löwig's work [L6] in his book *Inner Product Structures: Theory and Application* [Is].

Pietsch mentioned Löwig's work [L6] in the epilogue to his 1989 publication [HS]⁴⁷, as well as in his extensive monograph *History of Banach spaces and linear operators* [P] published in 2007:⁴⁸

The representation theorem for abstract non-separable Hilbert spaces had to wait for Löwig [1934a, p. 11] and Riesz [1934, p. 34]:

Für jede lineare Funktion $l(f)$ gibt es ein eindeutig bestimmtes erzeugendes Element g , so das $l(f) = (f, g)$. ([P], p. 31)

... formula (2.3.6.a) was already discovered by Löwig [1934a, p. 6]. ([P], p. 39)⁴⁹

In 2001, Löwig's work [L6] was cited in the book *Best Approximation in Inner Product Spaces* [De] by Frank Deutsch (born 1936); the information on Löwig's results is taken over from the monograph [DS].

In 2002, Vilmos Komornik (born 1954) cited Löwig's work [L6] in his textbook *Précis d'analyse réelle II. Analyse fonctionnelle, Intégrale de Lebesgue, Espaces fonctionnelles* [Ko].

⁴⁵ The reference (IXe) corresponds to Riesz's work [R3]. For the above quotation, see [Bo2]: p. 213 in the English version from 1994, p. 248 in the German version from 1971, or p. 224 in the Russian version from 1963, respectively. The reference to Rellich stands for his work [Re].

⁴⁶ See [DS], p. 372 and p. 373, or p. 407 and p. 414 in the Russian edition, respectively.

⁴⁷ *Nicht-separable Hilberträume sind erstmalig von Löwig (1934) und Rellich (1934) betrachtet worden.* ([HS], p. 287)

⁴⁸ See [P], p. 13, p. 31, and p. 39.

⁴⁹ The number (2.3.6.a) refers to Löwig's formula $Lt = Rt - iR(it)$.

In 2005, Femi O. Oyadare cited Löwig's work [L6] in his paper *Construction of higher orthogonal polynomials through a new inner product, $\langle \cdot, \cdot \rangle_p$ in a countable real L^p -space* [Oy].

Let us return to the quotation from Pietsch's monograph concerning Löwig's formula

$$Lx = Rx - iR(ix),$$

which describes the structure of a functional on a complex space.

In 1938, Henri F. Bohnenblust (1906–2000) and Andrew Sobczyk (1915–1981) came up with a similar idea; in their short paper *Extensions of functionals on complex linear spaces* [BS], they stated the following analogue of the Hahn-Banach theorem:

Theorem 1. Let l be any complex linear subspace of a normed complex linear space L . Let $f(x)$ be any complex linear functional defined on l , having a norm M . Then there always exists a complex linear functional $F(x)$ defined on L , which coincides with $f(x)$ in l , and which has the same norm M on L . ([BS], p. 91)

They expressed the complex functional f in the form

$$f(x) = f_1(x) + i f_2(x)$$

and noted that $f_2(x) = -f_1(ix)$. The extended functional F is then obtained by letting

$$F(x) = F_1(x) - i F_1(ix),$$

where F_1 is a real functional from the classical version of the Hahn-Banach theorem.

Moreover, they mentioned the existence of a similar proof given by Murray in the work *Linear transformations in \mathfrak{L}_p , $p > 1$* [Mu] in 1936. However, Murray was aware of Löwig's work [L6] at the time (or a few weeks later), because he cited it (together with von Neumann) in the article [MN].⁵⁰

Soon after, the above mentioned result of Bohnenblust and Sobczyk became generally known; for example, it was cited by Salomon Bochner (1899–1982) and Angus Ellis Taylor (1911–1999) in their paper *Linear functionals on certain spaces of abstractly-valued functions* [BT] from 1938.

Later, it became apparent that a similar result appeared in Suchomlinov's paper *O prodolženii linejnych funkcionalov v kompleksnom i kvaternionnom linejnom prostranstve* [Su] from 1938. Löwig's result dating back to 1934 was thus completely overlooked.

⁵⁰ The work [Mu] was presented on 23rd February 1935, the work [MN] was received by the journal on 3rd April 1935.

As late as 1975, John A. R. Holbrook wrote the following words in his paper *Concerning the Hahn-Banach theorem* [Ho]:

... the Hahn-Banach extension theorem for real spaces dates from papers by H. Hahn [3] in 1927 and S. Banach [1] in 1929, while the familiar trick deriving the theorem for other scalars by reduction to the real case was not forthcoming until 1938: H. F. Bohnenblust and A. Sobczyk [2] (complex scalars); G. A. Soukhomlinoff [9] (complex or quaternionic scalars). ([Ho], p. 322)⁵¹

In 2008, Lawrence Narici and Edward Beckenstein acknowledged the contribution of Löwig's work [L6] in their article *The Hahn-Banach theorem and the sad life of E. Helly* [NB2]. They remarked that the important moment was the discovery of the relation between the real and complex component of a linear functional f on a complex space, i.e. the relation

$$\Re f(ix) = \Im f(x).$$

Although usually credited to F. Murray [1936], H. Löwig discovered this in 1934. Murray reduced the complex case to the real case, then used the real Hahn-Banach theorem to prove the complex form for subspaces of $L_p[a, b]$ for $p > 1$. Murray's perfectly general method was used and acknowledged by Bohnenblust and Sobczyk [1938] who proved it for arbitrary complex normed spaces. They, incidentally, were the first to call it the Hahn-Banach theorem. Also by reduction to the real case, Soukhomlinov [1938] and Ono [1953] obtained the theorem for normed spaces over the complex numbers and the quaternions. ([NB2], p. 105)⁵²

In 2009, Barbara D. MacCluer mentioned Löwig's contribution concerning complex spaces in her book *Elementary Functional Analysis*:

The extension of the proof of the Hahn-Banach theorem from the real case to the complex case is outlined in Exercises 3.2 and 3.3. While it is not hard, historically there was a span of nearly ten years between the work on the real case by Banach and the extension to the complex case by H. Bohnenblust and A. Sobczyk in 1938. Perhaps not coincidentally, Banach's esteemed 1932 treatise *Opérations Linéaires* deals only with real Banach spaces. In the particular setting of $X = L^p$ the complex case appeared in 1936 in the work of F. Murray; see also the comment in Exercise 3.3 on an earlier contribution by H. Löwig.

⁵¹ The references correspond to the following works: H. Hahn: *Über lineare Gleichungssysteme in linearen Räumen*, Journal für die reine und angewandte Mathematik 157 (1927), 214–229, S. Banach: *Sur les fonctionnelles linéaires. I, II*, Studia mathematica 1 (1929), 211–216, 223–239, and finally [BS] and [Su].

⁵² The references correspond to [Mu], [BS], [Su], [O]. It is interesting that Narici and Beckenstein did not mention Löwig's work [L6] in their earlier paper *The Hahn-Banach theorem: the life and times* [NB1] from 1997 – they cited only [Mu], [BS], [Su] and [O], where the work [O] is cited incorrectly under the number [60]; on the other hand, they also recalled the work [Ho] from 1975. It is most probable that they learned about Löwig's result in the above mentioned monograph [P] by Pietsch.

The work of Bohnenblust and Sobczyk may be the first place that the result is referred to as the “Hahn-Banach Theorem.” ([Mc], p. 54)

* * *

Löwig’s article *Über die Dimension linearer Räume* [L7] consisting of six pages was published in the journal *Studia mathematica* in 1934; it was received by the journal on 17th March 1934.⁵³

Löwig’s point of departure were two important works published in 1932, namely the long paper *Zur Theorie der linearen metrischen Räume* [H1] by Felix Hausdorff, and the monograph *Théorie des opérations linéaires* [Ba2] by Stefan Banach. He also cited his previous paper *Komplexe euklidische Räume von beliebiger endlicher oder transfiniten Dimensionszahl* [L6].

He considered both real and complex normed linear spaces, and referred the reader to the concepts introduced by F. Hausdorff in his work [H1], e.g. *linearer metrischer Raum*, *lineare Hülle*, and *abgeschlossene lineare Hülle*. The article [L7] contains the following two definitions of a dimension:

- *The affine dimension of a linear space \mathfrak{R}* , which coincides with the classical dimension of a linear space \mathfrak{R} .

It is defined as the smallest cardinal number among the cardinalities of all sets whose linear span is the whole space \mathfrak{R} .

- *The metric dimension of a normed linear space \mathfrak{R}* .

It is defined as the smallest cardinal number among the cardinalities of all sets whose closed linear span is the whole space \mathfrak{R} . We feel Zermelo’s well-ordering theorem behind both definitions.

Definition 1. Unter der affinen Dimensionszahl eines linearen Raumes \mathfrak{R} verstehe man die kleinste Kardinalzahl \aleph von der Eigenschaft, daß es mindestens eine Teilmenge von \mathfrak{R} von der Mächtigkeit \aleph gibt, deren lineare Hülle mit \mathfrak{R} zusammenfällt. ([L7], p. 18)

Definition 2. Unter der metrischen Dimensionszahl eines linearen metrischen Raumes \mathfrak{R} verstehe man die kleinste Kardinalzahl \aleph von der Eigenschaft, daß es mindestens eine Teilmenge von \mathfrak{R} von der Mächtigkeit \aleph gibt, deren abgeschlossene lineare Hülle mit \mathfrak{R} zusammenfällt. ([L7], p. 18)

Löwig noted the following relation between the two concepts: the affine dimension is never smaller than the metric dimension.

Es ist klar, daß die affine Dimensionszahl eines linearen metrischen Raumes niemals kleiner sein kann als seine metrische Dimensionszahl. ([L7], p. 19)

⁵³ The journal *Studia mathematica* was founded by Stefan Banach in 1929 to deal mainly with the questions of functional analysis.

He introduced the notion of a *basis* of a linear space (a linearly independent spanning set). His wording is cumbersome by modern standards; he was trying to avoid saying that an infinite set is linearly independent.

Definition 3. Eine Teilmenge \mathfrak{M} eines linearen Raumes \mathfrak{R} heie eine Basis von \mathfrak{R} , wenn endlich viele Elemente von \mathfrak{M} stets linear unabhngig sind und die lineare Hlle von \mathfrak{M} mit \mathfrak{R} zusammenfllt. ([L7], p. 19)

He remarked that the fact that every linear space has a basis is well known. In the subsequent theorem, he proved that all bases of a given linear space have the same cardinality, because the cardinality of every basis equals the affine dimension introduced earlier. He made use of some basic facts concerning infinite cardinal numbers. Nowadays, this topic is usually classified as belonging to linear algebra.

Satz 1. Jede Basis eines linearen Raumes \mathfrak{R} hat eine Mchtigkeit, welche der affinen Dimensionszahl von \mathfrak{R} gleich ist. ([L7], p. 19)

In the next theorem, he compared the cardinality of a linear space with its affine dimension. Again, he needed the basic properties of infinite cardinal numbers.

Satz 2. Ein linearer Raum \mathfrak{R} , der nicht nur aus einem Nullelement besteht, hat die Mchtigkeit des Kontinuums, wenn seine affine Dimensionszahl kleiner als die Mchtigkeit des Kontinuums ist, und sonst eine Mchtigkeit, welche gleich dieser affinen Dimensionszahl ist. ([L7], p. 20)

For normed linear spaces, Lwig introduced the notion of a *fundamental set*, showed that it corresponds to the metric dimension, and that a subspace and its closure have the same metric dimension.

Definition 4. Eine Teilmenge \mathfrak{M} eines linearen metrischen Raumes \mathfrak{R} heie eine Grundmenge von \mathfrak{R} , wenn erstens kein Element von \mathfrak{M} der abgeschlossenen linearen Hlle der brigen Elemente von \mathfrak{M} angehrt und zweitens die abgeschlossene lineare Hlle von \mathfrak{M} mit \mathfrak{R} zusammenfllt. ([L7], p. 20)

Lwig compared his concept of a fundamental set with similar notions defined in the above mentioned Hausdorff's work [H1] as well as in Banach's monograph [Ba2]. He noted that in Hausdorff's notion of a *Grundmenge*, the first condition is replaced by linear independence⁵⁴, while Banach introduced his *ensemble fondamental* using the second condition only.⁵⁵ According to the

⁵⁴ Wenn A linear unabhngig ist, so heie sie eine Grundmenge ihrer abgeschlossenen linearen Hlle \bar{L} . . . ([H1], p. 295)

⁵⁵ Un ensemble $G \subset E$ s'appelle fondamental, lorsque l'ensemble de toutes les combinaisons linaires d'lments de G est dense dans E ; il s'appelle total, lorsque toute fonctionnelle linaire $f(x)$ qui s'annule pour chaque $x \in G$, s'annule aussi pour chaque $x \in E$. . . Thorme 7. Pour qu'un ensemble $G \subset E$ soit fondamental, il faut et il suffit qu'il soit total. ([Ba2], p. 58)

next theorem, the notion of a fundamental set is closely related to the notion of a metric dimension.

Satz 3. Jede Grundmenge eines linearen Raumes \mathfrak{R} besitzt eine Mächtigkeit, welche gleich der metrischen Dimensionszahl von \mathfrak{R} ist. ([L7], p. 21)

Satz 4. Eine lineare Mannigfaltigkeit \mathfrak{B} eines linearen metrischen Raumes und ihre abgeschlossene Hülle $\overline{\mathfrak{B}}$ haben stets die gleiche metrische Dimensionszahl. ([L7], p. 22)

Löwig then remarked that in a complete Euclidean space, a maximal orthonormal set (*vollständiges normiertes Orthogonalsystem*) is always a fundamental set, and its cardinality equals the metric dimension of the space.⁵⁶ He also noted that it is not clear whether a normed linear space \mathfrak{R} always possesses a fundamental set. Even if we consider a well ordering on the space \mathfrak{R} and the set of elements not belonging to the closed linear span of their predecessors, we do not necessarily obtain a fundamental set.

Ob jeder lineare metrische Raum eine Grundmenge im Sinne unserer Definition 4 besitzt, konnte nicht entschieden werden. (Man könnte von einer Wohlordnung des Raumes ausgehen und in dieser die Menge aller derjenigen Elemente betrachten, welche nicht der abgeschlossenen linearen Hülle der Menge ihrer Vorgänger angehören. Die so ausgesonderte Menge muss aber keine Grundmenge des Raumes im Sinne unserer Definition 4 sein.) ([L7], p. 22)

The paper [L7] ends with a remark that every normed linear space \mathfrak{R} can be extended into a complete normed linear space \mathfrak{R}^* , which coincides with the closed linear span of the space \mathfrak{R} in \mathfrak{R}^* (moreover, \mathfrak{R} and \mathfrak{R}^* have the same metric dimension).

* * *

Georg Aumann wrote a short report on Löwig's work [L7] for the journal *Jahrbuch über die Fortschritte der Mathematik*.⁵⁷

In 1936, Teichmüller cited the work [L7] in his paper *Operatoren im Wachstumschen Raum*.

In 1939, Vitold L. Šmul'jan (Shmulyan, 1914–1944) cited the work [L7] in his article *O nekotorych geometričeskich svojstvach ediničnoj sfery prostranstva tipa (B) [Š]*.⁵⁸

⁵⁶ The term Euclidean space refers to a real or complex linear inner product space, while a complete normed orthogonal system stands for a maximal orthonormal set; its existence follows from the axiom of choice.

⁵⁷ See JFM 60.1229.01.

⁵⁸ See [Š], p. 86, 93.

Casper Goffman (1913–2006) made a reference to Löwig’s work [L7] in his paper *On linear spaces which may be rendered complete normed metric spaces* [Go] published in 1943.

George Whitelaw Mackey (1916–2006) cited Löwig’s work [L7] in his 1943 paper *On infinite dimensional linear spaces* [Mac1], and also in the paper [Mac2] published under the same name two years later:

It has been shown by Löwig [15] that any two Hamel bases for the same linear space have the same cardinal number. This cardinal number we shall call the dimension of the space. And he added the following footnote: This is what Löwig calls the affine dimension. ([Mac2], pp. 157–158)

Löwig [15] has shown that whenever X is a linear space with more than C elements then its dimension is equal to the number of its elements. ([Mac2], p. 159)

In 1948, Schiek cited the work [L7] in his paper *Mengen mit affiner Anordnung* [Sc].

In 1950, Victor L. Klee (1925–2007) made a reference to Löwig’s work [L7] in his paper *Decomposition of an infinite-dimensional linear system into ubiquitous convex sets* [K11]; he recalled Löwig’s result on the existence of a (Hamel) basis of a linear space L and the invariance of its cardinality (i.e., the dimension of the space L).

In 1951, Halmos included the work [L7] in his textbook *Introduction to Hilbert space and the theory of spectral multiplicity* [H] (2nd edition: 1957); [L7] is the 27th item in his list of references containing 52 works. He wrote:

The elegant proof (in §16) of the uniqueness of dimension in the infinite case is due to Löwig, [27]. ([H], p. 110)⁵⁹

In 1954, the work [L7] was mentioned by D. T. Finkbeiner and Otton Martin Nikodým in their paper *On convex sets in abstract linear spaces where no topology is assumed (Hamel bodies and linear boundedness)* [FN], and by Nikodým in his paper *Limit-representation of linear, even discontinuous, linear functionals in Hilbert spaces* [Ni].

Taylor cited Löwig’s work [L7] in his textbook *Introduction to functional analysis* [T] from 1958; many subsequent editions followed.⁶⁰

Mahlon Marsh Day (1913–1992) cited the work [L7] in the second chapter

⁵⁹ In the above mentioned 16th paragraph entitled *Dimension* (pp. 29–31), Halmos presented the following three theorems:

1. *Any two bases of a subspace \mathfrak{M} have the same power.*
2. *A linear transformation U from a Hilbert space \mathfrak{H} to a Hilbert space \mathfrak{K} is an isomorphism if and only if is an isometry, mapping \mathfrak{H} onto \mathfrak{K} .*
3. *Two Hilbert spaces are isomorphic if and only if they have the same dimension.*

⁶⁰ Taylor makes a reference to Löwig’s theorem on the relation between cardinality and dimension (see p. 46).

of his book *Normed Linear Spaces* [Da] published in 1958 (subsequent editions followed in 1962 and 1973):

(8) *If M is a complete linear metric space with distance function d and if a new and always smaller [or always larger] distance function d' is introduced under which M becomes a complete linear metric space M' , the identity operator is an isomorphism of M with M' .*

(9) *The theorem of Löwig that every separable, infinite-dimensional, Banach space has a vector basis of the cardinal number of the continuum, and the fact that two such spaces may fail to be isomorphic (say $l^1(\omega)$ and $l^2(\omega)$), shows that some relation between the metric is needed in (8). ([Da], 1973, p. 42)*

In 1960, Klee cited Löwig's paper [L7] in his article *Mappings into normed linear spaces* [K12].

In 1982, Manuel Valdivia cited the work [L7] in his article *A class of locally convex spaces without C -webs* [V].

In 1996, a reference to the work [L7] appeared in the textbook *Functional Analysis* [Li] written by Balmohan Vishnu Limaye.

In 1997, Herbert Schröder cited the work [L7] in his book *Funktional-analysis* [Sch]:

... nichtseparable Hilberträume wurden erst 1934 von H. Löwig (Studia Math. 5(1934) S. 18–23) im Zusammenhang mit dem Dimensionsbegriff und von F. Rellich (Math. Ann. 110(1935) S. 342–356) im konkreten Beispiel des Raums der fastperiodischen Funktionen untersucht. ([Sch], p. 33)

Paul Howard and Jean E. Rubin cited the work [L7] in their 1998 monograph *Consequences of the axiom of choice* [HR]; the following theorem is mentioned in two places – *Löwig's Theorem: If B_1 and B_2 are both bases for the vector space V then $|B_1| = |B_2|$.* ([HR], p. 40, 78)

In 1999, Jürg Rätz cited the work [L7] in his article *Comparison of inner products* [Rä].

Löwig's work [L7] is also cited in Pietsch's monograph *History of Banach Spaces and Linear Operators* [P] published in 2007.⁶¹

It is also cited in the recent work of Wolfgang Arendt (born 1950) and Robin Nittka entitled *Equivalent complete norms and positivity* [AN] and published in 2009:

Let \mathfrak{B} be a Hamel basis of a vector space E . Then $\text{card}(E) = \max\{\text{card}(\mathfrak{B}), c\}$, see Löwig [16]. ([AN], p. 425)

Helemskii A. Ya. cited Löwig's work [L7] in his 2011 paper *Metric version of projectivity for normed modules over sequence algebras* [HI].

Let us finally quote from Ralph-Hardo Schulz's textbook *Repetitorium Bach-*

⁶¹ See [P], p. 5 and p. 44.

elor *Mathematik. Zur Vorbereitung auf Modulprüfungen in der mathematischen Grundausbildung* [Scu] published in 2009:

Der grundlegende Satz ist der Satz von Löwig über die Gleichmächtigkeit aller Basen eines Vektorraums: Ist V ein VR, und sind B und C Basen von V , so gibt es eine Bijektion von B auf C ; folglich gilt $|B| = |C|$. ([Scu], p. 8)

* * *

The short paper *Über allgemeine Spektralfunktionen* [L8] summarises the basic results of Löwig's presentation at the second congress of mathematicians from Slavic countries, which took place in Prague from 22nd to 28th September 1934.⁶² The congress was attended by numerous Czech and German mathematicians from universities, high schools and other institutions of the Czechoslovak Republic, but also by many foreign mathematicians (Pavel Sergejevich Aleksandrov, Dan Barbilian, Wilhelm Blaschke, Samuel Dickstein, Maurice Fréchet, Guido Fubini, Heinz Hopf, Witold Hurewicz, Jovan Karamata, Solomon Lefschetz, Stefan Mazurkiewicz, Karl Menger, Kyrille Poppoff, Petre Sergescu, Waclaw Sierpiński, Alfred Tarski, Lubomir Tchakaloff, Vladimir Varičak, George Neville Watson, Tadeusz Ważewski, Stanisław Zaremba, and others). The chairman of the organising committee was Karel Petr (1868–1950).

One of the most active attendees was Fréchet, who delivered the following three lectures:

Détermination de la classe la plus générale d'espaces vectoriels distanciés applicables vectoriellement sur l'espace concret de Hilbert,

Sur deux relations simples entre le „coefficient“ de corrélation et le „rapport“ de corrélation,

*Sur les précisions comparées de la valeur moyenne et de la valeur médiane.*⁶³

Apart from that, he also gave an opening speech on 23rd September as well as a closing speech on 28th September.⁶⁴

It is unclear whether Stefan Banach was present at the congress. His name is included in the list of congress delegates from the Uniwersytet Jana Kazimierza (Lwów), but not in the list of members and attendees.

Together with other contributed papers and additional congress materials, Löwig's short article [L8] was published in the journal *Časopis pro pěstování*

⁶² The first congress of mathematicians from Slavic countries took place in Warsaw in September 1929.

⁶³ In 1935, they were published in the journal *Časopis pro pěstování matematiky a fyziky* on pages 176–177, 209–210, 210–211. See also the independent publication *Zprávy o druhém sjezdu matematiků zemí slovanských*, JČMF, Praha, 1935.

⁶⁴ *Časopis pro pěstování matematiky a fyziky* 64 (1935), pp. xxxv, xlii–xliii.

matematiky a fyziky in 1935.⁶⁵ It contained a generalisation of earlier results by Ernst Hellinger (1883–1950) [Hel] from 1907 and Hans Hahn [Hah] from 1912 concerning the orthogonal equivalence of two quadratic forms of infinitely many variables.

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The author is grateful to Prof. Ivan Netuka and Dr. Antonín Slavík for their inspiring suggestions.

REFERENCES

- [L6] Löwig H., *Komplexe euklidische Räume von beliebiger endlicher oder transfiniten Dimensionszahl*, Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae. Sectio Scientiarum Mathematicarum, Szeged **7** (1934), 1–33.
- [L7] Löwig H., *Über die Dimension linearer Räume*, Studia mathematica **5** (1934), 18–33.
- [L8] Löwig H., *Über allgemeine Spektralfunktionen*, Časopis pro pěstování matematiky a fyziky **64** (1935), 153–154.

* * *

- [AG] Achiezer N. I., Glazman I. M., *Teorija linejnych operatorov v Gil'bertovom prostanstve*, Gosudarstvennoe izdatel'stvo tehniko-teoretičeskoj literatury, Moskva, 1950, 483 pp., 2nd edition: Nauka, Moskva, 1966, 543 pp.; German: *Theorie der linearen Operatoren im Hilbert-Raum*, Akademie-Verlag, Berlin, 1954, xiii+369 pp.; 2nd edition: 1958, xiii+369 pp., 1960, 469 pp., 1968, xvi+488 pp., 1977, 1981; English (translated by M. Nestell from Russian): *Theory of Linear Operators in Hilbert Space I, II*, Frederick Ungar Publishing Co, New York, 1961, 1963, xi+147+218 pp., reprint: 1981, xxxii+552 pp., Dover Publications, New York, 1993.
- [AN] Arendt W., Nittka R., *Equivalent complete norms and positivity*, Archiv der Mathematik (Basel) **92** (2009), 414–427.
- [Ba1] Banach S., *Sur les opérations dans les ensembles abstraits et leurs application aux équations intégrales (Thèse de doctorat)*, Fundamenta Mathematicae **3** (1922), 133–181, Oeuvres II, 305–348.
- [Ba2] Banach S., *Théorie des opérations linéaires*, Monografie matematyczne 1, Fundusz kultury narodowej, Warszawa, 1932, vii+254 pp., reprint: Chelsea Publishing Company, New York, 1943, 1955, 1963, 1978, xii+259 pp., reprint: Hafner Publishing Company, New York, 1949, vii+254 pp., Oeuvres II, 13–219; English: *Theory of Linear Operations*, Mathematical Library 38, North-Holland, Amsterdam, 1987, x+237 pp.; Ukrainian: *Kurs funkcional'noho analizu*, Kiev, 1948.
- [B1] Bečvář J., *Lineární algebra*, Matfyzpress, Prague, 2000, 435 pp., subsequent editions: 2002, 2005, 2010.
- [B2] Bečvář J., *Z historie lineární algebry*, series Dějiny matematiky, vol. 35, Matfyzpress, Prague, 2007, 519 pp.
- [Br1] Bernkopf M., *The development of function spaces with particular reference to their origins in integral equation theory*, Archive for History of Exact Sciences **3** (1966), 1–96.

⁶⁵ It also appeared in *Zprávy o druhém sjezdu matematiků zemí slovanských*.

- [Br2] Bernkopf M., *Vvedenie abstraktnych predstavlenij v teoriju funkcional'nych prostranstv*, Istoriko-matematičeskie issledovanija **18** (1973), 94–103, report from the 13th international congress in the history of science, Moscow, 1971.
- [BK] Birkhoff G., Kreyszig E., *The establishment of functional analysis*, *Historia mathematica* **11** (1984), 258–321.
- [BS] Bohnenblust H. F., Sobczyk A., *Extensions of functionals on complex linear spaces*, *Bulletin of the American Mathematical Society* **44** (1938), 91–93.
- [BT] Bochner S., Taylor A. E., *Linear functionals on certain spaces of abstractly-valued functions*, *Annals of Mathematics* **39** (1938), 913–944.
- [Bor] Borgwadt H., *Die historische Entwicklung der Funktionalanalysis zu einer selbständigen mathematischen Disziplin*, NTM – Schriftenreihe für Geschichte der Naturwissenschaften, Technik und Medizin **12** (1975), No. 1, 1–11.
- [Bo1] Bourbaki N., *Espaces vectoriels topologiques*, *Éléments de mathématique*, Livre V, Herman & C^{ie}, Éditeurs, Paris, 1953, 123 pp., 1955, 191 pp., 1966, 178 pp., 1981, vii+368 pp.; English: *Topological vector spaces*, Springer-Verlag, Berlin, 1987, vii+364 pp.; Russian: *Topologičeskie vektornye prostranstva*, Izdatel'stvo inostranoj literatury, Moskva, 1959, 410 pp.
- [Bo2] Bourbaki N., *Éléments d'histoire des mathématiques*, Herman & C^{ie}, Éditeurs, Paris, 1960, 276 pp., 2nd edition: 1969, 325 pp., 1974, 376 pp., Springer-Verlag, 2006, 376 pp., Masson, Paris, 1984, 2007, 376 pp.; Russian: *Očerki po istorii matematiki*, Izdatel'stvo inostranoj literatury, Moscow, 1963, 292 pp.; English: *Elements of the History of Mathematics*, Springer-Verlag, Berlin, Heidelberg, 1994, 1998, viii+301 pp.; German: *Elemente der Mathematikgeschichte*, Vandenhoeck und Ruprecht, Göttingen, 1971, 297 pp.; Italian: *Elementi di storia della matematica*, Feltrinelli, Milano, 1963, 270 pp.
- [Bro] Browder F. E., *The relation of functional analysis to concrete analysis in 20th century mathematics*, *Historia mathematica* **2** (1975), 577–590.
- [Ca] Calkin J. W., *Two-sides ideals and congruences in the ring of bounded operators in Hilbert space*, *Annals of Mathematics* **42** (1941), 839–873.
- [Da] Day M. M., *Normed Linear Spaces*, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1958, viii+139 pp., 2nd edition: Academic Press, New York, 1962, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1962, viii+139 pp., 3rd edition: Springer-Verlag, Berlin, Heidelberg, New York, 1973, viii+211 pp.; Russian: *Normirovannnye linejnye prostranstva*, Izdatel'stvo inostranoj literatury, Moskva, 1961, 232 pp.
- [De] Deutsch F., *Best Approximation in Inner Product Spaces*, Springer-Verlag, New York, Berlin, Heidelberg, 2001, xv+338 pp.
- [Di] Dieudonné J., *History of Functional Analysis*, *Mathematics Studies* 49, North-Holland Publishing Company, Amsterdam, New York, Oxford, 1981, vi+312 pp.
- [Do] Dorier J.-L. (ed.), *On the Teaching of Linear Algebra*, *Mathematical Education Library* 23, Kluwer Academic Publishers, 2000, xxii+288 pp.
- [Du] Duda R., *The discovery of Banach spaces*, in W. Więśław (ed.): *European Mathematics in the Last Centuries*, Lectures presented at the conference held at Będlewo, 26–30 April 2004, Stefan Banach International Mathematical Center, Institute of Mathematics Wrocław university, Wrocław, 2005, 37–46.
- [DS] Dunford N., Schwartz J. T., *Linear Operators I, II, III*, Interscience Publishers, New York, London, 1958, 1963, 1971, xiv+1–858, ix+859–1924, xix+1925–2592 pp., subsequent editions of the 1st volume: 1964, J. Wiley, New York, 1988; Russian: *Linejnye operatory*, Mir, Moskva, 1962, 1966, 1974, 895+1063+661 pp.

- [E] Ebbinghaus H.-D., *Ernst Zermelo. An Approach to His Life and Work*, Springer-Verlag, Berlin, Heidelberg, 2007, xiv+356 pp.
- [FN] Finkbeiner D. T., Nikodým O. M., *On convex sets in abstract linear spaces where no topology is assumed (Hamel bodies and linear boundedness)*, Rendiconti del Seminario della Università di Padova **23** (1954), 357–365.
- [F] Fréchet M., *Les espaces abstraits et leur théorie considérée comme introduction à l'Analyse générale*, Gauthier-Villars, Paris, 1928, reprint: 1989, 312 pp., xi+296 pp.
- [Go] Goffman C., *On linear spaces which may be rendered complete normed metric spaces*, Bulletin of the American Mathematical Society **49** (1943), 611–614.
- [G] Gray J. D., *The shaping of the Riesz representation theorem: a chapter in the history of analysis*, Archive for History of Exact Sciences **31** (1984), 127–187.
- [Hah] Hahn H., *Über die Integrale des Herrn Hellinger und die Orthogonalinvarianten der quadratischen Formen von unendlich vielen Veränderlichen*, Monatshefte für Mathematik und Physik **23** (1912), 161–224.
- [H] Halmos P. R., *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, Chelsea Publishing Company, New York, 1951, 114 pp., 2nd edition: 1957, 114 pp., reprint: 1998.
- [Ha] Hamel G., *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung: $f(x + y) = f(x) + f(y)$* , Mathematische Annalen **60** (1905), 459–462.
- [H1] Hausdorff F., *Zur Theorie der linearen metrischen Räume*, Journal für die reine und angewandte Mathematik **167** (1931/1932), 294–311, also in *Gesammelte Werke IV*, Springer-Verlag, Berlin, 2001, 269–288.
- [H2] Hausdorff F., *Grundzüge der Mengenlehre*, Veit & Comp., Leipzig, 1914, viii+476 pp., reprint: Chelsea, New York, 1949, 1965, 1978, see also *Gesammelte Werke II*, Springer-Verlag, Berlin, Heidelberg, New York, 2002, xviii+883 pp.
- [H3] Hausdorff F., *Mengenlehre*, Walter de Gruyter & Co., Berlin, 1927, 285 pp. (a second edition of the work *Grundzüge der Mengenlehre*), 3rd edition: 1935, 307 pp., reprint: Dover Publications, New York, 1944; English: *Set Theory*, 3rd German edition translated by J. R. Aumann et al., Chelsea Publishing Company, New York, 1957, 1962, 1978, 1991, 352 pp.; Russian: *Teoriya množestv*, Moskva, Leningrad, 1937.
- [He] van Heijenoort J. (ed.), *From Frege to Gödel. A Source Book in Mathematical Logic 1879–1931*, Harvard University Press, Cambridge, Mass., 1967, viii+664 pp., reprint: 1971.
- [Hl] Helemskii A. Ya., *Metric version of projectivity for normed modules over sequence algebras*, 2011, 19 pp.
- [Hel] Hellinger E., *Die Orthogonalinvarianten quadratischer Formen von unendlich vielen Variablen*, Dissertation, Göttingen, 1907.
- [Hu1] Heuser H., *Zur Ideengeschichte der Funktionalanalysis*, Mathematische Semesterberichte **35** (1988), 38–63.
- [Hu2] Heuser H., *Funktionalanalysis. Theorie und Anwendung*, B. G. Teubner, Stuttgart, 1986, 2nd edition, 696 pp.; *Functional Analysis*, J. Wiley & Sons, Chichester, New York, Brisbane, Toronto, Singapore, 1982, xv+408 pp.
- [Hi] Hilbert D., *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, 4. Mitteilung*, Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse, 1906, 157–227, also reprinted in [HS].

- [HS] Hilbert D., Schmidt E., *Integralgleichungen und Gleichungen mit unendlich vielen Unbekannten*, Teubner-Archiv zur Mathematik, Band 11, Leipzig, 1989, 316 pp.
- [Ho] Holbrook J. A. R., *Concerning the Hahn-Banach theorem*, Proceedings of the American Mathematical Society **50** (1975), 322–327.
- [Hor] Horváth J., *On the Riesz-Fischer theorem*, Studia Scientiarum Mathematicarum Hungarica **41** (2004), 467–478.
- [HR] Howard P., Rubin J. E., *Consequences of the axiom of choice*, Mathematical Surveys and Monographs 59, American Mathematical Society, Providence, 1998, 432 pp.
- [Ch] Chatterji S. D., *Commentary on [H 1931]*, in F. Hausdorff: Gesammelte Werke IV, 289–300; a commentary to the work [H1].
- [Is] Istratescu V. I., *Inner Product Structures: Theory and Applications*, D. Reidel Publishing Company, 1987, 895 pp.
- [J1] James R. C., *Inner products in normed linear spaces*, Bulletin of the American Mathematical Society **53** (1947), 559–566.
- [J2] James R. C., *Orthogonality and linear functionals in normed linear spaces*, Transactions of the American Mathematical Society **61** (1947), 265–292.
- [JN] Jordan P., von Neumann J., *On inner products in linear, metric spaces*, Annals of Mathematics **36** (1935), 719–723.
- [Ka] Katětov M., *Některé aspekty vývoje funkcionální analýzy*, DVT – Dějiny věd a techniky **1** (1968), 17–23, lecture given at the 11th international congress in the history of science (Krakow, 1965).
- [Ki] Kitagawa T., *The Parseval theorem of the Cauchy series and the inner products of certain Hilbert spaces*, Annals of Mathematics **40** (1939), 71–80.
- [K11] Klee V. L., *Decomposition of an infinite-dimensional linear system into ubiquitous convex sets*, American Mathematical Monthly **57** (1950), 540–541.
- [K12] Klee V. L., *Mappings into normed linear spaces*, Fundamenta Mathematicae **49** (1960), 25–34.
- [Kli] Kline M., *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1972, xvii+1238 pp.
- [Ko] Komornik V., *Précis d'analyse réelle II. Analyse fonctionnelle, Intégrale de Lebesgue, Espaces fonctionnelles*, Ellipses, Paris, 2002, Hungarian: *Valós analízis előadások II.*, 2010.
- [Kö1] Köthe G., *Eine axiomatische Kennzeichnung der linearen Räume vom Typus ω* , Mathematische Annalen **120** (1947/49), 634–649.
- [Kö2] Köthe G., *Topologische lineare Räume I*, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1960, xii+456 pp., 2nd edition: 1966; English: *Topological Vector Spaces I, II*, Springer-Verlag, New York, Heidelberg, Berlin, 1969, 1979, xv+456+xii+331 pp.; only the first volume appeared in German.
- [K1] Kreyszig E., *Introductory Functional Analysis with Applications*, J. Wiley & Sons, New York, Santa Barbara, London, Sydney, Toronto, 1978, xvi+688 pp., reprint: 1989.
- [K2] Kreyszig H., *Zur Entwicklung der zentralen Ideen in der Funktionalanalysis*, Elemente der Mathematik **41** (1986), 25–35.
- [K3] Kreyszig H., *Über die weitere Entwicklung der Funktionalanalysis bis 1932*, Elemente der Mathematik **41** (1986), 49–57.
- [K4] Kreyszig H., *Fridrich Riesz als Wegbereiter der Funktionalanalysis*, Elemente der Mathematik **45** (1990), 117–130.

- [Ku] Kuroš A. G., *Lekcii po obščej algebre*, Gosudarstvennoe izdatel'stvo fiziko-matematičeskoj literatury, Moskva, 1962, 396 pp.; German: *Vorlesungen über allgemeine Algebra*, Teubner, Leipzig, 1964, x+30 pp.; English: *Lectures on general algebra*, Chelsea Publishing Company, New York, 1963, 335 pp.
- [LS] Lengyel B. A., Stone M. H., *Elementary Proof of the Spectral Theorem*, Annals of Mathematics **37** (1936), 853–864.
- [Li] Limaye B. V., *Functional Analysis*, New Age International Limited Publishers, 1996, x+612 pp., reprint: 2004.
- [Mc] MacCluer B. D., *Elementary Functional Analysis*, Graduate Texts in Mathematics 253, Springer Science + Business Media, 2009, x+207 pp.
- [Mac1] Mackey G. W., *On infinite dimensional linear spaces*, Proceedings of the National Academy of Sciences of the U. S. A. **29** (1943), 216–221.
- [Mac2] Mackey G. W., *On infinite-dimensional linear spaces*, Transactions of the American Mathematical Society **57** (1945), 155–207.
- [Ma] Mazur S., *Über die Nullstellen linearer Operationen*, Studia mathematica **2** (1930), 11–20.
- [Me1] Medvedev F. A., *Osnovopoložniki funkcional'nogo analiza o ego rannej istorii*, Istoriko-matematičeskie issledovanija **18** (1973), 55–70.
- [Me2] Medvedev F. A., *Pervaja monografija po funkcional'nomu analizu*, Istoriko-matematičeskie issledovanija **18** (1973), 71–93.
- [Mo] Monna A. F., *Functional Analysis in Historical Perspective*, Oosthoek Publishing Company, Utrecht, 1973, viii+167 pp.
- [M] Moore G. H., *Zermelo's Axiom of Choice, its Origins, Development, and Influence*, Springer Studies in the History of Mathematics and Physical Sciences 8, Springer-Verlag, New York, Heidelberg, Berlin, 1982, xiv+410 pp.
- [Mu] Murray F. J., *Linear transformations in \mathfrak{L}_p , $p > 1$* , Transactions of the American Mathematical Society **39** (1936), 83–100.
- [MN] Murray F. J., von Neumann J., *On rings of operators*, Annals of Mathematics **37** (1936), 116–229, also in F. Bródy, T. Vámos (ed.): *The Neumann Compendium*, World Scientific Series in 20th Century Mathematics, Vol. 1, World Scientific, 1995, 703 pp.
- [NB1] Narici L., Beckenstein E., *The Hahn-Banach theorem: the life and times*, Topology and its Applications **77** (1997), 193–211.
- [NB2] Narici L., Beckenstein E., *The Hahn-Banach theorem and the sad life of E. Helly*, in J. M. Delgado Sánchez, T. Domínguez Benavides (eds.): *Advanced Courses of Mathematical Analysis III. Proceedings of the third international school, 3–7 September 2007*, World Scientific Publishing Co. Pte. Ltd., 2008, 208 pp., 97–110.
- [NS] Naylor A. W., Sell G. R., *Linear Operator Theory in Engineering and Science*, Holt, Rinehart and Winston, New York, 1971, xv+624 pp.; subsequent editions: Springer-Verlag, Berlin, Heidelberg, New York, 1982, 2000.
- [N0] von Neumann J., *Mathematische Begründung der Quantenmechanik*, Nachrichten von der Königliche Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse, 1927, 1–57.
- [N1] von Neumann J., *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Mathematische Annalen **102** (1930), 49–131.

- [N2] von Neumann J., *Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren*, *Mathematische Annalen* **102** (1930), 370–427.
- [N3] von Neumann J., *Mathematische Grundlagen der Quantenmechanik*, Springer-Verlag, Berlin, 1932, 262 pp., Dover Publications, New York, 1943, 266 pp.; English: *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, 1955, xii+445 pp.; Russian: *Matematičeskie osnovy kvantovoj mechaniki*, Moskva, Nauka, 367 pp.
- [N4] von Neumann J., *On infinite direct products*, *Compositio Mathematica* **6** (1939), 1–77.
- [N5] von Neumann J., *Invariant Measures. Notes by Paul R. Halmos*, Institute for Advanced Study, 1940 – 1941, cyclostyled version, 230 pp., printed version: American Mathematical Society, Providence, 1999, xv+134 pp.
- [Ni] Nikodým O. M., *Limit-representation of linear, even discontinuous, linear functionals in Hilbert spaces*, *Rendiconti del Seminario della Università di Padova* **23** (1954), 290–298.
- [O] Ono T., *A generalization of the Hahn-Banach theorem*, *Nagoya Mathematical Journal* **6** (1953), 171–176.
- [Oy] Oyadare F. O., *Construction of higher orthogonal polynomials through a new inner product, $\langle \cdot, \cdot \rangle_p$ in a countable real L^p -space*, *Americal Journal of undergraduate research* **4** (2005), N. 3, 13–26.
- [Ph] Phillips R. S., *A characterization of euclidean spaces*, *Bulletin of the American Mathematical Society* **46** (1940), 930–933.
- [P] Pietsch A., *History of Banach Spaces and Linear Operators*, Birkhäuser, Boston, Basel, Berlin, 2007, xxiii+855 pp.
- [Pl] Plesner A. I., *Spektral'naja teorija linejnyh operatorov*, *Uspechi matematičeskich nauk* **9** (1941), 3–125.
- [Rä] Rätz J., *Comparison of inner products*, *Aequationes Mathematicae* **57** (1999), 312–321.
- [Re] Rellich F., *Spektraltheorie in nichtseparabeln Räumen*, *Mathematische Annalen* **110** (1935), 342–356.
- [R1] Riesz F., *Les systèmes d'équations linéaires à une infinité d'inconnues*, Gauthier-Villars, Paris, 1913, vi+182 pp., *Oeuvres complètes II*, 829–1016.
- [R2] Riesz F., *Über die linearen Transformationen des komplexen Hilbertschen Raumes*, *Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae. Sectio Scientiarum Mathematicarum, Szeged* **5** (1930–1932), 23–54, *Oeuvres complètes II*, Akadémiai Kiadó, Budapest, 1960, 1103–1134.
- [R3] Riesz F., *Zur Theorie des Hilbertschen Raumes*, *Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae. Sectio Scientiarum Mathematicarum, Szeged* **7** (1934–1935), 34–38, *Oeuvres complètes II*, Akadémiai Kiadó, Budapest, 1960, 1150–1154.
- [RS] Riesz F., Szökefalvi-Nagy B., *Leçons d'analyse fonctionnelle*, Akadémiai Kiadó, Budapest, 1952, viii+448 pp., 2nd edition: Budapest 1953, viii+455 pp., 3rd edition: Budapest, Paris, 1955, viii+488 pp., 4th edition: 1965, viii+488 pp., 5th edition: Gauthier-Villars, Paris, 1968, viii+488 pp., 6th edition: Akadémiai Kiadó 1972, viii+488 pp.; German: *Vorlesungen über Funktionalanalysis*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956, xi+481 pp., 2nd edition: 1968, xi+482 pp., 3rd edition: 1973, xi+482 pp., 4th edition: 1982, 518 pp.; English: *Functional Analysis*, Frederick Ungar Publishing Company, New York, 1955, xii+467 pp., also 1971, 1978,

- Dover Publications, New York, 1990, xii+504 pp.; Russian: *Lekcii po funkcional'nomu analizu*, Izdatel'stvo inostrannoĭ literatury, Moscow, 1954, 499 pp., 2nd edition: Mir, Moscow, 1979, 587 pp.
- [Ru1] Rudin W., *Real and Complex Analysis*, McGraw-Hill, New York, 1966, xi+412 pp., 2nd edition: 1974, xii+452 pp., 3rd edition: 1987, xiv+416 pp.
- [Ru2] Rudin W., *Functional Analysis*, McGraw-Hill, New York, 1973, xiii+397 pp., 2nd edition: 1991, xv+424 pp.; Russian: *Funkcional'nyĭ analiz*, Mir, Moskva, 1975, 443 pp.
- [Sa] Saxe K., *Beginning Functional Analysis*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, Berlin, Heidelberg, 2002, xi+197 pp.
- [Sc] Schiek H., *Mengen mit affiner Anordnung*, Archiv der Mathematik **1** (1948), 473–479.
- [S] Schmidt E., *Über die Auflösung linearer Gleichungen mit unendlich vielen Unbekannten*, Rendiconti del Circolo Matematico di Palermo **25** (1908), 53–77.
- [Sch] Schröder H., *Funktionalanalysis*, Akademie Verlag, Berlin, 1997, viii+384 pp.
- [Scu] Schulz R.-H., *Repetitorium Bachelor Mathematik: Zur Vorbereitung auf Modulprüfungen in der mathematischen Grundausbildung*, Vieweg, Teubner, Wiesbaden, 2009, 380 pp.
- [Si1] Siegmund-Schultze R., *Die Anfänge der Funktionalanalysis und ihr Platz im Umwälzungsprozess der Mathematik um 1900*, Archive for History of Exact Sciences **26** (1982), 13–71.
- [Si2] Siegmund-Schultze R., *Eliakim Hastings Moore's "General Analysis"*, Archive for History of Exact Sciences **52** (1998), 51–89.
- [Sm] Smithies F., *The shaping of functional analysis*, Bulletin of the London Mathematical Society **29** (1997), 129–138.
- [St1] Stone M. H., *Linear transformations in Hilbert space*, Proceedings of the National Academy of Sciences U. S. America **15** (1929), 198–200, 423–425, **16** (1930), 172–175.
- [St2] Stone M. H., *Linear Transformations in Hilbert Space and their Applications to Analysis*, American Mathematical Society, New York, 1932, viii+622 pp., subsequent editions: ..., 1966, 1970.
- [Su] Suchomlinov G. A., *O prodolženii linejnykh funkcionalov v kompleksnom i kvaternionnom linejnom prostranstve*, Matematičeskij sbornik **3(45)** (1938), 353–358, short German abstract *Über Fortsetzung von linearen Funktionalen in linearen komplexen Räumen und linearen Quaternionräumen* on p. 358.
- [Š] Šmul'jan V. L., *O nekotorych geometričeskich svojstvach ediničnoj sfery prostranstva tipa (B)*, Matematičeskij sbornik **6(48)** (1939), 77–94, English resumé *On some geometrical properties of the unit sphere in the space of the type (B)* on pp. 90–94.
- [T] Taylor A. E., *Introduction to Functional Analysis*, J. Wiley & Sons, Inc., New York, 1958, xvi+423 pp., subsequent editions: 1961, 1964, 1967, 1980, xi+467 pp., 1986.
- [Te] Teichmüller O., *Operatoren im Wachsschen Raum*, Journal für die reine und angewandte Mathematik **174** (1936), 73–124.
- [V] Valdivia M., *A class of locally convex spaces without C-webs*, Annales de l'Institut Fourier **32** (1982), 261–269.
- [Ve] Vershik A. M., *The life and fate of functional analysis in the twentieth century*, in A. A. Bolibruch, Yu. S. Osipov, Yakov G. Sinai (Eds.): *Mathematical Events of the Twentieth Century*, Springer-Verlag, Berlin, PHASIS, Moscow, 2006, 545 pp., 437–447, 2nd edition 2010, 553 pp.

- [W] Wecken F., *Unitärinvarianten selbstadjungierter Operatoren*, *Mathematische Annalen* **116** (1939), 422–455.
- [Ze1] Zeidler E., *Applied Functional Analysis. Applications to Mathematical Physics*, *Applied Mathematical Sciences*, Vol. 108, Springer-Verlag, New York, 1995, xxix+481 pp.
- [Ze2] Zeidler E., *Applied Functional Analysis. Main Principles and Their Applications*, *Applied Mathematical Sciences*, Vol. 109, Springer-Verlag, New York, 1995, xvi+444 pp.
- [Z1] Zermelo E., *Beweis, daß jede Menge wohlgeordnet werden kann (Aus einem an Herrn Hilbert gerichteten Briefe.)*, *Mathematische Annalen* **59** (1904), 514–516, for English translation see [He], pp. 139–141.
- [Z2] Zermelo E., *Neuer Beweis für die Möglichkeit einer Wohlordnung*, *Mathematische Annalen* **65** (1908), 107–128, for English translation see [He], pp. 183–198.
- [Z3] Zermelo E., *Untersuchungen über die Grundlagen der Mengenlehre. I.*, *Mathematische Annalen* **65** (1908), 261–281, for English translation see [He], pp. 199–215.
- [Zo] Zorn M., *A remark on method in transfinite algebra*, *Bulletin of the American Mathematical Society* **41** (1935), 667–670.

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Never in the history of mathematics has a mathematical theory been the object of such vociferous vituperation as lattice theory. Dedekind, Jónsson, Kurosh, Malcev, Ore, von Neumann, Tarski, and most prominently Garrett Birkhoff have contributed a new vision of mathematics, a vision that has been cursed by a conjunction of misunderstandings, resentment, and raw prejudice.

The hostility towards lattice theory began when Dedekind published the two fundamental papers that brought the theory to life well over one hundred years ago. Kronecker in one of his letters accused Dedekind of “losing his mind in abstractions,” or something to that effect. . . .

It is a miracle that families of sets closed under unions and intersections can be characterized solely by the distributive law and by some simple identities. Jaded as we are, we tend to take Birkhoff’s discovery for granted and to forget that it was a fundamental step forward in mathematics.

G.-C. Rota ([Ro], pp. 1440, 1441)