made less than any assignable, the last proportion may be written $C-D : C'-D' :: R : R'$; whence by Euc. vi. 16,

rect. $(C, R')$ — rect. $(D, R')$ = rect. $(C', R)$ — rect. $(D', R)$.

The second rectangles on each side may become less than any assignable, and hence, by "Carnot's principle," we have *always*, whether $D$, $D'$ are large or small,

rect. $(C, R')$ = rect. $(C', R)$, and rect. $(D, R')$ = rect. $(D', R)$,

that is, *both* $C : C' :: R : R'$, *and* $D : D' :: R : R'$.

The last result was not wanted, (although it is useful to draw attention to it when $D$ is large,) and hence, as soon as we had stated the proportion as $C-D : C'-D' :: R : R'$, we might have inferred $C : C' :: R : R'$, which was all of the truth we wanted, although not the whole truth.

(iii.) Apply the reduced process to find the ratio $A : A'$ of the areas of the circles, $Q : Q'$ being that of the areas of the similar polygons, which is the same as that of sq. on $R$ : sq. on $R'$. $E$ and $E'$ being the varying differences between the invariable areas of the circles and of the variable areas of the polygons, which differences may become less than any assignable areas, the constant proportion $Q : Q'$ :: sq. on $R$ : sq. on $R'$ can be expressed as

$A-E : A'-E'$ :: sq. on $R$ : sq. on $R'$,

and hence by "Carnot's principle" we infer

$A : A'$ :: sq. on $R$ : sq. on $R'$,

and also, if required, $E : E' :: A : A'$. This solves Art. 3. v.

(iv.) It is obvious that the application of this principle in higher algebraical geometry, the differential calculus, &c., is impossible unless we assume that we can deal with incommensurable expressions by the ordinary laws of commutative algebra. I have never seen any attempt to prove the justifiability of this condition, which seems to be taken as an axiom. Yet it is evident that if the limit of a convergent series is incommensurable, we cannot, from conclusions drawn from the (commensurable) sum of any finite, or ever increasing (infinite) number of its terms, conclude an exact relation which depends solely on its limit, until we know what are the laws by which we may calculate with incommensurables. The ordinary algebraical proof that $\sqrt{2} . \sqrt{3} = \sqrt{6}$, by "squaring each side," is absurd if we do not know the *meaning* of *multiplying* $\sqrt{6}$ by $\sqrt{6}$ or of multiplying $\sqrt{2}$ by $\sqrt{3}$. To this question, then, the next Tract is devoted.

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## III. The Laws of Tensors, or the Algebra of Proportion.

12. *Proportion expressed by Tensors.*—(i.) Euc. v. and vi. are assumed. It is also assumed that no ratio is known till two straight lines have been found having that ratio. And only known ratios are here dealt with.

(ii.) $OI$, fig. 3, is a straight line continued indefinitely beyond $I$, on which

$I$ is a fixed known point, $OI$ being the standard length to which all others are referred. All other points mentioned, as $A$, $B$, $C$, &c., are supposed to lie on $OI$ and on the $I$ side of $O$. The length, position, and direction of $OI$ are quite arbitrary. But for ulterior purposes I shall assume $OI$ to be horizontal, and drawn from right to left.

(iii.) Given two points $A$ and $B$, find a third point $C$, so that $OI : OA :: OB : OC$. This is a perfectly simple known geometrical operation (art. 5. iii.), not in the slightest degree involving, but also not excluding, commensurability. As this is an operation performed on $OB$, through the instrumentality of $OA$, (it is not necessary to consider the invariable $OI$,) I designate it by the small letter $a$, of which $A$ is the capital, (a relation between forms of letters constantly observed,) and call it a *tensor*, (the name is borrowed from Sir W. R. Hamilton,) and I write $OC = a \,.\, OB$ (read $a \,.\, OB$ as "$a$ ante $OB$",) to shew that $OC$ is the *result* of performing the *operation* called the tensor $a$, upon the *operand* $OB$. Hence this equation has no other meaning than the original proportion $OI : OA :: OB : OC$.

(iv.) Since $OI : OA :: OI : OA$, we must have $OA = a \,.\, OI$, and similarly $OB = b \,.\, OI$, $OC = c \,.\, OI$, so that the fundamental equation in (iii.) may be written $c \,.\, OI = a \,.\, (b \,.\, OI)$, observing *order* of letters.

13. *Commutative Multiplication of Tensors.* (i.)—Now let $ab$ be a symbol which has the same resultant meaning as $c$, but shews that $c$ has been reached by the two operations $a$, $b$, performed in the order of the equation in (art. 12. iv.), then $c = ab$ (read $ab$ as "$a$ ante $b$"), represents the effect of that equation without the use of $OI$, and defines what may be termed *the multiplication of tensors*, the *order* of the symbols being observed. It must be remembered that no knowledge of arithmetical operations is assumed.

(ii.) But $OI : OA :: OB : OC$ gives by Euc. v. 16, $OI : OB :: OA : OC$; and hence, by precisely the same process as before, $c = ba$, and hence $ab = ba$, or the *multiplication of tensors is commutative.*

(iii.) A product of tensors represents a compound ratio; compare art. 19. iii.

14. *Division of Tensors.*—Given any two of the three points $A$, $B$, $C$, the third may be found by well-known tensor operations. Express them by writing $a = \frac{c}{b}$, $b = \frac{c}{a}$, (read $\frac{c}{b}$ as "$c$ super $b$").

Then $$\frac{c}{b} \,.\, b = ab = c, \quad a = \frac{c}{b} = \frac{ab}{b},$$
which are the two laws of *division* of tensors.

15. *Associative Multiplication of Tensors.*—(i.) Let

| | | |
|---|---|---|
| | $OI : OM :: OA : OB$ | (1), |
| and | $OI : OM :: OC : OD$ | (2), |
| whence | $OA : OB :: OC : OD$ | (3), |
| and also | $OA : OC :: OB : OD$ | (4), |
| and | $OB : OA :: OD : OC$ | (5). |

Then, by art. 14, $m = \frac{b}{a}$, $m = \frac{d}{c}$, and $\frac{b}{a} = \frac{d}{c}$ ...............(6);
and hence equation (6) becomes the tensor expression of the proportion (3); and as this involves (4) and (5), which may be similarly expressed, we find that

$$\text{if} \quad \frac{b}{a} = \frac{d}{c}, \quad \text{then} \quad \frac{a}{b} = \frac{c}{d}, \quad \frac{c}{a} = \frac{d}{b}, \quad \text{and} \quad \frac{a}{c} = \frac{b}{d} \text{.........(7)};$$

whence follow various results, and among others,

$$\text{if} \quad \frac{b}{a} = \frac{d}{c}, \quad \text{then} \quad \frac{b}{a} . c = \frac{d}{c} . c = d, \quad \text{and} \quad \frac{c}{a} . b = \frac{d}{b} . b = d,$$

$$\text{or} \qquad \frac{b}{a} . c = \frac{c}{a} . b \text{ ............................ (8)},$$

which is, in fact, the great law of *association of tensors in multiplication*, which may be put in the usual form thus:

(ii.) Given $m = ab$, find $n = bc$; then $a = \frac{m}{b}$, $c = \frac{n}{b}$, and

$$ab . c = m . c = m . \frac{n}{b} = n . \frac{m}{b} = n . a = bc . a,$$

$$\text{or} \qquad ab . c = a . bc,$$

the usual form of this law, the extreme importance of which, and its occasional independence of the law of commutation, is well shewn by the laws of quaternions.

16. *Results for Multiplication and Division of Tensors.*—(i.) The following are immediate results of these laws:

If $p$ is any tensor,

$$b . \frac{pa}{pb} = \frac{b . pa}{pb} = \frac{pb . a}{pb} = a, \quad \text{or} \quad \frac{pa}{pb} = \frac{a}{b},$$

$$\text{and} \qquad \frac{a}{b} . \frac{c}{d} = \frac{a . \frac{c}{d}}{b} = \frac{a . \frac{c}{d} d}{bd} = \frac{ac}{bd},$$

$$\text{and} \qquad \frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} . bd\right) \div \left(\frac{c}{d} . bd\right) = \frac{ad}{bc};$$

$$\text{and if} \quad \frac{a}{b} = \frac{c}{d}, \quad \text{then} \quad \frac{a}{b} . bd = \frac{c}{d} . bd, \quad \text{or} \quad ad = bc, \quad \&c.$$

(ii.) The whole of the laws of commensurable fractions, so far as multiplication and division are concerned, therefore, hold for (indifferently, commensurable or incommensurable) tensors, and each equation represents a geometrical relation between points on the line $OI$, that is, between pure lengths.

17. *Addition and Subtraction of Tensors.*—(i.) To find the point $C$ from $A$ and $B$, by the addition of $OA$ to $OB$ in either order, as

$$OA + OB = OC = OB + OA,$$

is a geometrical commutative operation. It must now be shewn (v.), that under these circumstances we can find $B$ from $A$ and $C$, or $A$ from

$B$ and $C$, so that $OA = OC - OB$, $OB = OC - OA$,
and $(OC - OB) + OB = OC$, $(OA + OB) - OB = OA$.

(ii.) Expressed by means of tensors, these equations give

$$a \,.\, OI + b \,.\, OI = c \,.\, OI = b \,.\, OI + a \,.\, OI,$$
$$a \,.\, OI = c \,.\, OI - b \,.\, OI, \quad b \,.\, OI = c \,.\, OI - a \,.\, OI,$$
$$(c \,.\, OI - b \,.\, OI) + b \,.\, OI = c \,.\, OI, \quad (a \,.\, OI + b \,.\, OI) - b \,.\, OI = a \,.\, OI.$$

(iii.) Taking then the symbol $a+b$ to mean $c$, as derived from $OC$, thus related to $OA$ and $OB$, and similarly for $c-b$, we have

$$a+b = c = b+a \quad \text{.............................}(1),$$
$$a = c-b, \quad b = c-a,$$
and
$$(c-b)+b = c, \quad (a+b)-b = a.$$

(iv.) From these definitions it follows that in $c-b$ the $c$ is always greater than $b$, that is, derived from a line $OC$ longer than $OB$. No meaning can be given to $b-c$, because $OB - OC$ has no meaning in the geometry of magnitude. The algebra of tensors has therefore, as regards subtraction, the same defect as the ordinary algebra of commensurables. The resultant difficulty of negatives and consequent imaginaries is overcome in the next section.

(v.) To form $OC - OB$, we set off a line $CA = OB$ from $C$ towards $O$, and find $A$ such that $OA = OC - OB$. Now although $CA$ could not project beyond $O$, the point $A$ might lie on $O$ itself, in which case $B$ lies on $C$, and we may shew this by writing $OO = OC - OC$, wherever $C$ may lie. Now from the point $OO$ we can form no tensor, because $OI$, $OO$ can have no ratio, as no multiple of $OO$ has any length (art. 1. ii.). But the operation indicated by $OC - OC$ is that of reducing any length to a point, forming one of its extremities; and if we represent this by $o$, we may inquire what are the laws of $o$.

(vi.) First, by the nature of $o$, whatever be $C$, $o \,.\, OC = OO$, that is, $o \,.\, (c \,.\, OI) = o \,.\, OI$, so that we may write $o \,.\, c = o$, whatever be $c$. This equation, wherever it arises, shews that $c$ is indeterminate. It also shews that if any conditions require $o \,.\, c = b$, where $b$ is not $= o$, those conditions are impossible. That is, there is no point $C$ at all which will answer the condition that, when you measure from $O$ to $C$ and then back to $O$, you should stop at a point $B$ short of $O$. This is the one geometrical impossibility of simultaneous coincidence and separation (Appendix I.). $C$ is not "*at* infinity," as usually stated. And with this the whole of the conceptions deprecated in art. 6. i. fall to the ground, to be replaced by intelligibilities. See the observations and citations in Appendix I.

(vii.) Then from the proved associative character of geometrical addition and subtraction, namely, that

$$(OA \pm OB) \pm OC = OA \pm (OB + OC),$$

taking all the upper or all the lower signs, we find for tensors

$$(a \pm b) \pm c = a \pm (b + c),$$

provided always that the subtractions be possible. And thence we obtain the further laws of $o$, $a \pm o = a$, whatever be $a$.

(viii.) Since $i$, $o$, answer to the arithmetical 1, 0, the laws of addition give $i+i$ as 2, $2+i$ as 3, and so on. But in these Tracts the geometrical operations $i$, $o$ will be kept distinct from 1, 0. Thus geometry and arithmetic will be completely separated. The figures 2, 3, &c. represent certain arithmetical operations so that $2OA = OA+OA$, $2a = a+a$, $2i = i+i$, $\frac{2}{3}a = \frac{2i}{3i}a$, &c. Again, $a^2 = a\,.\,a$, $i^2 = i\,.\,i = i$, and so on. While $o\,.\,OA = OA-OA$, $o\,.\,a = a-a$, $o\,.\,i = i-i$.

18. *Distributive Character of Tensors.*—(i.) Let $A, B, C, D, P$ be such points that $\quad pa = b,\ pc = d,$ whence $\frac{a}{c} = \frac{b}{d}$,

or $\quad OA : OC :: OB : OD$, by Art. 15. i. (6.).

Then Euc. v. 17 and 18, supposing $OA > OC$ for the lower sign,

$$(OA \pm OC) : OC :: (OB \pm OD) : OD, \quad \text{or} \quad \frac{a\pm c}{c} = \frac{b \pm d}{d},$$

whence $\quad b\pm d = \frac{d}{c}\,.\,(a\pm c),\quad$ or $\quad pa \pm pc = p\,.\,(a\pm c)$,

that is, *tensors are distributive.*

(ii.) And taking, therefore, $p\,.\,(a-a)$ to mean $pa-pa$, we have $p\,.\,o = o$, as well as $o\,.\,p = o$ (art. 17. vi.), or $o$ is commutative with any tensor.

19. *Applications of Tensors.*—(i.) This completes all the laws that need be adduced; powers with integral coefficients follow immediately, roots become intelligible as expressed either geometrically (for square roots, and for any roots by "Peaucellier's cell"), or as tensor limits to converging sums of tensors or fractions (art. 11. iv.). The whole of arithmetical algebra has been shewn to hold for tensor algebra, which also includes incommensurable algebra.

(ii.) The application to all the numerous cases for which merely quantitative geometry is used, is too evident to need explanation. And if, when $OM = AB$, we agree to represent the tensor $m$ by $AB$ in calculations, the whole of the usual algebra of quantitative geometry becomes exact, without any trace of limits. Moreover it would be possible to take proportion immediately after the theory of parallels, and prove by tensors numerous propositions which, although usually (and very properly) otherwise proved, it will be a useful exercise to prove by tensors.

(iii.) The whole of Euc. ii. admits of this treatment. The algebraical proofs of these propositions, so much eschewed, now become strictly geometrical. Thus $\quad (AC+CB)^2 = AC^2+CB^2+2AC\,.\,CB$, with the conventional notation of (ii.), is a tensor relation.

Referring each term to $OI$, it becomes a relation of lengths only, and as such should be *drawn* by the learner, that he may fully feel its meaning. Thus take $OM, ON, OP, OQ$ so that in lengths $OI : AC :: AC : OM$, $OI : CB :: CB : ON$, $OI : AC :: CB : OP$, and $OI : AB :: AB : OQ$. The meaning of the equation is that $OQ = OM + ON + 2OP$.

Referred to the square on $OI$, it becomes the relation of rectangular areas, which is figured in Euc. ii. 4; for (fig. 4.)

since $\quad OI : OM ::$ rect. $(OI, OP)$ : rect. $(OM, OP)$,

we have $\quad$ rect. $(OM, OP) = m\,.\,$rect. $(OI, OP)$.

Similarly, on taking $OJ$ on $OP$ of the length $OI$, so that

$$\text{rect.}\ (OI,\ OJ) = \text{square on } OI,$$

we have $$OJ : OP :: \text{rect.}\ (OI,\ OJ) : \text{rect.}\ (OI,\ OP),$$

so that, on using $p$ for the tensor of $OP$,

$$\text{rect.}\ (OI,\ OP) = p\,.\,\text{square on } OI,$$

whence $\text{rect.}\ (OM,\ OP) = mp\,.\,\text{square on } OI = OM\,.\,OP\,.\,\text{square on } OI$, on using the ordinary notation. The proof is here conducted by proportion only, and quite independently of commensurability, so that the objections to the "algebraical proof"—really, commensurable proof—of Euc. ii. no longer hold.

Referred to cube on $OI$, it becomes a relation of rectangular solids, having one constant dimension $OI$.

(iv.) The demonstration of all these relations flows at once from the laws of tensors. But there is no room for negatives or imaginaries in an algebra derived from the geometry of magnitude only. The laws of both are obtained at once from the geometry of direction, as follows.

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## IV. THE LAWS OF CLINANTS, OR THE ALGEBRA OF SIMILAR TRIANGLES LYING ON THE SAME PLANE.

20. *Data from the Geometry of Direction.*—(i.) The following propositions are borrowed from the geometry of *direction*, as opposed to that of *ratio*, or magnitude only. See fig. 5.

(ii.) $AB$, without further limitation, always represents the line $AB$, as respects both magnitude and direction, considered as the trace of the motion of a point along a straight line *from $A$ to $B$*, $A$ being its *initial* and $B$ its *final* point. When length is considered without direction, write len $AB$, and read "length of $AB$."

(iii.) The equation $AB = CD$ implies that $AB$ and $CD$ are the opposite sides of the parallelogram $ABDCA$, the points lying in this order.

(iv.) *Directional Addition* is defined by the equation

$$AB + AC = AB + BD = AD,$$

or the sum of two adjacent sides of a parallelogram measured from their point of intersection is the included diagonal. And since in this case

$$AC + AB = AC + CD = AD,$$

*directional* addition (which is quite different in its results from quantitative or *rational* addition) is commutative. It is also easily shewn to be associative.

(v.) *Directional Subtraction* is defined by the equation

$$AB - AC = AB + CA = AB + BE = AE = CB,$$

or the directional difference of two adjacent sides of a parallelogram is the transverse (non-included) diagonal, measured from the final extremity of the subtrahend to that of the minuend.

(vi.) $IOI'$, $JOJ'$ (fig. 6 and 8) are diameters of a *standard* (or unit) circle, drawn at right angles. The position of centre $O$ (taken as *origin*) and radius $OI$, (taken as the *standard* of *length* and of *original* direction,) and the direction of right angle $IOJ$, (taken as standard of angular rotation,) are arbitrary, but once fixed remain throughout the problem, and deter-