BOLZANO'S "FUNCTIONENLEHRE"

The publication of Bolzano's writings by the Royal Bohemian Learned Society started with Volume I including a not yet published (and, until recently, completely unknown) Bolzano's work "Functionenlehre"[1]). It is such an extraordinary work that we cannot but regret that, as an unpublished manuscript, it had not the opportunity to influence the development of mathematics in his own time. In Bolzano's days (1781–1848) the theory of functions was already considerably developed, its main concepts, however, lacked sharp contours and the principal theorems were not upheld by exact proofs. And it is in the very foundations of the theory of functions that Bolzano's "Functionenlehre" represents a virtual milestone, unfortunately a milestone overgrown with the moss of ignorance. Among Bolzano's contemporaries, only Gauss, Abel and Cauchy manifested the same sense for the proper construction of the foundations of the theory of functions. Two of them, Gauss and Abel, presented masterpieces of exact mathematical methods, but did not deal with these fundamental problems systematically. The last of them, Cauchy, in his works "Cours d'Analyse" (1821), "Résumé des leçons ... sur le Calcul Infinitésimal" (1823), "Leçons sur le Calcul différentiel" (1829) based the main branches of the theory of functions (algebraical analysis, differential and integral calculus) on firm foundations (or let us say more carefully on firmer foundations) in a systematic way[2]). However, Bolzano goes in his efforts even beyond Cauchy's achievements[3]). Cauchy usually contented himself with building the foundations to a level necessary for his further deductions; unlike him, Bolzano was more of a philosopher, interested particularly in the fundamental problems of mathematics[4]). We shall see later how rigorously Bolzano introduces his definitions, how critically

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1) Functionenlehre, published 1930 by the Royal Bohemian Learned Society, printed by the Society of Czechoslovak Mathematicians and Physicists in Prague; pp. XX + 183 + + 24 + IV. Including a foreword by Prof. K. Petr and comments by Prof. K. Rychlík.

2) Bolzano knew at least some of these works; the origin (or at least the completion) of Bolzano's manuscript is to be dated not sooner than 1831 (in Sec. II, § 90 the author quotes a book by Young from 1831).

3) Actually, Cauchy himself made some errors; some of them will be mentioned later in more detail.

4) See e.g. in Petr's foreword (p. IX) an eloquent quotation from Bolzano's "Beiträge zu einer begründeteren Darstellung der Mathematik, 1. Lieferung" (1810): Seit etwa

he dissects his concepts, with what deep interest and thoroughness he discusses all logically possible cases regardless of their greater or lesser importance for concrete mathematical problems.

It was this philosophical interest that had predestined Bolzano to outstanding achievements in the sphere of foundations of mathematical analysis. And his "Functionenlehre" is indeed a really pioneering work, a work so modern that we have to wait several decades for other works of analogous character. On the other hand, Bolzano's philosophical background as well as the pioneering character of his investigation burdened his book unfavourably in several directions. First of all, Bolzano was no skilled mathematician, therefore the considerations requiring only "pure thinking" are in most cases more perfectly executed than those which require more "craftsmanship". Secondly, due to the novelty of the problems considered no suitable terminology at all was available to Bolzano[5]); he had to write up every detail in extenso and no wonder that sometimes, wading through the deluge of words, he slipped and made an error in his reasoning. There was one more circumstance that interfered with Bolzano's work. In his time, only the greatest mathematical minds could have understood his pioneering work; what understanding could Bolzano expect in the Austrian monarchy of that time where mediocrity, uniformity and reactionary conservatism were raised by the State to the status of the highest civil virtues? Alone, surrounded by total lack of interest and understanding, Bolzano performed his work; certainly many a defect of his work would have disappeared had he had the opportunity to discuss his problems with a congenial mind.

In this paper my desire is to call the reader's attention to the problems and methods of this work of Bolzano; the reader will see how many concepts, problems and methods that now belong to the most essential parts of mathematical analysis, appear in their definitive form in Bolzano. Of course, my intention is rather to offer some comments than to give a definitive evaluation and criticism of Bolzano's work; the latter has to be left to a historian of mathematics.

Bolzano's "Functionenlehre" which was intended as part of a more extensive work on mathematics[6]) consists of an introduction (Einleitung: Verhältnisse zwischen veränderlichen

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15 Jahren ... ist diese Wissenschaft (Mathematik) immer eines von meinen Lieblingsstudien gewesen; doch vornähmlich nur nach ihrem speculativen Teile, als Zweig der Philosophie und Uibungsmittel im richtigen Denken. (For some 15 years this science (Mathematics) has been my favourite subject; but above all its speculative part, as a branch of philosophy and exercise in correct thinking.)

[5]) Altogether, reading Bolzano's book may create an impression (in a mathematician of the *present* period) that Bolzano was no adroit stylist (in science); of course only a historian could draw the right conclusions in this matter.

[6]) The introduction to "Functionenlehre" appears in the manuscript under the heading "Fünftes Hauptstück"; manuscripts of some other parts have survived, too.

Zahlen – Introduction: Relations between variable numbers – pp. 1–12) and two main sections (Erster Abschnitt: Stetige und unstetige Functionen – First Section: Continuous and discontinuous functions – pp. 13–79; Zweiter Abschnitt: Abgeleitete Functionen – Second Section: Derivatives – pp. 80–183). The introduction includes several more or less formal considerations, and I will not deal with it in more detail. On the other hand, I will discuss both the main sections more thoroughly.[7])

Erster Abschnitt
Stetige und unstetige Functionen

(First Section. Continuous and Discontinuous Functions)

At the very beginning of my review I face a certain difficulty. Bolzano is very particular about definitions; nevertheless, the definition of the concept of a function is missing – from the very beginning of the introduction he freely uses the word "function" as a current term, without giving any definition. Taking into account Bolzano's carefulness in introducing new concepts I should believe that Bolzano might have defined the term in another – perhaps unknown – part of his manuscript. However, Prof. Rychlík has not been able to provide any information on this matter. Bolzano also frequently uses the words "einförmige Function"; Rychlík interprets the word "einförmig" as "univalent" or "one-valued" and many parts of Bolzano's work seem to imply that Bolzano applied the words "einförmige

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[7]) In order not to tire the reader by lengthy and often peculiar formulations of Bolzano, I shall "translate" his statements into modern language. For example, in Sec. I, § 20 Bolzano says: "Wenn die unendlich vielen Werthe, die eine Function $F(x)$ annimmt, indem ihre Veränderliche x alle von $x = a$ bis $x = b$ einschliesslich vorkommenden Werthe erhält, von solcher Beschaffenheit sind, dass sich zu jeder messbaren Zahl irgend einer aus ihnen ausfinden lässt, der diese Zahl übertrifft, so ist diese Function gewiss nicht für alle Werthe von $x = a$ bis $x = b$ einschliesslich stetig." This statement will be formulated briefly as follows: "a function unbounded in an interval $[a, b]$ cannot be continuous in $[a, b]$" or even better equivalently as "every function continuous in $[a, b]$ is bounded in $[a, b]$". (Moreover, it is apparent from the proof that Bolzano intended to say "die der absolute Betrag einer Function $F(x)$ annimmt" instead of "die eine Function $F(x)$ annimmt"; such evident slips are often corrected by me without further notice in the sequel.)

Function" in the present sense of the word "function" (or nearly in this sense). Therefore I shall translate his term "einförmige Function" simply by the word "function" and I shall understand a function (e.g. a function of one variable) in the present current sense: The variable y is a function of the variable x defined in a domain M if every value x from the set M is assigned a certain (single) value of the variable y.[8]) Sometimes (mainly in Sec. II in the definition of derivative) Bolzano omits the adjective "einförmig"; however, it does not seem that this should express anything particular; only in §§ 47 – 48 (Sec. I) is this matter doubtful and causes certain difficulties (see my comments on these §§ below).

After an introductory § 1, in § 2 Bolzano defines continuity of a function of one variable in the same way as we do it now: a function $f(x)$ is continuous at a point a if for every $\varepsilon > 0$ there is a number $\eta > 0$ such that for all x satisfying $|x - a| < \eta$ we have $|f(x) - f(a)| < \varepsilon$.[9]). Bolzano's concept of continuity is thus quite modern; probably only Cauchy preceded him in defining the continuity of a function in a similar way; however, Cauchy does not reach the concept of a function continuous at a *point* but stops at the concept of continuity in an *interval* (his concept of a function "continuous in a neighbourhood of a point a" means a function continuous in an interval $(a - \varepsilon, a + \varepsilon)$, $\varepsilon > 0$).[10]) Bolzano's approach based on continuity at a point means a considerable step forward. Moreover, Bolzano develops his concept even further by introducing immediately (in the now current way) continuity from the right and from the left.[11])

If this definition rouses our admiration, then even more admiration is deserved by the next §§ 3 – 33, which include an excellently built theory of the continuous functions of one variable. After § 3, which offers an interesting criticism of former definitions of continuity, a proof of the continuity of the simplest functions (up to rational functions) follows in §§ 4 – 8. The next §§ 9 – 11 show the thoroughness of Bolzano's study of all possibilities which may occur with regard to continuity and discontinuity of functions; thus he constructs

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[8]) The whole book deals with functions of a real variable. Of course, I have no intention to claim that Bolzano would have been able to present that definition in such a perfect form as we do today.

[9]) Note that Bolzano speaks about continuity only if the function $f(x)$ is defined at the point a and in its neighbourhood.

[10]) Cauchy, Cours d'Analyse (1821), I[ère] partie, chap. II, § 2. It should be noticed that already in 1817 in his paper "Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine Wurzel der Gleichung liegt" Bolzano introduces the notion of continuity similarly to Cauchy (i.e., continuity in an interval).

[11]) Here we can see the difficulties Bolzano had to face in his formulations: he spent 23 lines for his definition of continuity.

a function defined for x and continuous at precisely one point in § 10. An interesting § 12 gives evidence of Bolzano's acute and critical mind:

Let $f(x)$ be a function continuous from the right for all x. Let us choose a number a; then[12])

$$(1)\quad \lim_{h\to 0+} (f(a + h) - f(a)) = 0\,.$$

Similarly,

$$(2)\quad \lim_{h\to 0+} (f(a - k + h) - f(a - k)) = 0$$

for an arbitrary k. If we might put $k = h$ in the last identity, then we should obtain

$$\lim_{h\to 0+} (f(a) - f(a - h)) = 0\,,$$

i.e. we should obtain that the function $f(x)$ is continuous from the left at the point a. Bolzano warns explicitly against such an incorrect reasoning; as the reason of inadmissibility of such an argument he quite rightly gives the fact that k in (2) has to be viewed as a fixed number; if we want to satisfy the inequality $|f(a - k + h) - f(a - k)| < \varepsilon$ with a positive ε we have just to choose $0 < h < \eta$, where the positive number η, however, may depend not only on ε *but on k as well.* In modern terms: the incorrectness of the argument is due to the fact that the continuity from the right need not be "uniform". Bolzano presents an example ($f(x) = x^2$ for $x < 2$, $f(x) = x^3$ for $x \geqq 2$) which makes the incorrectness of the above argument apparent.

The next § 13 concerns again the notion of uniformity; Bolzano presents the following example: The function $1/(1 - x)$ is continuous in the open interval $(0, 1)$.[13]) Nevertheless, given a positive real ε, then for any positive δ it is possible to find a number x_0 $(0 < x_0 < 1)$ such that in order to have

$$\left|\frac{1}{1 - x} - \frac{1}{1 - x_0}\right| < \varepsilon \quad (0 < x < 1)$$

it is *necessary* to take $|x - x_0| < \delta$. Let us notice here that the property formulated by Bolzano (both in a general theorem and in the example just mentioned) is no logical negation of uniform continuity (in the present sense); thus the theorem from § 13 is not equivalent to the theorem "a function continuous in (a, b) need not be uniformly continuous in (a, b)";

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[12]) $\lim_{x\to a+} g(x)$ is the limit of a function $g(x)$ at a point a from the right; the analogous symbol with $a-$ is used for the limit from the left.

[13]) The following notation is used: the set of all x satisfying $a < x < b$ or $a \leqq x \leqq b$ is denoted by (a, b) or $[a, b]$ and called an open or closed interval, respectively.

nonetheless it is very close to it (in fact, it contains the latter theorem). It is clear that Bolzano realized the importance of a concept with features similar to those of today's "uniformity" (uniform continuity, uniform convergence, etc.); however, he did not succeed in his attempts to formulate this notion properly. Consequently — as we shall see later — he often neglects uniformity altogether or, in other cases, works with a concept with similar features but not suitably introduced. This causes some mistakes and defects in Bolzano's works; we shall later pay attention to some of them in more detail.

Paragraphs 14—16 deal with the so called "indefinite expressions" on the basis of the following theorem: If $f(x)$ is continuous in (a, b) and if two numbers m, M $(a < m < b)$ have the property that for any pair of numbers $\varepsilon > 0$, $\eta > 0$ there is a number x such that $|x - m| < \eta$, $|f(x) - M| < \varepsilon$, then $f(m) = M$. § 17 contains the proof of the following theorem: If $f(x)$ is continuous in (a, b) and if for a certain value M the roots of the equation $f(x) = M$ are everywhere dense in the interval (a, b) then $f(x) = M$ identically in (a, b). § 18 presents a conversion of the theorem from § 17; here, however, Bolzano commits his *first substantial error*. Namely, Bolzano argues as follows: "Thus, if conversely $f(x)$ is continuous in (a, b) and if $f(x)$ depends on x (which should probably mean that $f(x)$ is not constant) and if M is a number, then the equation $f(x) = M$ can have infinitely many roots in (a, b) but there exists a subinterval in (a, b) in which the equation has no root. In other words, *for every root of the equation $f(x) = M$ we can find a root closest to it*". The last sentence is obviously wrong even if we make the assumption (probably made by Bolzano, even if not quite consciously) that $f(x)$ is not constant in any subinterval. Indeed, under this assumption, in any subinterval it is possible to find a point α such that $f(\alpha) \neq M$; if the equation $f(x) = M$ has at least one root to the left as well as to the right from α, then it is possible to find an interval (y_1, y_2) containing the point α and such that $f(x) \neq M$ for $y_1 < x < y_2$ but $f(y_1) = M$, $f(y_2) = M$; then y_1, y_2 really represent two "adjacent" roots of the equation $f(x) = M$; however — and this is the error in Bolzano's argument — we need not obtain all roots of the equation $f(x) = M$ in this way (even if we vary the number α). This error is the more embarrassing in view of the fact that Bolzano constructs (in § 70) a function that after a slight modification would have provided an example disproving his own assertion (and even this modification is carried out in Sec. II, § 26).

On the other hand, this error shows how difficult, formidable and pioneering was the task that Bolzano undertook. Only the immense difficulty, total novelty and strangeness of the subject can account for the fact that Bolzano, who in the next paragraphs demonstrated his utmost precision and perfection in the most delicate considerations, was able to commit such a blunder in § 18. Unfortunately, Bolzano did not avoid the harmful effect of § 18 upon some of his later considerations, and the application of the incorrect theorem from § 18 affected especially some parts of Bolzano's study of extrema.

Nonetheless, the next §§ 29—30 deserve our unconditional admiration and respect. Here Bolzano proves in detail and without the least gap the basic theorems on continuous functions. The proofs are fully "arithmetized", they proceed purely deductively starting from the definition of continuity, without any reference to intuition (anyhow, this is a characteristic feature of the whole book). Bolzano proves the following theorems:

I. If

$$\limsup_{x \to c} |f(x)| = \infty$$

then the function $f(x)$ is not continuous at the point c.

II. A function continuous in a closed interval $[a, b]$ is bounded in it.

III. If a function $f(x)$ is continuous in $[a, b]$ and if there exists a sequence $x_1, x_2, x_3, \ldots$ $\ldots (a \leqq x_n \leqq b)$ such that

$$\lim_{n \to \infty} f(x_n) = C$$

then the function $f(x)$ assumes the value C in the interval $[a, b]$.

IV. If a function $f(x)$ is continuous in $[a, b]$ then this function assumes in $[a, b]$ both its greatest and its least value.

V. Let $f(x)$ be continuous in (a, b); let $a < x_1 < x_2 < b$, $f(x_1) \neq f(x_2)$; then in the interval (x_1, x_2) the function $f(x)$ assumes all values between $f(x_1)$ and $f(x_2)$.[14]

Bolzano not only proves these theorems but even critically examines the meaning of the assumptions made; thus he shows that Theorems III and IV are invalid if the closed interval $[a, b]$ is replaced by the open one (a, b). The proofs of these theorems are carried out with such a brilliance that they might be used, except for certain imperfections in verbal formulations, without the least change in any modern textbook of calculus. Just one remark should be made: according to the present terminology, continuity in a closed interval $[a, b]$ means continuity at each inner point, together with continuity from the right and from the left at the points a and b, respectively. Bolzano expresses this property by the construction "von $x = a$ bis $x = b$ einschliesslich stetig". This seems to indicate that Bolzano requires "both-sided" continuity at the endpoints of the interval, while one-sided continuity is sufficient. It is of course possible that Bolzano was aware of this fact and that it was due only to his difficulties in formulating his ideas if he was not quite explicit about it.[15]

The proofs are based mainly on two theorems whose formulation is also due to Bolzano. The first, the so-called *theorem on the least upper bound*, is formulated by Bolzano as follows: If not all real numbers possess a property B but all reals less than a certain number U do, then there exists a number A which is the greatest of all numbers x with the following property: all numbers less than x have the property B.[16]

The other is the so-called *Bolzano-Weierstrass theorem*: Every bounded sequence has at least one point of accumulation.[17]

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[14]) The last theorem had been proved by Bolzano already in "Rein analytischer Beweis ..." (1817). It was proved also by Cauchy in Cours d'Analyse, Note III$^{\text{ème}}$ (1821).

In the next paragraphs Bolzano deals with composite functions; in § 31 he presents the theorem on the continuity of composite functions $f(\varphi(x))$ and formulates it even for one-sided continuity. Of course, when studying functions continuous only from one side we have to take into account very complex situations which may occur; it is not clear whether Bolzano was aware of all these possibilities. Nonetheless, both the theorem and its proof are correct, provided we show just a little indulgence in judging Bolzano's rather intricate formulations.

In §§ 34–36 Bolzano deals with the simplest theorems on the continuity of functions of several variables; however, this part of his work is based on an incorrect theorem. In fact, in § 38 Bolzano defines the continuity of a function $f(x, y)$[18]) as follows: $f(x, y)$ is continuous at a point (x_0, y_0) if there exists a positive number δ such that (i) $f(x, y_0 + k)$ is a continuous function of the variable x at the point x_0 provided $|k| < \delta$; (ii) $f(x_0 + h, y)$ is a continuous function of the variable y at the point y_0 provided $|h| < \delta$. In § 39 he attempts to prove: If $f(x, y)$ is continuous at a point (x_0, y_0), then

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x, y) = f(x_0, y_0) . \tag{3}$$

This theorem is obviously invalid: there exist functions that are continuous both in x and

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[15]) Some later considerations seem to support this opinion; e.g. in § 70 Bolzano constructs a function defined only in the interval $[0, 1]$ (so that we can speak at most about *one-sided* continuity at the endpoints); nonetheless, he states that this function is "von $x = 0$ bis $x = 1$ einschliesslich stetig" (continuous from $x = 0$ to $x = 1$ inclusively). § 75 contains the same statement about the so-called "Bolzano's function". Similarly in some other places.

[16]) This theorem had been presented by Bolzano already in his (unpublished) manuscript "Zahlenlehre" and in the paper "Rein analytischer Beweis ..." (1817). In this paper Bolzano proves the theorem by reducing it to a certain condition (the so-called Bolzano-Cauchy condition) for the convergence of a sequence; of course, Bolzano's attempt to prove the Bolzano-Cauchy condition (in the same paper) failed due to the non--availability of a theory of real numbers, which was created only several decades later. Today we prefer to proceed in the opposite direction: first we prove the theorem on the least upper bound and only then the Bolzano-Cauchy condition.

[17]) When using this theorem Bolzano refers to his (unpublished) manuscript "Lehre von der Messbarkeit der Zahlen". However, neither Jašek nor Rychlík have found it there. Rychlík observes an interesting detail, that this theorem has not been found even in the published works of Bolzano though it has been known under the name of "Bolzano-Weierstrass theorem".

[18]) Bolzano does it for functions of n variables and deals with one-sided continuity as well.

in y but do not satisfy the identity (3). Bolzano's error is quite easily discovered: it is

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) =$$
$$= (f(x_0 + h, y_0) - f(x_0, y_0)) + (f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)).$$

By our assumptions, the first bracket on the right hand side approaches zero with $h \to 0$; the second bracket approaches zero as well provided $k \to 0$ with h fixed. Bolzano overlooked the fact that there is no guarantee for the second bracket to approach zero when both h and k tend to zero *simultaneously*. This is caused by the fact that the continuity of the function $f(x, y)$ with respect to the variable y need not be uniform with respect to the variable x. It is rather surprising that Bolzano should commit here the same error against which he warned so pronouncedly in § 12. This is the *second serious mistake* of Bolzano; the same erroneous reasoning occurs in Cauchy[19]) and it is quite possible that Bolzano took it from him without profound analysis.

Paragraphs 47–48 are very interesting. Already in § 29 Bolzano showed that every function $f(x)$ continuous in (a, b) has the following property:

Property (A). If $a < x_1 < x_2 < b$, $f(x_1) \neq f(x_2)$, then the function $f(x)$ assumes in the interval (x_1, x_2) all values between $f(x_1)$ and $f(x_2)$.

Now Bolzano propounds the following question: Can it be true that the property (A) is characteristic of continuous functions? He answers the question quite correctly in the negative: there even exist functions which have the property (A) but are discontinuous at every point. This is certainly a wonderful result for his time if we realize that even Lebesgue[20]) complains that in some French schools, continuity is introduced via the property (A). Moreover, Bolzano attempts to prove the assertion; it is, however, difficult to judge this attempt correctly without precise knowledge of Bolzano's definition of a function.[21])

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19) Cours d'Analyse, Ière partie, chap. II, § 2. The present definition of continuity of functions of several variables originated probably only in the time of Weierstrass, about 1870.

20) Lebesgue, Leçons sur l'intégration (1904), p. 89.

21) In fact, if we introduce the notion of a function in the sense current at present (each value x from a certain domain is assigned one and *only one* value of y) then we are forced to regard Bolzano's proof as a total failure. However, in these §§ 47–48 Bolzano *does not attach* the adjective "einförmig" to the noun "Function"; hence it is quite possible that Bolzano had in mind some "multivalued" functions (cf. my comments on Bolzano's concept of a function); then perhaps it would be possible (after extending sufficiently the concept of a function) to consider Bolzano's proof correct (at least partly); nevertheless, the theorem itself would lose much of its interest. Thus §§ 47–48 either contain a very interesting theorem with an incorrect proof or a less interesting theorem with a (at least partly) correct proof.

In §§ 49—59 Bolzano develops a very interesting theory of monotone functions. I take the liberty of briefly sketching the contents of these paragraphs in order to illustrate the exactness and precision of Bolzano's arguments.

§ 49. There exist functions $f(x)$ that have the following property for all x (or at least for all x from a certain open interval): if $x_1 < x_2$ then $f(x_1) < f(x_2)$. Proof: The function cx $(c > 0)$ has this property as $x_1 < x_2$ implies $cx_1 < cx_2$. Analogously, if $c < 0$, then we obtain a function for which $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

§ 50. Such functions will be called increasing (either everywhere increasing or increasing in the interval (a, b)) or, respectively, decreasing.

§ 51. If $f(x)$ is not increasing in (a, b) then there necessarily exist numbers μ, ϱ such that $a \leqq \mu < \varrho \leqq b$, $f(\mu) \geqq f(\varrho)$; several other remarks of similar character.

§ 52. The sign of $f(x + h) - f(x)$ is the same as that of h for an increasing function; opposite for a decreasing function.

§ 53. An increasing (decreasing) function assumes any value at most once (with a detailed proof).

§ 54. For an increasing function, the inequality $f(\varrho) - f(\mu)$ implies the inequality $\varrho > \mu$; the latter inequality is converse for a decreasing function (with a proof).

§ 55. If a function $f(x)$ has the property (A) in an interval (a, b) and if there exist three numbers ε, μ, ϱ $(a < \varepsilon < \mu < \varrho < b)$ such that neither $f(\varepsilon) < f(\mu) < f(\varrho)$ nor $f(\varepsilon) > f(\mu) > f(\varrho)$, then $f(x)$ assumes some of its values at least twice. Proof: If two of the numbers $f(\varepsilon)$, $f(\mu)$, $f(\varrho)$ coincide then the assertion is obvious. Thus we have to discuss two cases: (i) $f(\varepsilon) < f(\mu) > f(\varrho)$, (ii) $f(\varepsilon) > f(\mu) < f(\varrho)$. If (i) holds then either (α) $f(\varepsilon) >$ $< f(\varrho) < f(\mu)$ or (β) $f(\varrho) < f(\varepsilon) < f(\mu)$. If (α) then $f(\varrho)$ is between $f(\varepsilon)$ and $f(\mu)$ and hence there exists (by the property (A)) a number λ $(\varepsilon < \lambda < \mu)$ such that $f(\lambda) = f(\varrho)$; as $\lambda \neq \varrho$, the assertion is proved. The other cases are discussed analogously.

§ 56. Consequently, if a function $f(x)$ has the property (A) in an interval (a, b) and assumes there each value at most once, then $a < \varepsilon < \mu < \varrho < b$ implies either $f(\varepsilon) <$ $< f(\mu) < f(\varrho)$ or $f(\varepsilon) > f(\mu) > f(\varrho)$.[22])

§ 57. Analogously for more numbers $\varepsilon < \mu < \varrho < \lambda < \nu < \dots$

§ 58. If a function $f(x)$ has the property (A) in an interval (a, b) and assumes there each value at most once, then it is either increasing or decreasing in (a, b). Proof: Let us choose two numbers μ, ϱ $(a < \mu < \varrho < b)$; certainly $f(\mu) \neq f(\varrho)$. Let us deal e.g. with the

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[22]) Note that, till now, we have proved only that each such triplet of numbers ε, μ, ϱ satisfies *one* of the relations $f(\varepsilon) < f(\mu) < f(\varrho)$, $f(\varepsilon) > f(\mu) > f(\varrho)$; however, we have not yet proved that all such triplets satisfy *the same* of the two relations. If the latter assertion were already proved then $f(x)$ would be obviously either increasing or decreasing in (a, b). This is in fact true, but it will be proved only in § 58.

case $f(\mu) < f(\varrho)$; I assert that in this case the function $f(x)$ is increasing in (a, b). This means: let $a < x_1 < x_2 < b$; we have to prove $f(x_1) < f(x_2)$. The discussion depends on the mutual position of the points x_1, x_2, μ, ϱ; Bolzano carries out this discussion in full detail.[23]) E.g., if $\mu < x_1 < \varrho < x_2$ then by § 57 necessarily either $f(\mu) < f(x_1) < < f(\varrho) < f(x_2)$ or $f(\mu) > f(x_1) > f(\varrho) > f(x_2)$; the latter case is excluded since $f(\mu) < < f(\varrho)$; thus the former inequalities hold and hence $f(x_1) < f(x_2)$.

§ 59. If $f(x)$ is increasing (or decreasing) in an interval (a, b) and has the property (A) in this interval, then $f(x)$ is continuous in (a, b). A wonderful (for Bolzano's time) supplement to §§ 47–48! Proof: Let $f(x)$ be, say, increasing and let us prove continuity from the right at a point x_0 $(a < x_0 < b)$. Let us choose h_0 so that $x_0 < x_0 + h_0 < b$; then $f(x_0 + h_0) - f(x_0) = D > 0$. Given $\varepsilon > 0$, let us choose μ $(0 < \mu < 1)$ such that $\mu D < \varepsilon$. Then $f(x_0) < f(x_0) + \mu D < f(x_0 + h_0)$ and hence by the property (A) there exists h_1 $(0 < h_1 < h_0)$ such that $f(x_0 + h_1) = f(x_0) + \mu D$; hence $0 < f(x_0 + h_1) - f(x_0) < \varepsilon$ and (since $f(x)$ increases), a fortiori $0 < f(x_0 + h) - f(x_0) < \varepsilon$ for $0 < h < h_1$. An admirable passage! Bolzano's reasoning in §§ 49–59 is really flawless.

After discussing monotone functions Bolzano passes to the theory of relative extrema. Both subjects are connected by the following theorem (§ 61): If $a < b < c$ and if a function $f(x)$ is (i) increasing in (a, b); (ii) decreasing in (b, c); (iii) continuous at the point b[24]), then there exists $\delta > 0$ such that $f(b) > f(b + h)$ for $0 < |h| < \delta$. The definition of relative extrema follows in § 62: A function $f(x)$ has a relative maximum at a point b if there exists a number $\delta > 0$ such that $f(x) < f(b)$ for $0 < |x - b| < \delta$; analogously for a relative minimum. Bolzano introduces also one-sided extrema. E.g. a relative maximum from the right occurs at b according to his definition if first $f(b + h) < f(b)$ for sufficiently small positive h and secondly $f(b - h)$ either does not exist or equals $f(b)$ for sufficiently small positive h. It was Bolzano's erroneous idea on the distribution of roots of the equation $f(x) = M$, based on the incorrect § 18, that led him to this – not very suitable – definition of one-sided extrema. The effect of § 18 upon the further development of the theory of maxima and minima manifests itself in several incorrect theorems and proofs. Nonetheless, the paragraphs devoted to these problems include a number of interesting points; I shall mention some of them in more detail. First of all, Bolzano deals rather extensively with functions that exhibit "infinitely many oscillations" (§§ 65–75). Thus in § 65 he demonstrates (by constructing a suitable example) that it is possible to construct a function $f(x)$ continuous in (a, b) which possesses the following property: there are two numbers m, M

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[23]) The discussion contains some minor incidental omissions.

[24]) Bolzano says "um den Wert b herum stetig" (continuous about the point b); actually in the proof he uses only the continuity at the point b, not in a *neighbourhood* of the point b. The proof, even if essentially correct, is not quite perfect in details.

and a sequence $x_1, x_2, x_3, \ldots$ $(a < x_1 < x_2 < x_3 < \ldots < b)$ such that

(4) $m < M$[25]$, \quad f(x_{2n-1}) \geqq M, \quad f(x_{2n}) \leqq m.$

In § 69 Bolzano shows that functions $f(x)$ with these properties cannot be continuous in the closed interval $[a, b]$. At the same time the example in § 65 provides an example of a function continuous in (a, b) which has the following property: there is a number μ (say, $\mu = (m + M)/2$) and a sequence $x_1, x_2, x_3, \ldots$ $(a < x_1 < x_2 < x_3 < \ldots < b)$ such that

(5) $f(x_{2n-1}) > \mu, \quad f(x_{2n}) < \mu.$

It is evident that the property (5) says *less* than the property (4) and in § 70 Bolzano shows, in contradistinction to the property (A), that even a function continuous in a *closed* interval $[a, b]$ can possess the property (5). This difference between the properties (4) and (5) consists, as Bolzano mentions correctly, in the fact that – roughly speaking – the inequalities (5) admit the existence of the limit $\lim_{n\to\infty} f(x_n)$ while the inequalities (4) exclude it. Bolzano presents some more comments of similar character; it would just distract the reader if we mentioned all of them. However, it should be noticed that till now I have neglected § 63, § 64 and § 68. Of these, § 63 and § 68 contain wrong results based on the incorrect § 18. In § 64 Bolzano proves that a function continuous in $[a, b]$ and increasing in (a, b) satisfies $f(a) < f(c) < f(b)$ for $a < c < b$. To prove this assertion, Bolzano proceeds as follows: Let us choose h such that $a < a + h < c$; then $f(a + h) < f(c)$, $\lim_{h\to 0+} f(a + h) = f(a)$; this is claimed to imply $f(a) < f(c)$. For a general function we might of course conclude merely $f(a) \leqq f(c)$; for an increasing function the conclusion is true. However, Bolzano, when passing to the limit, does not point out that the function $f(x)$ is monotone – this seems to indicate that he made a mistake at this point. I ought to complete my comments by observing that the formulations in §§ 64–74 are not always perfect; apparently these are no mistakes or oversights but mere stylistic difficulties.

It is already in these paragraphs (§§ 65, 70, 73) that Bolzano constructs several remarkable examples of functions whose (bounded) definition domain cannot be divided into a finite number of intervals of monotonicity; however, the most magnificent example of this kind is given in § 75. Indeed, Bolzano constructs here a function $f(x)$ continuous in $[a, b]$ which is monotone in no subinterval. Later (Sec. II, § 19), Bolzano shows that the points at which $f(x)$ has no derivative are everywhere dense in the interval $[a, b]$.[26] Even

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[25] Bolzano omitted this condition though he evidently had it in mind.

[26] Today we know that this function has a derivative at no point at all; see M. Jašek, On Bolzano's function (Czech), Časopis pěst. mat. 51 (1922), pp. 69–76; M. Jašek, Aus dem handschriftlichen Nachlass B. Bolzanos, Věstník Král. české spol. nauk 1920–21, Vol. I, pp. 1–32; K. Rychlík, Über eine Funktion aus Bolzanos handschrift-

the fact that it occurred to Bolzano at all that such a function might exist deserves our respect; the fact that he succeeded in actually constructing such a function is even more admirable. This function is the famous "Bolzano's function". It is well known from literature so that a brief exposition is sufficient.

Bolzano defines in an interval $[a, b]$ a sequence of continuous functions

$$(6)\quad f_1(x), f_2(x), f_3(x), \ldots;$$

here the curve $y = f_n(x)$ is a polygonal line consisting of a finite number of line segments (with this number tending to infinity together with n). The points at which the function $f_n(x)$ changes its character from an increasing to a decreasing function or vice versa (this can apparently occur solely at the vertices of the polygonal line) are distributed in the interval $[a, b]$ as densely as we desire (this means: for every $\varepsilon > 0$ there is n_0 such that the function $f_n(x)$ is monotone in no interval of the length ε provided $n > n_0$). Moreover, all vertices of a curve $y = f_n(x)$ belong also to the subsequent curves $y = f_m(x)$ $(m > n)$. Evidently, if the limit

$$(7)\quad \lim_{n\to\infty} f_n(x) = f(x)$$

exists, then f is monotone in no subinterval of $[a, b]$; indeed, at all vertices of the curve $y = f_n(x)$ $(n = 1, 2, \ldots)$ we have obviously $f(x) = f_n(x)$[27]; those vertices of the curve, at which the curve $y = f_n(x)$ changes from an ascending one to a descending one or vice versa, are distributed in the interval $[a, b]$ for n sufficiently large as densely as we desire. So far Bolzano's reasoning (which I have consciously shifted a little) is correct. Still it remains to be proved that the limit (7) exists and is a continuous function of x in the interval $[a, b]$. Bolzano's proof of the existence of this limit is essentially correct; it is based virtually on the Cauchy-Bolzano condition[28]. On the other hand, he commits a substantial error when proving the *continuity* of the function $f(x)$. He argues simply as follows: the function $f(x)$ is the limit of continuous functions, and *the limit of a convergent sequence of continuous functions is always a continuous function.* Of course the last assertion is wrong; it becomes true, however, if we require e.g. *uniform* convergence. Here Bolzano's error is the same as in § 39: he disregards the necessity of an assumption of uniformity

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lichen Nachlasse, Věstník Král. české spol. nauk 1921–22, Vol. IV, pp. 1–20; V. Jarník, On Bolzano's function (Czech), Časopis pěst. mat. 51 (1922), pp. 248–264 (English translation in this volume, pp. 67–81.)

[27]) As these points satisfy $f_m(x) = f_n(x)$ for $m > n$ and hence also $f(x) = \lim_{m\to\infty} f_m(x) = f_n(x)$.

[28]) Concerning this condition see footnote 16.

(or something similar).[29]) Again this is essentially the same error against which he warned in § 12. This is the *third important error* of Bolzano; after all, we find the same mistake in Cauchy.[30])

Paragraphs 76–78 include several results on the alternation of maxima and minima; however, they are heavily affected by the incorrect § 18.

The rest of Sec. I is devoted to discontinuous functions. However, a perfect investigation of this topic would require a more profound knowledge of the theory of sets; no wonder that Bolzano's considerations are hardly satisfactory. In § 79 Bolzano presents an incorrect theorem asserting that there exist increasing functions which are discontinuous at every point. In §§ 80–81 another incorrect theorem is presented concerning the existence of one-sided limits for a certain type of discontinuous functions. The proofs in §§ 79–80 are not only incorrect but even formally so imperfect that they make the impression of mere sketches. Theorems given in the last § 82 are in the main correct: there exists a function increasing in $[a, b]$[31]) which is discontinuous only at infinitely many isolated points of the interval (a, b). The sum of "jumps" of such a function is necessarily a convergent series; the last condition is dropped if monotonicity is not required. The proof is essentially correct if not perfect in details.

This is the end of Sec. I; let us have one more look back at it! It is Bolzano's genuinely modern, arithmetizing approach which above all attracts our attention in Bolzano's work. Bolzano fully realized the inadmissibility of intuition in mathematical proofs and avoided it in his considerations with remarkable consistence and success (it should be observed that this was actually the first systematic and consistent attempt in this direction).[32])

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[29]) When proving convergence of the sequence (6) Bolzano deduced an estimate for $|f(x) - f_n(x)|$ *independent of* x so that he did prove the uniformity of convergence implicitly. On the basis of this estimate, the direct proof of continuity of the function $f(x)$ would be a matter of several lines. However, I believe that Bolzano was not aware of it; first, he would have certainly pointed out such an important fact, secondly, he had committed the same error already in § 39.

[30]) Cours d'Analyse, I^ère^ partie, chap. VI, § 1. Abel was probably the first who observed the incorrectness of this theorem and proceeded correctly in a special case (in his famous study on binomial series). The general theorem "the limit of a uniformly convergent sequence of uniformly continuous functions is a continuous function" appeared about 1847–48. Cauchy, later, recognized his mistake, too, and proved this theorem in 1853. Cf. Pringsheim, Grundlagen der allgemeinen Functionenlehre, Enzyklop. der math. Wissenschaften, II A 1.

[31]) The assumptions concerning the endpoints are not formulated quite clearly.

[32]) Bolzano's viewpoint in this matter is surely closely connected with his philosophical

Bolzano deserves respect for his very definition of continuity in § 2, for his introducing continuity at a *point* instead of the former continuity in an *interval* (Cauchy); his construction of a function continuous at a *single* point in § 10 proves that he fully recognized the importance of the fact. The further development of the theory of continuity of functions of one variable includes comprehensive passages without the least flaw; let us just recall §§ 19–30 (containing perfect proofs of fundamental theorems on continuous functions), or the really beautiful theory of monotone functions (§§ 49–59), or eventually §§ 65–75, where Bolzano so thoroughly discusses all possibilities that may occur for functions with infinitely many oscillations. Just the one example from § 75 (Bolzano's function) is sufficient for the reader to form a high opinion of Bolzano's talent for fundamental problems of mathematical analysis; however, I am not sure whether the perfect construction of the whole as in §§ 19–30 or 49–59[33]) should not be appreciated even higher then this dazzling detail.

Bolzano's remarkable sense for purity of the method manifests itself in the main contours as well as in details; this is especially clear in his examples demonstrating the various possibilities that may occur with continuous functions. Let us consider for example § 65 where Bolzano constructs a function continuous in $[0, 1)$[34]) which infinitely often and "alternately" assumes the values 0 and 1/2. Bolzano constructs the function as follows:

$$f(0) = 0\,,\quad f(\tfrac{1}{2}) = \tfrac{1}{2}\,,\quad f(\tfrac{3}{4}) = 0\,,\quad f(\tfrac{7}{8}) = \tfrac{1}{2}\,,\quad f(\tfrac{15}{16}) = 0\,,\quad f(\tfrac{31}{32}) = \tfrac{1}{2}, \ldots;$$

for $1 - 2^{-n} \leqq x \leqq 1 - 2^{-(n+1)}$ $(n = 0, 1, 2, \ldots)$ let $f(x)$ be linear. In § 66 Bolzano observes: the function $g(x) = \sin \log (1 - x)$, for example, has analogous properties. Bolzano was certainly aware of the fact that the function $\sin \log (1 - x)$ would not be conclusive enough, for in the book the functions $\sin x$, $\log x$ were never properly introduced; therefore, in a way certainly rather unusual for his days, he first constructs the function $f(x)$ and only then incidentally mentions the function $g(x)$; analogous situations occur in §§ 70 and 71.

We have mentioned already at the relevant places how particular Bolzano was about definitions and how clearly he realized the import of individual assumptions. We have seen also how carefully Bolzano discusses all possibilities that may occur in connection with continuity and discontinuity of functions; a number of existence theorems verified directly by examples (mainly in §§ 65–75) give evidence of his efforts in this direction. Such existence theorems accompany also some definitions; e.g. before defining an increasing function Bolzano demonstrates that there exist functions that virtually possess the property to be introduced.

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opinions about the substance and methods of mathematics; see, for example, the expressive quotation in Petr's foreword, pp. XI–XII.

[33]) Let us notice in these paragraphs, for example, the fact how fully Bolzano realized the effect of closedness or openness of the interval.

[34]) I have added the point $x = 0$ for the sake of simplicity.

On the other hand, these positive aspects of Bolzano's work are accompanied also with several errors. Let us omit the obscure §§ 47–48, the non-essential § 64 and the last paragraphs (namely §§ 79–81), where Bolzano tackles problems for which the time had not yet ripened; then we actually face only three substantial errors: in §§ 18, 39, 75.

The mistake in § 18 makes the impression of a sudden decay of Bolzano's ingenuity; it is of course peculiar that Bolzano should not have realized the incorrectness of this paragraph at least later, especially when (in § 70 and Sec. II, § 26) he constructed a function evidently disproving his assertion.

The second and the third mistake (§ 39 and § 75) have a common ground: namely, the omission of a requirement of *uniformity*; § 39 lacks the assumption of uniform continuity, the proof in § 75 the assumption of uniform convergence. We have also seen that the errors in §§ 39, 75 are of the same character as the erroneous consideration against which Bolzano himself warned pronouncedly in § 12 (cf. § 13); we shall see later (Sec. II, § 27) that he realized the importance of the assumption of uniformity (or of a similar notion) in connection with another problem. Thus Bolzano recognized the necessity of a concept of this character in some places; however, he did not succeed in formulating it suitably. This is why he argues mostly correctly when he discovers the incorrectness of such considerations that proceed without justification as if the continuity considered were uniform (see especially § 12); on the other hand, he either did not fulfil his task quite satisfactorily or neglected the significance of uniformity altogether when he ought to have either made use of uniformity or assumed it (§§ 39, 75, Sec. II, § 27 and others). We can expect that if Bolzano had had the opportunity to discuss his work with an interested colleague or a bright student, he would have realized his mistake either by himself or with their help. Of course, the question remains whether he would have found suitable substitutes for these considerations (this being doubtful above all in § 39 – continuity of functions of several variables; the proof of continuity of Bolzano's function in § 75 could have been done directly without making a detour via the general theorem on uniform continuity).[35])

Bolzano was surely aware of the revolutionary novelty of his work; an interesting passage in this regard is in the Introduction (Einleitung, § 2); being not of much topical interest, it is the more interesting from the psychological viewpoint. It reads approximately

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[35]) I have mentioned at the very beginning that Bolzano was no skilled mathematician. This is why, at less important points, he sometimes commits such naive omissions that are simply inconceivable. I have not mentioned them, for they are of no consequence for the whole; nevertheless, let me present at least two examples of such reasoning: in § 35 Bolzano asserts that the function $x^2 + 1/(1 - y) + 1/(2 - y) + 1/(3 - y) + 1/(4 - y) + \ldots$ is a continuous function of x for each y except $y = 1, 2, 3, \ldots$; in Sec. II, § 12 he states that the fact that continuity of a function does not imply existence of its derivative is seen from the example of the function $1/(1 - x)$ for $x = 1$.

as follows: Functions that can be described by a single expression valid for all x will be called functions of the first kind (e.g. $f(x) = 3x + 5$ or $f(x) = \sin x$ etc.); those for which such a representation is impossible are functions of the second kind (e.g. $f(x) = x$ for $x < 1$, $f(x) = 2$ for $1 \leqq x \leqq 2$, $f(x) = x^3$ for $x > 2$). It is of course very important to distinguish between functions of the first and second kinds. At the end of § 2, Bolzano says: we shall deal mostly, if not exclusively, only with functions of the first kind. I observe: (i) It is apparent from the whole Sec. I that the difference between functions of the first and second kinds is of no consequence at all for Bolzano's considerations. (ii) If Bolzano mentions functions of the first and second kinds in the sequel, he does so solely to point out that his considerations are valid for functions of both kinds. (iii) It seems doubtful to me that Bolzano who was so exacting in his definitions would have been satisfied with concepts qualified so inadequately as "functions of the first and second kinds". The whole paragraph seems[36]) rather to indicate the embarrassment Bolzano felt when seeing the newness and strangeness of his problems and methods. Fortunately, however, Bolzano seems to forget this embarrassment very soon, which enabled him to create a work really new, original, pioneering and in many respects perfect.

Zweiter Abschnitt
Abgeleitete Functionen

(Second Section. Derivatives)

As is evident from the heading, this section deals with the elements of differential calculus. Bolzano's basic viewpoint is the same as in Sec. I; however, the difference in the performance is substantial. While in Sec. I Bolzano fulfilled his task with remarkable success, in Sec. II we find a varied mixtuıe of correct considerations and essential failures. This is quite understandable, for the proofs that Bolzano attempts to bring out are mostly considerably more difficult than those in Sec. I; to carry them out without defects would require – in addition to a bright and profound mind – also a good deal of "craftsmanship" and we have observed above that Bolzano was not much of a routineer; after all, it is difficult to demand routine from somebody who had to develop his investigation all by him-

[36]) This is of course merely my personal impression; it would require a historian to find the actual significance of this paragraph in the whole work of Bolzano.

self with painstaking efforts from the very beginning.[37]) These notes are not intended to detract from Bolzano's admirable work; Section II is a pioneering and extremely significant piece of work just as Section I. Bolzano shows here on what grounds the differential calculus is to be built and which methods are to be used. The basic viewpoint, definitions, main ideas of the proofs are adequate and without defects; only the execution — sometimes too difficult — is often unsatisfactory. This is why I will not go through this section commenting the paragraphs one by one; the result would be a confusing mixture of favourable and unfavourable opinions, complicated by our efforts to follow the mutual interaction of individual errors and vague formulations. Moreover, I do not feel qualified for this difficult task: proofs employing sound ideas cannot be considered worthless just on the grounds of certain defects; their value can be estimated only relatively, taking into account the time and other circumstances of their birth. However, only a specialist in the history of mathematics is competent to pronounce a judgement. Therefore, I shall restrict my comments to several passages that I believe to be of special interest. Perhaps my review of Sec. I helped the reader to have a more complete idea of the progress of Bolzano's work.

Section II is again divided into paragraphs (§§ 1—99). The definition of the derivative is introduced in § 2, where Bolzano defines (in the same way as foi continuity) not only "both-sided" derivative but also the derivatives from the right and from the left. Further, he discusses the consequences of this definition in detail. Thus in § 12 he proves: if a function $f(x)$ has a derivative, say, from the right, at x_0, then $f(x)$ is a continuous function from the right at the point x_0.[38]) The converse theorem is not valid: a function continuous at a point x_0 need not have a derivative at x_0. In § 19 Bolzano even demonstrates that the set of points at which the continuous "Bolzano's function" (constructed in Sec. I, § 75) has no (both-sided) derivative is everywhere dense in the interval $[a, b]$. The subsequent passages on differential calculus include many interesting points; as I have mentioned above, I will not go into details. Only as an example I present here the following assertion from the theory of relative extrema (§§ 76—77). The following cases may occur for a function $f(x)$ at a point x_0: (i) The function $f(x)$ has derivatives at x_0 both from the right and from the left. (ii) $f(x)$ has no derivative either from the right or from the left (or neither of them) at x_0. The case (i) admits the following subcases: (i_1) both derivatives (from the right and the left) have the same sign; (i_2) they have opposite signs; (i_3) at least one of them is zero. Bolzano proves: in the case (i_1) no extremum can occur; in (i_2) there is an extremum; in (i_3) and (ii) an extremum can but need not occur.

We shall be interested mainly in Bolzano's approach to the fundamental theorems of differential calculus: the Mean Value Theorem and the Taylor Theorem.

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[37]) The lack of suitable symbols and terminology certainly hampered Bolzano in more complex considerations (e.g. even for the absolute value Bolzano introduces no symbol but only points out repeatedly in the text when it is the absolute value that is concerned).

[38]) Bolzano deals (almost everywhere) with finite derivatives.

A modern way of formulating and proving the Mean Value Theorem proceeds as follows: first we prove the Rolle Theorem which yields by an easy transformation the Mean Value Theorem in the following form: Let $f(x)$ be continuous in $[a, b]$ and let it have the derivative in (a, b)[39]; then there is a number c $(a < c < b)$ such that

$$f(b) - f(a) = (b - a) f'(c) \,.$$ [40]

Bolzano's proof proceeds quite differently; it is similar to Cauchy's proof[41] but it differs from it in one important point. Cauchy proceeds roughly speaking as follows: Let us assume that $f(x)$ is continuous and has a continuous derivative in $[a, b]$. *Since*

$$(1) \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \,,$$

then for an arbitrary positive number ε it is possible to choose a positive δ so that

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \varepsilon$$

for all x from the interval $[a, b]$ and for all h satisfying $0 < |h| < \delta$. The argument contained in the last sentence is evidently incorrect: in fact, Cauchy asserts here that the convergence of the expression $h^{-1}(f(x+h) - f(x))$ to $f'(x)$ is *uniform*; though this is correct ($f'(x)$ being continuous) it is by no means obvious and it is necessary to prove this uniformity[42]. The next course of the proof is essentially correct and can be roughly described as follows: For a number $\varepsilon > 0$ let us find a number $\delta > 0$ as mentioned above and let us divide the interval $[a, b]$ by points $x_0, x_1, x_2, \ldots, x_n$ $(a = x_0 < x_1 < x_2 < \ldots < x_n = b)$ so that $x_i - x_{i-1} < \delta$. Let A and B be the least and the greatest value, respectively, of the function $f'(x)$ in the interval $[a, b]$; then

$$A - \varepsilon \leqq f'(x_{i-1}) - \varepsilon < \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} < f'(x_{i-1}) + \varepsilon \leqq B + \varepsilon \,,$$

hence

$$(A - \varepsilon)(x_i - x_{i-1}) < f(x_i) - f(x_{i-1}) < (B + \varepsilon)(x_i - x_{i-1}) \,;$$

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[39] This derivative is admitted to be infinite.

[40] The present form of the proof is due to Bonnet.

[41] Résumé des leçons sur le calcul infinitésimal, 7ème leçon.

[42] Cauchy introduces the assumption on continuity of the derivative only a little later, so that even his statement is untrue; similarly, he has strictly speaking no right to speak later about the least and the greatest value of the function $f(x)$.

summing these inequalities for $i = 1, 2, 3, \ldots, n$ we obtain

$$(A - \varepsilon)(b - a) < f(b) - f(a) < (B + \varepsilon)(b - a)$$

and consequently, as $\varepsilon > 0$ was arbitrary,

$$A(b - a) \leqq f(b) - f(a) \leqq B(b - a),$$
$$f(b) - f(a) = M(b - a) \text{ (where } A \leqq M \leqq B).$$

Since A and B are values of a continuous function $f'(x)$ and $A \leqq M \leqq B$, there exists necessarily such a c $(a \leqq c \leqq b)$ that $f'(c) = M$ and hence

$$f(b) - f(a) = (b - a) f'(c) \; (a \leqq c \leqq b).$$

In §§ 28–29 Bolzano takes over this proof in its main lines from Cauchy. Nevertheless, he realized Cauchy's error and tried to set it right in § 27. Thus the aim of § 27 was, roughly speaking, a proof of the above-mentioned uniform convergence, i.e. a proof of the following statement:

Theorem A. If both $f(x)$ and $f'(x)$ are continuous in $[a, b]$[43]) then for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$(2) \quad \left| \frac{f(x + h) - f(x)}{h} - f'(x) \right| < \varepsilon$$

provided $a \leqq x \leqq b$, $a \leqq x + h \leqq b$, $0 < |h| \leqq \delta$.

Bolzano actually applies this theorem in § 28; however, the theorem presented instead in § 27 is very complicated and rather vague; it is possible either to conclude that Bolzano desired to present Theorem A, or to interpret the theorem from § 27 in the following way:

Theorem B. If both $f(x)$ and $f'(x)$ are continuous in $[a, b]$[43]) then for every $\varepsilon > 0$ there is $\delta > 0$ such that for every x $(a \leqq x \leqq b)$ there exists at least one number h with $|h| \geqq \delta$ such that

$$\left| \frac{f(x + h) - f(x)}{h} - f'(x) \right| < \varepsilon \; (a \leqq x + h \leqq b).$$

Theorem B is evidently weaker than Theorem A, for Theorem A guarantees the inequality (2) for all $|h| \leqq \delta$[44]) while B does so eventually for a single h, $|h| \geqq \delta$, which can moreover depend also on x. Bolzano's proof in § 27 (provided we interpret some minor vaguenesses on Bolzano's behalf) demonstrates correctly Theorem B but not Theorem A. Definitely, Bolzano's proof of the Mean Value Theorem is not quite correct: if the theorem from § 27

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[43]) Bolzano assumes only one-sided derivatives at the points a, b.

[44]) Hence also for $|h| = \delta$.

is interpreted as Theorem A, then the proof in § 27 fails; if it is interpreted as Theorem B, then § 28 is incorrect, for Bolzano uses Theorem A there. In spite of this, Bolzano's attempt in § 27 deserves our respect; here we have one more occasion when Bolzano met with "uniformity" and recognized that such a notion would be necessary. On the other hand, we had an opportunity to see (Sec. I, § 39 and § 75) that in other cases he simply neglected the necessity of such considerations. Bolzano's proof in § 27 is indirect. Let me introduce it here (formally re-arranged) on account of its interest.

Let a function $f(x)$ satisfy the assumptions of Theorem B but not its assertion. Then there exists a number $\varepsilon > 0$ and a sequence $x_1, x_2, \ldots$, so that

$$(3) \quad \left|\frac{f(x_n + h) - f(x_n)}{h} - f'(x_n)\right| \geqq \varepsilon$$

for all h, $|h| \geqq 1/n$.[45] Let ξ be an accumulation point of the sequence $x_1, x_2, \ldots$ Let us choose $\delta \neq 0$ such that

$$\left|\frac{f(\xi + \delta) - f(\xi)}{\delta} - f'(\xi)\right| < \tfrac{1}{2}\varepsilon .$$

The assumption of continuity implies existence of a number $\eta > 0$ such that

$$\left|\frac{f(x + \delta) - f(x)}{\delta} - f'(x)\right| < \varepsilon$$

for $|\xi - x| < \eta$. Let us choose n so that $1/n \leqq |\delta|$ and $|\xi - x_n| < \eta$; then

$$\left|\frac{f(x_n + \delta) - f(x_n)}{\delta} - f'(x_n)\right| < \varepsilon$$

though $|\delta| \geqq 1/n$, which contradicts (3).[46]

In spite of Bolzano's remarkable efforts, his proof of the Mean Value Theorem is not – as mentioned above – quite perfect; and neither is its formulation. The Mean Value Theorem is expressed by the identity

$$f(b) - f(a) = (b - a)f'(c)$$

where $f(x)$ is continuous in $[a, b]$ and $f'(x)$ exists in (a, b); c is a suitable number from the interval (a, b). Unlike this modern version, continuity of the derivative as well as its exist-

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[45]) For the sake of simplicity I omit the inequalities which say that we must stay within the interval $[a, b]$ – the reader supplies them easily himself if required.

[46]) This proof – in an even less satisfactory form – requires more than two printed pages in Bolzano; one more example of the difficulties Bolzano had to face.

ence at the endpoints is assumed by Bolzano (he attempts to get rid of the latter assumption in § 31 but fails); the most important defect however is that he obtains only the inequality $a \leqq c \leqq b$ instead of $a < c < b$. This causes certain difficulties, e.g. in § 80 (relative extrema). Let, for example, $f'(x_0) = 0, f''(x_0) > 0$ and let $f''(x)$ be continuous in a neighbourhood of the point x_0. Then there is a relative minimum at the point x_0 which can be proved on the basis of the modern version of the Mean Value Theorem: If $|h|$ is sufficiently small, $h \neq 0$, then

$$f(x_0 + h) - f(x_0) = hf'(x_0 + \vartheta h)\,(0 < \vartheta < 1)\,;$$
$$hf'(x_0 + \vartheta h) = h[f'(x_0 + \vartheta h) - f'(x_0)] = \vartheta h^2 f''(x_0 + \vartheta\vartheta' h) > 0$$
$$(0 < \vartheta' < 1)\,.$$

Bolzano argues in the same way; however, he has no right to do so, for his version of the Mean Value Theorem implies only $0 \leqq \vartheta \leqq 1$, and $\vartheta = 0$ would yield $f(x_0 + h) - f(x_0) = 0$.

The Taylor Theorem is presented by Bolzano in the following version: If a function $f(x)$ and all its derivatives up to the n-th are continuous in the interval $[a, a + h]$ then

$$f(a + h) = f(a) + \sum_{i=1}^{n-1} \frac{h^i}{i!} f^{(i)}(a) + \frac{h^n}{n!} f^{(n)}(a + \Theta h)\,,$$

where $0 \leqq \Theta \leqq 1$. Bolzano's proof proceeds by induction (§ 82): For $n = 1$, this is simply the Mean Value Theorem. Let us assume that the theorem holds for $n = k - 1$ and let us prove it for $n = k$ (for simplicity, let $h > 0$). The assumptions yield

$$f'(a + y) = f'(a) + \sum_{i=1}^{k-2} \frac{y^i}{i!} f^{(i+1)}(a) + \frac{y^{k-1}}{(k-1)!} f^{(k)}(a + \Theta' y)$$

for $0 \leqq y \leqq h$ $(0 \leqq \Theta' \leqq 1)$; thus if $f^{(k)}(p)$, $f^{(k)}(q)$ are, respectively, the least and the greatest value of $f^{(k)}(x)$ for $a \leqq x \leqq a + h$ then

$$f'(a + y) - \sum_{i=0}^{k-2} \frac{y^i}{i!} f^{(i+1)}(a) - \frac{y^{k-1}}{(k-1)!} f^{(k)}(p) \geqq 0\,,$$

$$f'(a + y) - \sum_{i=0}^{k-2} \frac{y^i}{i!} f^{(i+1)}(a) - \frac{y^{k-1}}{(k-1)!} f^{(k)}(q) \leqq 0\,.$$

The primitive function to the left hand side is therefore nondecreasing. On the other hand, choosing a suitable integration constant, this primitive function is

$$f(a + y) - f(a) - \sum_{i=1}^{k-1} \frac{y^i}{i!} f^{(i)}(a) - \frac{y^k}{k!} f^{(k)}(p)\,;$$

for $y = 0$ this function equals zero, hence for $y = h$ it is nonnegative:

$$f(a+h) - f(a) - \sum_{i=1}^{k-1} \frac{h^i}{i!} f^{(i)}(a) - \frac{h^k}{k!} f^{(k)}(p) \geqq 0 .$$

Analogously we conclude

$$f(a+h) - f(a) - \sum_{i=1}^{k-1} \frac{h^i}{i!} f^{(i)}(a) - \frac{h^k}{k!} f^{(k)}(q) \leqq 0$$

and by virtue of continuity of the function $f^{(k)}(x)$ we have finally for a suitable $\Theta(0 \leqq \leqq \Theta \leqq 1)$

$$f(a+h) - f(a) - \sum_{i=1}^{k-1} \frac{h^i}{i!} f^{(i)}(a) - \frac{h^k}{k!} f^{(k)}(a + \Theta h) = 0 .$$

Bolzano attempts to weaken the assumptions at the endpoints; however, his reasoning is not correct[47]). Conditions guaranteeing the possibility of developing a function into an *infinite* Taylor series are given correctly in the main, even if not quite in the most suitable form.

In Sec. II Bolzano often deals with functions of several variables. I do not intend to comment upon these parts, for they are mostly incorrect. The error Bolzano had made already when studying continuity of functions of several variables (Sec. I, § 39) repeats almost regularly, namely the omission of the uniformity requirement. It is interesting to trace how Bolzano realizes the necessity of such a requirement here and there but after several lines he plainly drops it. For example, in § 33 he tries to prove the following (incorrect) theorem: If $\lim_{y \to 0} f(x, y) = 0$ and if $\partial f(x, y)/\partial x$ exists for $a < x < b$, then also $\lim_{y \to 0} \partial f/\partial x = 0$ (the range of the assumptions is not quite clearly indicated). Here Bolzano states: we have

$$\frac{f(x+\omega, y) - f(x, y)}{\omega} = \frac{\partial f(x, y)}{\partial x} + \Omega ,$$

where $\lim_{\omega \to 0} \Omega = 0$ provided x, y *are fixed*; at the same time, the left hand side converges to zero with $y \to 0$ provided x, ω *are fixed*. However, this does not imply $\lim_{y \to 0} \partial f/\partial x = 0$. This is quite correct, but Bolzano replaces this argument immediately by another one which is as erroneous as the one he rejected: we have

$$\frac{f(x+\omega, y) - f(x, y)}{\omega} = \frac{\partial f(x + \mu\omega, y)}{\partial x} \quad (0 \leqq \mu \leqq 1) ;$$

[47]) Similarly to the Mean Value Theorem in § 31.

hence (let us add: for x and ω fixed)

$$(4)\quad \lim_{y \to 0} \frac{\partial f(x + \mu\omega, y)}{\partial x} = 0\,.$$

Simultaneously, as $\partial f/\partial x$ is a continuous function of the variable x[48]),

$$(5)\quad \lim_{\omega \to 0} \frac{\partial f(x + \mu\omega, y)}{\partial x} = \frac{\partial f(x, y)}{\partial x}$$

(let us add: for y fixed); (4) and (5) together yield (according to Bolzano)

$$\lim_{y \to 0} \frac{\partial f(x, y)}{\partial x} = 0$$

– and this is roughly speaking the same mistake as above!

I hope that these comments have enabled the reader to have a certain idea of Sec. II of Bolzano's book as well; the detailed study as well as the definitive assessment of the book is an interesting but by no means easy task for a historian of mathematics.

Prague, March 18, 1931.

[48]) It is not clear to me how Bolzano arrived at this strange statement (see p. 119, the last three lines in Bolzano's book); nevertheless, let us take it for a correct statement – possibly as an assumption.