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1. Among the numerous articles of Vojtěch Jarník on Analysis and Number Theory there is a small set of papers written in the decade from 1939 to 1949 belonging to Geometric Number Theory, consisting of the papers (65, 72, 74, 76, 77, 78)[1]. In the following the main results from these papers will be described and discussed, but before doing so some personal remarks seem appropriate.

One of my first papers deals with the product of non-homogeneous linear forms. Being a self-confident young man then, I submitted it to Acta Arithmetica. After some months I was shocked to get a big envelope from the editor and thought that the paper was refused. Hours later I dared to open it and was very pleased to see that it contained a friendly report of the referee along with a list of his suggestions for changes. They were readily inserted into the manuscript. Most probably the referee was Professor Jarník. He pointed out many minor errors and masterly corrected *my* German. Several years later I wrote a paper on successive minima. Before submitting it I consulted very carefully the Zentralblatt für Mathematik and the Mathematical Reviews and found to my considerable disappointment that one main result had been published by Jarník (76) some 20 years earlier and so my paper never was published. (Incidentally, a more general version of my second result in this paper was published later on by Barnes.) Then in November 1967 I had the privilege to meet Professor Jarník. In a lecture at the Institute of Mathematics of the University of Vienna, entitled "Über einige Ergebnisse der Gitterpunktlehre", he gave an outline of results of Břetislav Novák on the number of integer lattice points in ellipsoids. I still remember his clear and concise way of delivering the lecture, his precise use of German with a slight Czech accent and his dry humor.

[1] Parentheses refer to Jarník's articles, brackets to our list of references.

2. The Geometry of Numbers is a branch of Mathematics situated so to speak between Geometry and Number Theory. In order to give the reader a first idea of it some basic definitions and results will be formulated. This will also serve as an introduction to the work of Jarník in the Geometry of Numbers. For more information see [5, 11, 19].

A *lattice* L in the n-dimensional Euclidean space $\mathbb{R}^n$ is the set of all integer linear combinations of n linearly independent vectors. The volume of the parallelepiped generated by these vectors is the *determinant* $d(L)$ of L. A *convex body* in $\mathbb{R}^n$ is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. Now the *fundamental theorem of Minkowski* can be formulated as follows: Let K be a convex body in $\mathbb{R}^n$ which is symmetric with respect to the origin o and let L be a lattice. If the volume of K satisfies the inequality $V(K) \geqslant 2^n d(L)$, then K contains at least one pair of points $\pm l$ of L different from o.

If K is a convex body, symmetric in o, and L a lattice, define the *first successive* or *homogeneous minimum* $\lambda_1 = \lambda_1(K, L)$ of K with respect to L by

$$\lambda_1 = \inf\{\lambda > 0\colon\ \lambda K \cap L \text{ contains a point } \neq o\},$$

or, equivalently,

$$\lambda_1 = \sup\{\lambda > 0\colon\ \{\tfrac{\lambda}{2}K + l\colon\ l \in L\} \text{ is a packing}\},$$

where $\lambda K + l = \{\lambda x + l\colon\ x \in K\}$ and a *packing* in $\mathbb{R}^n$ is a family of non-overlapping convex bodies. The fundamental theorem is equivalent to the inequality

$$\text{(i)} \qquad \lambda_1^n V(K) \leqslant 2^n d(L).$$

The convex bodies K for which equality in (i) is attained are polytopes of a very special type, so-called *parallelohedra* or *space-filling polytopes.* These are polytopes P such that for a suitable lattice the system of translates of P by the lattice vectors covers $\mathbb{R}^n$ without overlappings. Parallelohedra have been investigated intensively since the times of Fedorov, Minkowski and Voronoi around the turn of the 19th century up to the present.

Because of its many applications and its intrinsic interest, the fundamental theorem has been generalized and refined in many different ways. One such refinement with important recent applications in Diophantine Approximations is based on the concept of successive minima. Let K be a convex body, symmetric in o, and L a lattice. For $i = 1, 2, \dots, n$, the *i-th successive minimum* $\lambda_i = \lambda_i(K, L)$ of K with respect to L is defined by

$$\text{(ii)} \qquad \lambda_i = \inf\{\lambda > 0\colon\ \dim((\lambda K) \cap L) \geqslant i\},$$

where dim stands for the dimension of the linear hull. Clearly $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n < +\infty$. Thus *Minkowski's theorem on successive minima*, also called the *second fundamental theorem*,

$$\text{(iii)} \qquad \lambda_1 \lambda_2 \ldots \lambda_n V(K) \leqslant 2^n d(L),$$

refines the fundamental theorem (i). Later proofs of this result are due to Davenport, Weyl, Estermann, Bambah, Woods and Zassenhaus and to Danicic, to mention some of them.

3. While working on a problem in Diophantine Approximations, Jarník saw that he could obtain an easy proof using a result of Mahler [14] on successive minima. His proof (see (65)) also produced as a by-product a so-called transference theorem in the Geometry of Numbers. The first acquaintance with successive minima seems to have aroused a great deal of interest in Jarník since all of his further contributions to the Geometry of Numbers are related to successive minima. Apart from the transference theorem in (65) and an application of the second fundamental theorem to the product of linear forms in (72), Jarník's results on successive minima are of three different types: (a) Refinements of Minkowski's theorem on successive minima depending on the shape of the convex body K are given in (76). These refinements are analogous to the refinements of the fundamental theorem (i) due to van der Corput and Davenport [4]. (b) In (78) Jarník determines the convex bodies K for which equality is attained in the second fundamental theorem. His proof is based on a careful analysis of Estermann's proof [6] of the second fundamental theorem and essentially simplifies earlier considerations of Minkowski (10, p. 235–236). (c) The remaining papers (72, 74, 77) deal with extensions of the definitions (ii) of successive minima to more general sets and with the corresponding extensions of the second fundamental theorem. (Concerning (77) I have to rely on [11] and on reviews because of my lacking knowledge of Jarník's language.)

4. Let K be a convex body, symmetric with respect to o, and let L be a lattice in $\mathbb{R}^n$. The *inhomogeneous minimum* $\mu = \mu(K, L)$ of K with respect to L is defined by

$$\mu = \inf\{\lambda > 0\colon\ \{\lambda C + l\colon\ l \in L\} \text{ covers } \mathbb{R}^n\}.$$

In (65) and (72) Jarník proved a so-called transference theorem which he formulated as the following inequality relating the successive minima and the inhomogeneous minimum of K with respect to L:

$$\tfrac{1}{2}\lambda_1 \leqslant \mu \leqslant \tfrac{1}{2}(\lambda_1 + \ldots + \lambda_n).$$

This result then is used in a proof of a result on Diophantine Approximations; see (65). For a survey on transference theorems the reader is referred to [11], Ch. 41.

5. The convex bodies K which are symmetric in o and for which equality holds in (i) are special convex polytopes. Thus (i) may be improved for non-polytopal convex bodies K. Quantitative versions of this are due to van der Corput and Davenport [4], Fejes Tóth [7] and Groemer [9]. Since the convex bodies K for which equality holds in (iii) are also polytopes (see Sect. 6 below), it seems plausible that similar refinements can be achieved for the theorem on successive minima, too.

For $n = 2$ Jarník (76) gave a refinement of (iii) for convex bodies K with $V(K) = 4$, the boundary of which has continuous curvature bounded below by some constant. The refinement in case $\lambda_1 < \lambda_2$ is even better. As a tool for his proof Jarník uses the case $n = 2$ of the following result: Let K be a convex body and L a lattice in $\mathbb{R}^n$ such that K is symmetric in o. Then there exists a polytope P which is symmetric with respect to o, contains K, has the same successive minima with respect to L as K and has only a 'small' number of facets.

6. The main result of Jarník's paper (78) reads as follows: If K is a convex body, symmetric in o, and L a lattice such that equality holds in (iii) then K is a direct sum of lower-dimensional parallelohedra. Conversely, if a convex body K is the direct sum of lower-dimensional parallelohedra, then there is a lattice L such that there is equality in (iii).

For more results on parallelohedra the reader is referred to [5, 11] and to Venkov [20] and McMullen [15].

7. The remaining papers (72, 74, 77) of Jarník and of Jarník and Knichal, respectively, in the Geometry of Numbers deal mainly with different definitions of successive minima for more general sets than convex bodies and with estimates for products of these minima.

Let M be a measurable set in $\mathbb{R}^n$ of finite positive measure $V(M)$. Its *difference set* $D(M)$ is defined by

$$D(M) = \{x - y, x, y \in M\}.$$

If M is a convex body, symmetric in o, then $D(M) = 2M$. Jarník considers the following concepts of successive minima of M with respect to a lattice L:

$$\lambda_i = \inf\{\lambda > 0\colon\ \dim((\lambda M) \cap L) \geqslant i\},$$
$$\gamma_i = \inf\Big\{\lambda > 0\colon\ \dim\Big(\Big(\bigcup_{0<\mu\leqslant\lambda} \lambda D(M)\Big) \cap L\Big) \geqslant i\Big\},$$

$$\kappa_i = \inf\{\lambda > 0\colon\ \dim((\lambda D(M)) \cap L) \geqslant i\},$$
$$\nu_i = \inf\{\lambda > 0\colon\ \dim((\mu D(M)) \cap L) \geqslant i\} \text{ for all } \mu \geqslant \lambda\},$$
$$\pi_i = \inf\Big\{\lambda > 0\colon\ \dim\Big(\Big(\bigcap_{\mu \geqslant \lambda} \mu D(M)\Big) \cap L\Big) \geqslant i\Big\}.$$

Clearly

$$\gamma_i \leqslant \kappa_i \leqslant \nu_i \leqslant \pi_i,$$

but there is no inequality relating the λ_i's to the quantities γ_1, κ_i, ν_i, π_i, unless additional assumptions on M are made. In analogy to (iii) Jarník (72) proved that

(iv) $$\kappa_1 \dots \kappa_n V(M) \leqslant 2^{n-1} d(L).$$

The constant 2^{n-1} can be replaced by $2^{n-(3/2)}$ as later on shown by Jarník and Knichal (74). Rogers [17] proved that it can be replaced by $2^{(n-1)/2}$. It is not known whether this is the best possible constant. A different refinement of (iv) with 2^{n-1} replaced by $2^{(n-1)/2}$ was proved independently by Rogers [18] and Chabauty [3]. Further inequalities for λ_i, γ_i, κ_i, μ_i, π_i were derived by Jarník (77) and Jarník and Knichal (74). For these the reader is referred to the original papers and to [11], Ch. 11, 18.

8. Jarník's results in the Geometry of Numbers are mainly concerned with the important concept of successive minima. His work exhibits a clear feeling for missing parts of the theory; he studied them and in several cases succeeded in supplementing them. On the other hand he explored systematically the unknown territory.

9. Readers who are interested in more modern developments dealing with successive minima are referred to the surveys of Kannan [13], Gritzmann and Wills [8] and Gruber [10] and to the articles of Henk [12] and Banaszczyk [1, 2] and the references cited there.

10. For their help in the preparation of this note I am obliged to Professors Groemer and Schnitzer.

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