LIFE AND SCIENTIFIC ENDEAVOUR OF BERNARD BOLZANO

There have been few mathematicians whose lives were determined by such significant components as those that were shaping the fate and work of Bernard Bolzano. On the one hand,their consequence was an almost total isolation, multiplied by the fact that during his lifetime the crucial results of his mathematical research were not published at all. On the other hand, his evaluation of the significance and importance of various domains of his work was developing under the influence of his own life career; thus the mathematical efforts were somewhat suppressed by his other interests. It is all the more necessary to admire the amazing force of his mathematical invention, which had its roots in an extraordinarily strong mathematical talent as well as in his ability to approach the fundamental problems of mathematics of his time in a new, nontraditional way; invention, which thrust the way – both methodically and topically – in the directions that mathematical research started to follow only in the last third of the 19th century. At the same time, we have to keep in mind that Bolzano's unfinished mathematical works represent only a fraction of his work, which covers a number of topical problems of logic, philosophy, Utopian socialism as well as theological disputes, etc. And, moreover, within the general framework of Bolzano's endeavour mathematics served no practical purpose but the perfection of man; the features of this conception inevitably manifested themselves also in the choice of direction of Bolzano's mathematics as well as in his results.

1. Life story

Soon after the middle of the 18th century an educated dealer in works of art, Bernard Pompeius Bolzano (1737–1816), from a village Nesso on Como Lake in northern Italy, chose Prague as his new abode and by marrying Maria Cecilia Maurer (1753–1821) in 1776 established his home there. At the time of his arrival in Bohemia, the enlightened absolutism of Maria Theresa and Joseph II, together with a diffident echo of the Enlightenment tendencies of the pre-revolutionary France, was making itself felt in education and in the cultural and social life of Bohemia. It was on 5 October 1781, the year of the abolition of serfdom and of the Toleration Patent that guaranteed religious freedom to non-Catholics, that the fourth child – a son Bernardus – was born in Bolzano's family. There were twelve children; but apart from Bernard, only the eldest son Jan (1777–1859), who was his younger brother's resort throughout his life, lived to reach maturity. After a few years of private education, both brothers entered the piarist grammar school in the New Town of Prague. While the elder brother did not finish his studies and devoted himself to the family's business, which he took over after father's death in 1816, Bernard graduated from the secondary school with honours in 1796 and entered his philosophical studies at Charles University. This was already at the moment when the decade of Josephian Enlightenment was being replaced by the growing fear of the Austrian monarchy with respect to the influence of the French Revolution. The hardening of reaction, repression and censorship and the efforts to defend the feudal state against "rebellious" ideas by a police regime became a constituent part of life in Austria till the bourgeois revolution in 1848. One of the most active institutions in the anti-revolutionary propaganda was the Catholic Church. This situation accompanied Bolzano throughout his life and many times complicated his personal career.

During his university studies, Bolzano was deeply involved in philosophy, logic and mathematics. Paradoxically enough, it was logic nad mathematics in which at the beginning he needed the help of his colleagues. In that period he was more attracted by physics; nonetheless, already this science gave him the opportunity to find that the majority of its empirical theorems were not "deduced immediately from experience" and that they inevitably required a substantiation of proofs and the delimitation of domains of their validity.[1]) The lectures on elementary mathematics delivered by S. Vydra did not inspire him too much, and his teacher actually was the teacher of Gauss, A. G. Kästner, through his comprehensive textbook "Anfangsgründe der Mathemaιik" [69]. In this book Bolzano appreciated the author's very effort to determine the principles used to deduce and to prove consequences. After completing his studies of philosophy he devotιd the whole year 1799 to 1800 to the study of mathematics, mostly with Prof. Gerstner; he payed, however, equal attention to the study of Kant and of both his critics and followers. In 1800 he decided to study theology; nevertheless, this did not mean at all that he would lose interest in mathematics. On the contrary, at that very time he wrote his first mathematical treatise "Betrachtungen über einige Gegenstände der Elementargeometrie" [1], which in the year 1804 won him the attention of the Prague mathematicians. In 1804 he was still hesitating between mathematics and theology. He took part in a competition for the professorship of elementary mathematics after Prof. Vydra as well as in a competition for a post in the then established department of "religious science". In March 1805, Bolzano was appointed temporary religious teacher at this department, the task of which was to paralyze the revolutionary ideas among the university students. The establishment of the department – as Bolzano

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[1]) For example, this is demonstrated by one of his early unfinished physical manuscripts "Aetiologie" [3], which contains an attempt at the axiomatization of mechanics in a way analogous to mathematics. There Bolzano tried to make the relations between notions — and hence the whole exposition of the foundations of this science — more exact.

recalls in his autobiography [25] – was received by the students with considerable indignation. After his appointment and in a quick succession Bolzano was graduated, ordained and he started to lecture. While the mathematician Bolzano was writing his second treatise "Beyträge zu einer begründeteren Darstellung der Mathematik" [2], the Catholic priest Bolzano was enlisting the university students for the Catholic faith in his Sunday exhortations. Seemingly, Bolzano entered a simple life career. However, the break was soon to come. It was already in 1805 that the first denunciation appeared. On its ground, his appointment was changed to temporary and two years elapsed before owing to interventions he was confirmed in his office. However, he was ordered to lecture on religion according to the official textbook [56] and was forbidden to use his own curriculum. Nevertheless, he did not abandon his own commentaries and from 1811 he was allowed again to prepare and deliver his own lectures.[2]) In 1813 he published his Sunday exhortations ([4;] cf. footnote 7), which he held regularly for the students and which gained wide popularity: they were recorded, copied and circulated. In 1815 he was even elected Dean of the Faculty of Philosophy and became member of the Royal Bohemian Learned Society. In the years 1815–1816 Bolzano was ill and delivered no lectures. It seems, however, that he devoted himself intensively to mathematics. This is corroborated by the fact that in 1816 there appeared his treatise "Der binomische Lehrsatz ..." [8], in the next year "Rein analytischer Beweis ..." [9] and "Die drey Probleme ..." [10], in which he introduced and threw light on many fundamental notions of mathematical analysis as well as of geometry.

After Bolzano resumed his lectures, the new director of philosophical studies F. Wilhelm accused him again – in 1817 – of lecturing on the basis of his own texts and not abiding by the official textbook. Bolzano justified his actions by a vehement attack against the compulsory textbook whose author, however, was the court chaplain and the Emperor's confessor Frint. In spite of the fact that neither a special committee nor the Faculties of Theology and Philosophy of Vienna University found anything irregular in Bolzano's lectures and that Bolzano's colleagues and even the Bohemian gubernatorial office pleaded for him, the assaults against Bolzano did not cease and the so-called "Bolzano's suit" assumed a political character, though the commentary to the Court Edict by which Bolzano was dismissed from his post at the university on 24 December 1819 stated that he "grossly offended against his duties of a priest, professor of religion and teacher of the young generation". (The edict was served on Bolzano only on 19 January 1820.) Undoubtedly, one of the last causes of this decision was a direct intervention of the Papal Curia, which resulted from another denunciation of Bolzano by his Prague enemies directly in Rome.[3]) The

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[2]) Later (1834) these lectures were anonymously published by Schneider, Dittrich and Mrs. Hoffmann in three volumes under the title "Lehrbuch der Religionswissenschaft" [22].

[3]) The following quotation demonstrates clearly the character of this denunciation: "At the Prague university the religious doctrine is lectured upon by a certain priest, Bernard Bolzano by name, who is famous throughout Bohemia for his new and irregular way of teaching religion and for his sophistic scheming ... An example of his blasphemy is his rejection of reliable and irrefutable

objections against Bolzano's teaching, which were the cause of his examination by the Church authorities lasting five more years, attacked Bolzano not only for his rationalism and replacing religion by ethics, but also for his adopting modern scientific results and, last but not least, especially for his theses that had an anti-feudal, mostly bourgeois, but also socialist character: the assertions on perpetual progress and on the equality of all people, the rejection of hereditary aristocracy with all their privileges and possessions, the exposure of the origins of private property, the pillorying of the immoral relation of the rich to the poor and their parasitism ("he who would wish to live only on others' work and would not give anything to them himself does not deserve to live"), the criticism of the imperfections of the State and social system, even the declaration of the right of citizens to armed resistance, and so on. Thus "he publicly and from a sacred place professed principles threatening the peace of the State". All this seems to indicate that the categorical reaction of the most conservative circles of the Catholic Church in Austria, which forced the authorities of the monarchy to isolate Bolzano totally from the contemporary intellectual circles, was aroused mainly by Bolzano's Sunday exhortations, which, though dealing with points of faith, assumed rather an ethical, moralizing as well as abstractly political character.[4])

Bolzano was ordered to recant his ideas, analogously to one of his most radical pupils, the administrator of the theological seminar at Litoměřice, M. J. Fesl, who had been forced to do so already in 1824. Bolzano wrote a new analysis, which showed numerous signs of inconsistency and contradictions in his controversial opinions. However, the answer did not satisfy the Church authorities. At that moment, a prominent Bohemian Slavonist and advocate of Enlightenment, J. Dobrovský, intervened on behalf on Bolzano by threatening both Archbishop Chlumčanský and the Emperor's chaplain Frint in personal letters with publishing Bolzano's reply in Lusatia. It was only due to this intervention that Bolzano was saved from imprisonment in a convent and from a full recantation.

From 1825 Bolzano was allowed to live in relative peace, taking advantage of the hospitality of the Hoffmann family, in their mansion-house at Těchobuz, or in the house of the lawyer Pistl at Radič south of Prague and later at Liběchov with A. Veith or at Jirny near Úvaly in the house of Veith's sister. Towards the close of his life he again lived in Prague with his brother. J. Hoffmann even expressed his opinion that the whole suit helped to prolong Bolzano's life by relieving him of the strenuous duty of lecturing and by making it possible for him to devote himself to scientific work under the solicitous care of his friends. Bolzano did not avoid tuberculosis, which had caused the death of his brothers and sisters;

Catholic doctrines ... This man, the coryphaeus of our false prophets, has from the very beginning had a number of pupils and followers who, being scattered all over the country, continue to spread among the mob the principles by which they were infected".

4) It is of interest that later on (1840) Bolzano considered the possibility of clearing the intellectual contents of his exhortations of their religious form.

he suffered from blood-spitting already in the 20's and later the malady frequently seized him again (1828, 1832, 1838, 1843).

Bolzano's application for the post of professor of mathematics at Prague university was rejected; he himself declined the professorship of religious dogmatics offered to him by Tübingen university. His relations to the Royal Bohemian Learned Society were becoming looser. Not only was he denied the publication of his works in Austria but even his manuscripts were confiscated. Therefore, he negotiated their publication abroad, with the help of his students. He aimed at formulating his main ideas, believing that his students would continue his work. However, apart from "Wissenschaftslehre" [13], a number of his works, particularly those of mathematical character, remained forgotten as manuscripts till the 20th century. Likewise, the compilation of his Utopian-socialist ideas "Von dem besten Staate" [18], dedicated to Mrs. Hoffmann in 1831, remained in manuscript for a whole century, though several copies of it were circulating.

Bolzano lived modestly, even ascetically, throughout his life. He received the means for buying books from some of his aristocratic followers, in particular from L. Thun. From the 20's his activities, correspondence and connections were under continuous police surveillance.

After his return to Prague (1841), which was in connection with the illness of Mrs. Hoffmann, Bolzano was not able to cope with the new relations and conditions that had formed in Czech society any more, and though he took deep interest in the social and political problems, he was not able to orientate himself in the events of the revolutionary year 1848 – the last year of his life – and he died in a new isolation from the progressive parts of the society of that time, to which he did not find his way, as well as from his former aristocratic protectors, whom he had alienated by some of his statements.

His Utopian socialism estranged him from the radical democrats – the most revolutionary component of the bourgeois revolution of 1848 – both in practice (for they surpassed him by their revolutionary activities) and in theory (for they were far from understanding, let alone converting into facts his socialist ideas). After his death (18 December, 1848) Bolzano soon became a "martyr": a symbol of an ideal, irreproachable man, persecuted for his progressive ideas by both the Church and the State authorities, and thus a shield of the whole coming generations of Czech progressive circles, which invoked him in their fight against the Habsburg monarchy as well as for social justice.

2. Scientific endeavour

There is no doubt that the course of Bolzano's life to a great degree affected his own point of view on the priorities of scientific work. We have already mentioned that after the 20's he tried to ensure the publication of his works; apparently, this above all concerned those of his results that he and his friends considered the most important – and these were

Erster Abschnitt. Einige Lehren aus der allgemeinen Aetiologie.

§. 1.

Zweck und Inhalt dieses Abschnittes.

In Rücksicht auf die so wichtigen Begriffe von Grund und Folge gibt es mancherley Lehrsätze, von denen Einige freylich jedem gesunden Menschenverstande von selbst einleuchten, Andere dagegen von der Art sind, daß sie nur durch ein methodisches Nachdenken erst gefunden und erwiesen werden können. Es wäre der Mühe werth, diese und zwar etwas vollständiger, als es in den gewöhnlichen Lehrbüchern der Philosophie, und insbesondere der Metaphysik oder der Ontologie* bisher geschehen ist, zu sammeln; und wären sie zahlreich genug, um einen eigenen Titel zu verdienen, so könnte man sie mit dem Nahmen der Aetiologie belegen.

*z. B. selbst in Wolfii Ontologia.

First page of Bolzano's manuscript of his treatise "Aetiologie" the works of religious-ethical character.[5]) From his other extensive works it is only "Wissenschaftslehre" [13] and later several treatises in the Transactions of the Royal Bohemian Learned Society. The latter, however, appeared only in the last period of his life and, moreover, among them there are only four mathematical-physical treatises from which a single one, "Versuch einer objectiven Begründung..." [32] was allegedly written as early as in 1815.[6]) If we take this fact into account, we see that – apart from the early period – from the main spheres of his interest, namely, mathematics, logic, religion and philosophy and social reforms, it is mathematics that was mostly neglected, and the main mathematical results long remained solely in manuscripts. And it is clear that already in Bolzano's time his mathematical ideas were the most progressive and could have affected the world development in this domain to a greater extent. On the other hand, Bolzano's conception of the Utopian socialism can be only confronted with similar tendencies of a number of his predecessors and contemporaries and, after all, even they have remained – except for the publication of "Erbauungsreden ..."[7]) – only in manuscripts.[8]) Thus the main attention of the "Bolzanists" was attracted by the works that were to demonstrate the injustice of the consequences of Bolzano's suit and actually represented the most passing topics.[9])

Bolzano's conception of mathematics

Going through all Bolzano's mathematical results known so far, we see that almost all problems on which he worked during his lifetime had their roots already in his early

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5) The following works appeared as a result of these negotiations: 1827 [11, 12]; 1831 [19]; 1834 [22, 23]; 1835 [24]; 1836 [25]; 1837 [26, 27]; 1838 [28a]; 1839 [29]; 1840 [30]; 1845 [39]; 1847 [43, 44].

6) Apart from two biographies by Gerstner [29] and Krombholz [40] we have only four treatises of mathematical-physical character: 1842 [31]; 1845 [32]; 1843 [35]; 1847 [41]. The other three are devoted to the logical problems of Exner's nominalism and realism [34], to the concept of the beautiful [33] and to general philology [42].

7) "Erbauungsreden für Akademiker" (cf. [4]), 1st ed. Prague 1813 (pp. XX + 354), 2nd ed. Sulzbach (pp.xvi + 376). Further editions appeared only after Bolzano's death (1849—50, 1851, 1852, in four volumes), then as Vol. I of Bolzano's Gesammelte Schriften (Vienna 1882) and immediately in Czech (Řeči vzdělávací akademické mládeži, Prague 1882—87).

8) This holds particularly for his crucial work in this respect "Von dem besten Staate" [18], written in 1831 but published for the first time in 1930.

9) In this connection we ought to mark out the attention payed to Bolzano's mathematical legacy by Czech professors M. Jašek [65—67], V. Jarník [61—64] and K. Rychlík [81—83], who at the beginning of the 20's of this century began to analyse and publish Bolzano's main results in a systematic way, above all those obtained in mathematical analysis and the theory of numbers. It was on their initiative that the Royal Bohemian Learned Society started to publish Bolzano's writings (Vol. I [20], II [21], III [18], IV: Der Briefwechsel Bernard Bolzano's und F. Exner, Prague 1935, V: Geometric writings, Prague 1948 [Ed. J. Vojtěch]). From earlier critical reviews of Bolzano's published works let us mention [60], [85], [86].

Dr. B. Bolzanos

Wissenschaftslehre.

Versuch

einer ausführlichen und größtentheils neuen Darstellung

der

Logik

mit steter Rücksicht auf deren bisherige Bearbeiter.

Herausgegeben

von

mehren seiner Freunde.

Mit einer Vorrede

des

Dr. J. Ch. A. Heinroth.

Erster Band.

Sulzbach,

in der J. E. v. Seidelschen Buchhandlung,

1837.

Title page of Bolzano's work "Wissenschaftslehre" (1837) works. He returned to them, so to say, in a more general context, and with the desire to build a "perfect" conception of the whole mathematics – that is, to build mathematics from its very foundations. In his time this trend actually meant to include in mathematics, in the present terminology, a certain part of metamathematics; that is, to extend mathematics even deeper down to the foundations. It was already in the introduction to his first mathematical paper (1804) that Bolzano explicitly formulated his programme of revision of the foundations: "First of all, I decided to observe the following rule: no obviousness of an assumption will force me to neglect the duty to seek its proof till I see clearly that no proof can be required and why it cannot be required." Bolzano applied this rule when studying, in chronological order:

– the foundations of geometry,
– the foundations of mathematical proofs and of the building up of a scientific theory,
– the foundations of mathematical logic,
– the foundations of mathematical analysis,
– the foundations of the theory of sets.

During the first period – till the 20's – he investigated minor problems[10]), after 1820 he started a systematic compilatory research.

In his "Wissenschaftslehre" [13] Bolzano gives a systematic treatment of the method of presentation of scientific theory; within this framework some parts deal with mathematical logic. This work should have been followed upon by "Grössenlehre" (the theory of quantities) which was never completed, though its individual parts were already given a relatively definitive form (there exist not less than three manuscript versions of some parts) and formed the basis of the first edition in the Writings of Bernard Bolzano ("Functionenlehre" [20], "Zahlentheorie" [21]). Bolzano counted – unfortunately in vain – upon the help of his students in the completion of this work. Consequently, with his last resources he resumed his study of the problems of the foundations of infinite processes and in his "Paradoxien des Unendlichen" [38] achieved significant results in the theory of sets.

All this shows that Bolzano as a mathematician, unlike a number of mathematicians of the 18th century, had no intention to write an all-comprising textbook – a compendium of mathematics, but a conceptual synthesis – an exposition on the building up of mathematics.

Mathematical logic

The only field of Bolzano's non-religious interest in which his results were published relatively soon after they had been obtained, was logic, the problems of which he treated in "Beyträge ... [2] for the first time and then again in "Wissenschaftslehre" [13] in the

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[10]) See [9], p. 26n where he says that already in 1804 he decided not to start the publication of the complete exposition of any theory but to publish his ideas, provided they differ from the current ones, in separate papers.

Dr. Bernard Bolzano's

Paradoxien des Unendlichen

herausgegeben

aus dem schriftlichen Nachlasse des Verfassers

von

Dr. Fr. Přihonsky.

Je suis tellement pour l'infini actuel, qu'au lieu d'admettre, que la nature l'abhorre, comme l'on dit vulgairement, je tiens qu'elle l'affecte par-tout, pour mieux marquer les perfections de son Auteur. (*Leibniz, Opera omnia studio Ludov. Dutens, Tom II part 1. p. 243.*)

Leipzig,
bei C. H. Reclam sen.
1851

Title page of Bolzano's work "Paradoxien des Unendlichen" (1851) years 1820–1829.[11]) This was characteristic of Bolzano, for what he missed most in all branches of science were the general rules according to which individual statements should be arranged so as to facilitate their proper and convincing interpretation and proof. And, although Bolzano considered mathematics an important tool for an exercise in correct thinking, he often found fault with the ways it was presented. He tried to improve the situation and remove the defects. For example, he refused to regard intuition based on graphical representation as a method of demonstration. In his preface to "Rein analytischer Beweis ..." [9, p. 6] he said: "The most current proofs make use of truths taken over from geometry... There are no possible objections whatsoever against the correctness nor against the clarity of the geometrical proposition. However, it is equally evident that it contradicts the correctness of the method if the truths of the pure (or general) mathematics (that is, arithmetic, algebra or analysis) are deduced from considerations which apply only to its special (or applied) part, namely, to geometry ... If we accept the view that scientific proofs should not be mere assurances but, contrariwise, substantiations, that is, presentations of objective reasons that corroborate the truth sought, then we see that a pure scientific proof, or the objective basis of a certain truth that holds for all quantities ... cannot be contained in a truth that holds only for spatial quantities ... For even if geometric truth we are referring to ... is most evident and thus requires no proof to assure us, it needs substantiation anyhow. It is evident indeed that it consists of notions too complicated for any admission that they might belong to those simple truths which we call fundamental propositions or fundamental truths, for they are fundamentals but not consequences of other propositions; hence it is a consequential proposition or truth, that is, such a truth that has its roots in some other truths, and hence even in science must be deduced from them as their result."[12])

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[11]) According to his diary ("Adversaria"), Bolzano engaged in logical studies already in 1812 when he wanted to publish a treatise "Attempt at a new logic, which shall cause a revolution in all sciences". In 1814 he wrote "Logische Vorbegriffe" [5] and soon after he completed the manuscript "Etwas aus der Logik" [7], which appeared for the first time in Czech version in the journal Krok (1831, pp. 553—578).

[12]) In Part 4 of "Wissenschaftslehre" he analogously pointed out how Kant's Critique of Pure Reason could in the same direction harmfully affect the scientific construction of mathematics: "For, if already earlier the mathematicians were inclined to refer in their proofs ... to facts which can be seen by simply looking at a figure, now they are beginning to believe ... that they have full right to proceed in this way ... However, if the exposition ... is to attain the highest scientific rigour, then I consider it my duty to conclude nothing from a simple look at a figure, from the so-called observation ..., but simply to proceed in the same way as when proving purely philosophical truths. The fact that this was often not achieved and that most of those who have attempted to prove mathematical propositions philosophically have obtained just rubbish does not prove that it is impossible."

2.^tens die Gleichung $x^3 - Ax + B = 0$ mit der

$$x^3 - 3mnx - mn(m+n) = 0,$$

welche mit $x = \sqrt[3]{m^2n} + \sqrt[3]{mn^2}$ hervorgeht, identificirt, und m und n durch A und B ausdrückt, nämlich

$$mn = \frac{A}{3}\ , \quad m+n = \frac{B}{3A} \quad \text{u.s.w.}$$

3.^tens die Gleichung $x^4 - Ax^2 - Bx - C = 0$ mit

(α) ..... $x^4 - 4(mn+mp+np)x^2 - 4mnpx - mnp(m+n+p) = 0$

vergleicht, indem er $mn+mp+np = \frac{A}{4}$, $mnp = \frac{B}{4}$, $m+n+p = \frac{4C}{B}$ setzt, u. nun m, n, p aus der cubischen Gleichung

$$y^3 - \frac{4C}{B}y^2 + \frac{A}{4}y - \frac{B}{4} = 0$$

finden läßt, und endlich voraussetzt, daß es ein Leichtes sey, die Gleichung (α) aufzulösen, wodurch man dann glücklich auch die gegebene Gleichung aufgelöst hätte.

4.^tens ebenso verfährt er mit der Gleichung des 5.^ten Grades

$$x^5 - Ax^3 - Bx^2 - Cx - D = 0,$$ die er gleichsetzt der

(β) ... $x^5 - 5(mn+mp+mq+np+nq+pq)x^3 - 5(mnp+mnq+mpq+npq)x^2$
$- 5mnpqx - mnpq(m+n+p+q) = 0,$

indem er

$$mn+mp+mq+np+nq+pq = \frac{A}{5}$$
$$mnp+mnq+mpq+npq = \frac{B}{5}$$
$$mnpq = \frac{C}{5}$$
$$m+n+p+q = \frac{5D}{C},$$

setzt, wo dann m, n, p, q sich aus der Gleichung des 4.^ten Grades

$$y^4 - \frac{5D}{C}y^3 + \frac{A}{5}y^2 - \frac{B}{5}y + \frac{C}{5} = 0$$

ergeben. Sind aber m, n, p, q gefunden, so, meint er, ist die Gleichung (β), weil sie so symmetrisch ist, leicht auflösbar, und mit ihr die gegebene des 5.^ten Grades.

Und so, meint er, gehe es bey allen höheren Gleichungen fort. Zur Unterstützung dieses gewagten Schlusses bringt er nun einige ganz ungenügende Gründe bey, die von der symmetrischen Gestalt dieser Gleichungen hergenommen sind. Aus den blossen Analogien vom 2ten, 3ten, dritten Grade meint er schließen zu dürfen, daß die Wurzel einer Gleichung vom n.^ten Grade

$x^n + Ax^{n-1} + Bx^{n-2} + \ldots$ die Form

$$x = \frac{A}{n} + \sqrt[n]{W_1} + \sqrt[n]{W_2} + \sqrt[n]{W_3} + \ldots$$

haben müsse; wo $W_1, W_2, W_3, \ldots$ aus A, B, C, ... und Wurzeln einer Gleichung vom Grade (n-1) zusammengesetzt sind. Man sieht von selbst, daß auch dieser Versuch als ein ganz mißlungener anzusehen ist.

Part of Bolzano's review of a competition work sent to the Royal Bohemian Learned Society in 1834

These considerations show at which aim Bolzano directed his energy in the development of logic.[13]) In "Beyträge ..." [2] he dealt primarily with mathematical notions and demonstrated with examples that many of those are only intuitive and require a more exact definition, just as in the development of a theory it is necessary to proceed consistently from simpler notions to the more complex ones. "Simplicity" was understood by Bolzano to mean an objective property of notions and statements and, according to him, the role of a mathematician was to find out notions and propositions that are fundamental for the given domain. Bolzano refused Kant's classification of notions as empirical and a priori and believed that the verity of our knowledge is based on various forms of our experience. (See K. Petr, [77], p. 14.)

Bolzano's aim in "Wissenschaftslehre" was – in analogy to mathematics – to build a logically perfect methodology of presentation of an arbitrary science. In Part 2 he studies the properties of propositions and their relations. For this purpose he employs the notion of variable taken over from mathematics. He introduces the notion of the propositional function as a proposition whose one or more parts are considered variable while the copula remains fixed, being always expressed by the word "has" (hat) in contradistinction to the more current "is". He also discusses the probability of the truth value of a statement that contains a variable idea as its part. Then he employs his results in the theory of deduction as well as in the definition of the analytical proposition as a proposition that is true for any variable idea for which it has sense. One of Bolzano's most valuable results is his introduction of the "relation of deducibility", which in modern terminology means the implication in a sense similar to that introduced later by Carnap [51] and Tarski. This facilitates the formulation of logical equivalence as well as the analysis of the other possible relations between propositions. (Cf. [76], p. 142n.)

Part 3 of Bolzano's work is devoted to the methods of discovering the truth, to the theory of questions and to the theory of definition. The last subject appears again in the next part, where the theory of proofs is dealt with as well.

The reasons why there were no immediate continuators of this work consist undoubtedly in the considerable extent of the book (2397 pages), in its insufficient publicity as well as in Bolzano's isolation and in the different philosophical background of the work. On the other hand, the fact reflects also the logic of the development of this branch of science, in which Bolzano's conception historically preceded the epoch of English logicians Boole

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[13]) Also in his letter to J. P. Romang from 1 May, 1847 he wrote: "If I should indicate ... the points in which both my philosophical and theological notions differ from [the conceptions of the] others, then I should say that it is in the fact that more carefully than was the general practice I strive to make all that I consider as clear and evident to myself as possible ... it is only through exact definitions of notions that I attained all those original theories and ideas you meet in my writings (including the mathematical ones). If anybody desires to understand and judge the reasons on which these statements, original as they are, rest, he necessarily must start from Logic (that is, Wissenschaftslehre)." Cf. [46].

[48, 49] and de Morgan [74] with their algebra of logic, developing, however, on an intuitively semantic level characteristic only of the 30's of this century; in the latter period Bolzano was again mentioned and referred to. (Cf., e.g., [53].)

Mathematical analysis

In the introduction to the preceding section we recalled the reasons why Bolzano devoted his main effort to the development of logic. At the same time, in his argumentation we can clearly see a new, consciously arithmetical approach to the foundations. It was this approach that enabled him – and in the same period also Cauchy [52] – to overcome the problems that the old Eulerian conception was unable to solve.

This is already seen in his early mathematical papers from the years 1816–1817. Here we find exact definitions of the limit, derivative, continuity of a function, of the interval and the greatest lower bound. But even "Grössenlehre", which was written in the years 1833–1841, considerably extended Bolzano's significant results from the theory of functions. Here the definition of continuity was given in a more exact form, the concept of right and left continuity was introduced, the closed and open intervals were clearly distinguished, the notions of the right and left derivatives were introduced together with their consequences and the theorem that a function continuous in a closed interval is bounded in it was established. Moreover, here Bolzano rejected the Eulerian identification of the function with an analytic expression, introducing the notion of the function in terms of mutual correspondence of values from two sets. This enabled Bolzano to construct "geometrically unrepresentable" functions with complicated properties. The most interesting example is the construction of a function defined and continuous on an interval, which is not monotone in any subinterval. Further, Bolzano proved that in spite of its continuity, the function "has no derivative for so many values of its variable that between any two of them there is another one for which it has no derivative either". (Cf. [20].) Already in "Rein analytischer Beweis ..." [9, §§ 6, 7] Bolzano formulated the necessary and sufficient condition for the convergence of a sequence (the Bolzano-Cauchy condition of convergence) but, as he could not base his arguments on the theory of real numbers, he did not succeed in proving its sufficiency. It is probable that it was such problems that in 1830–1835 led Bolzano to his attempt to create an arithmetic theory of real numbers in his work "Unendliche Zahlen-(Grössen)-begriffe (-ausdrücke)" [15]. Here Bolzano anticipated the method of Cantor's fundamental sequences (1869). Among other theorems, he proved here that the set of real numbers is everywhere dense and reconsidered the necessary and sufficient condition of convergence (§ 102). The theorem on the greatest lower bound had been formulated already in "Rein analytischer Beweis" [9, § 13]; here Bolzano dealt with it again and explicitly formulated the theorems on supremum and infimum. These theorems were necessary for the formulation of a more general assertion, which is included in "Functionenlehre" [20, §§ 19–23]: every bounded infinite sequence has a point of accumulation.

Even this brief summary of the main results shows that Bolzano's work could have served as one of the foundation stones for building up modern mathematical analysis. That nobody used this basis to build on is a paradox of the social development in Bohemia and a consequence of Bolzano's scientific and human isolation, as well as of the general dynamics of the development of mathematics.

General geometrical considerations

Bolzano's approach to mathematical problems was characterized by his ability to find new, non-traditional methods, and to use them to deal with problems that until then had withstood all attempts at solution. This approach manifests itself in geometry as well. Bolzano's first mathematical treatise "Betrachtungen ..." [1] was aimed at the solution of the then popular problem of parallels. It is not essential that Bolzano solved the problem via a very general concept of similarity, but rather that already in this work he subjected to criticism the contemporary (mostly traditionally Euclidean) interpretation of elementary geometry. In Part 2 of his treatise he tried to define the straight line and the plane, starting from and studying the properties of the simplest geometric object, a pair of points. Thus he defined the notion of the direction of a pair of points, its distance, and in essence constructed geometrically the vector space, indicating also its threedimensional analogue [55]. In this way he arrived at a result analogous to that obtained in 1799 by C. Wessel [88] in his geometrical interpretation of complex numbers or later (1844) H. Grassmann [59] in a much more general setting.

In the other works concerning geometry Bolzano desired to grasp the geometric essence of geometric notions and, by means of the methods developed for the arithmetization of analysis, he attained to an almost topological interpretation of geometric notions. This feature appeared already in 1817 in his paper "Die drey Probleme ..." [10], where he defined the notions of the curve, surface and body in a way recalling Menger's definition of the regular curve as a continuum [73]. Bolzano even anticipated Jordan's theorem for simple closed curves, defining the simple closed curve by the following two properties: (i) any point of the curve has exactly two neighbours for sufficiently small distances (in Bolzano's terminology, "neighbour" is a point of the curve at the given distance); (ii) the distance of any two points of the curve does not exceed a certain maximal value.

In the 30's in his manuscript "Versuch einer Erklärung ..." [17], a little later in "Anti-Euclid" [16] and in 1843–1844 in "Über Haltung, Richtung ..." [36] and in "Geometrische Begriffe ..." [37] these problems appeared in a more exact form. In his treatise "Versuch einer Erklärung" [17] he used the notion of a circular neighbourhood introduced by himself to study the notion of the extended spatial object ("ausgedehntes Raumding"), which in his interpretation actually means a point set dense in itself "whose each point is united ('verbunden') with the others, that is, it has neighbours in this object that are as close to it as desired". Immediately he indicated how to define an isolated point.

Explicitly he did so in his work "Über Haltung, Richtung ..." [36, p. 143] and in the same work he formulated (pp. 144–145) the concept of extension ("Ausdehnung") based on the spherical neighbourhood of a point: "An extended spatial object ('Ausgedehntes') whose each point for any arbitrarily small distance has only so many neighbours that their union for any of these distances considered just in itself does not represent any extended spatial object, is called 'an extended spatial object of one or simple extension' ('Ausdehnung') or a curve." (For details see Johnson [68].) Similarly Bolzano introduced the notion of double or triple extension, in this way attaining close to the modern definition of topological dimension.

Also these notions, which the mathematicians started to study only in the 70's of the last century, demonstrate the extraordinary extent of Bolzano's new methodical approaches to the objects of mathematical research, though even Bolzano himself did not exploit all of them in his subsequent studies.

Theory of sets

"Paradoxien des Unendlichen" [38], published in 1851 from Bolzano's legacy by F. Přihonský, is actually Bolzano's single result that found explicit continuators, as was admitted by the creator of the theory of sets G. Cantor who became acquainted with this book before he engaged himself in the theory of sets [50]. In his work Bolzano desired to remove the difficulties arising from the inaccurate definition of the notion of "infinity", which nonetheless was frequently used in mathematics. He discussed the notion of the set which alone can be assigned the property of "finiteness" or "infiniteness". A set is infinite if the "succession of natural numbers that should exhaust all elements of the set has no last term". This method of assignment actually gets close to the modern definition of the equivalence of sets, to the relation between a set and its proper part and so on. Bolzano concluded that from the viewpoint of the quantity of their elements not all infinite sets are equivalent. Thus he got close to the notion of the cardinal number and attempted to develop the "calculus of infinities"; he anticipated the notion of continuum and its properties. We find here again the notion of the set dense in itself, the idea of dimension and so on.

Bernard Bolzano spent all his life in Bohemia and considered it his home. From his several exhortations it is clear that he realized the social roots of the national oppression of the Czechs and, within the framework of his Utopian-humanistic conceptions, he favoured the oppressed, while trying with the same means to check the emerging bourgeois nationalism that raised nationalistic intolerance. Bolzano never left Bohemia even for a short time. The bulk of his scientific legacy remained in Prague. It is just sheer coincidence that his mathematical manuscripts should have been kept abroad. Throughout his life Bolzano hoped that among his students he would find a mathematician who would be able to complete his mathematical endeavour. Since his return to Prague (1841) he placed his hopes in the talented Robert Zimmermann (1824–1898) whom he several times mentioned favourably in his letters. This was the reason why he left his mathematical legacy to him. When Zimmermann became professor of philosophy at Vienna university (1850), he brought Bolzano's mathematical legacy to Vienna with him, where it has remained till now.

The fact that Zimmermann did nothing for the development of Bolzano's legacy represents the last bitterness of Bolzano's fate. Nonetheless, a new interest in Bolzano's mathematical results sprung up in the 20's of this century among Bolzano's compatriots – Czech mathematicians. Let us hope that the bicentennial of Bernard Bolzano will again attract attention to the evaluation of those aspects of his mathematical legacy that have still remained unknown or little known.

JAROSLAV FOLTA

3. From Bolzanian Literature

A) *Bolzano's works referred to in the article*)*

[1] 1804 — Betrachtungen über einige Gegenstände der Elementar-Geometrie. Prague 1804.
[2] 1810 — Beyträge zu einer begründeteren Darstellung der Mathematik. Erste Lieferung. Prague 1810.
[3] after 1810 — Aetiologie, in: Bernard Bolzano — Gesamtausgabe, II.A.5.c. [J. Berg, ed.] Stuttgart — Bad Cannstatt 1977.
[4] 1813 — Erbauungsreden für Akademiker. Prague 1813 (2nd edition Sulzbach 1839).
[5] 1814 — Logische Vorbegriffe, in: Bernard Bolzano — Gesamtausgabe, II.A.5.g. [J. Berg, ed.] Stuttgart — Bad Cannstatt 1977.
[6] 1815 — Versuch einer objectiven Begründung der Lehre von den drei Dimensionen des Raumes. Prague 1843.
[7] after 1815 — Etwas aus der Logik, in: Bernard Bolzano — Gesamtausgabe, II. A.5.i. [J. Berg. ed.] Stuttgart — Bad Cannstatt 1977.
[8] 1816 — Der binomische Lehrsatz, und als Folgerung aus ihm der polynomische, und die Reihen, die zur Berechnung der Logarithmen und Exponentialgrössen dienen, genauer als bisher erwiesen. Prague 1816.
[9] 1817 — Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege. Prague 1817.
[10] Die drey Probleme der Rectification, der Complanation und der Cubirung, ohne Betrachtung des unendlich Kleinen, ohne die Annahme des Archimedes, und ohne irgend eine nicht streng erweisliche Voraussetzung gelöst; zugleich als Probe einer gänzlichen Umstaltung der Raumwissenschaft, allen Mathematikern zur Prüfung vorgelegt. Leipzig 1817.
[11] 1827 — Athanasia oder Gründe für Unsterblichkeit der Seele. Sulzbach 1827 (2nd edition Sulzbach 1838).
[12] Schreiben eines katholischen Geistlichen an den Verfasser des Buches: die Katholische Kirche Schlesiens. Sulzbach 1827.
[13] 1829 — Wissenschaftslehre. Versuch einer ausführlichen und grösstentheils neuen Darstellung der Logik mit steter Rücksicht auf deren bisherige Bearbeiter. Sulzbach 1837.
[14] Bolzano's Wissenschaftslehre und Religionswissenschaft in einer beurtheilenden Uebersicht. Eine Schrift für Alle, die dessen wichtigste Ansichten kennen zu lernen wünschen. Sulzbach 1841.
[15] 1830—35 — Unendliche Zahlen- (Grössen)-begriffe (-ausdrücke), in: Rychlík, K.: Theorie der reellen Zahlen im Bolzanos handschriftlichen Nachlasse. Prague 1962.
[16] after 1830 — Anti-Euclid [K. Večerka, ed.], in: Sborník pro dějiny přírodních věd a techniky 11 (1966), 203—216.
[17] Versuch einer Erklärung der Begriffe von Linie, Fläche und Körper, in: Bernard Bolzano — Gesamtausgabe, II.A.11.c. [J. Berg, E. Morscher, eds.] Stuttgart — Bad Cannstatt.
[18] 1831—46 — Von dem besten Staate [A. Kowalewski, ed.], in: Spisy B. Bolzana, vol. 3, Prague 1930.
[19] 1831 — Ein Vorschlag zur Verbesserung einiger Armenanstalten. Gotha 1831.
[20] 1833—41 — Functionenlehre [K. Rychlík, ed.], in: Spisy B. Bolzana, vol. 1, Prague 1930.

*) The year before an entry indicates the approximate date of the origin of the work.

[21] 1833—41 — Zahlentheorie (Verhältnis der Theilbarkeit unter den Zahlen) [K. Rychlík, ed.], in: Spisy B. Bolzana, vol. 2, Prague 1931.
[22] 1834 — Lehrbuch der Religionswissenschaft, ein Abdruck der Vorlesungshefte eines ehemaligen Religionslehrers einer katholischen Universität, von einigen seiner Schüler gesammelt und herausgegeben. Sulzbach 1834.
[23] Ansichten eines freisinnigen katholischen Theologen über das Verhältnis zwischen Kirche und Staat; entwickelt in einer Kritik der Aphorismen des Herrn A. Gengler über denselben Gegenstand in den dritten Hefte des Jahrganges 1832 der Tübinger Quartalschrift. Sulzbach 1834.
[24] 1835 — Religionsbekenntnisse zweier Vernunftfreunde, nähmlich eines protestantischen [J. F. Röhr] und eines katholischen Theologen [B. Bolzano]. Sulzbach 1835.
[25] 1836 — Lebensbeschreibung des Dr. B. Bolzano mit einigen seiner ungedrückten Aufsätze und dem Bildnisse des Verfassers. Sulzbach 1836.
[26] 1837 — Sendschreiben an Se. Hochwürden Herrn Dr. Joh. Friedr. Röhr, betreffend die aus seiner kritischen Prediger-Bibliothek hier abgedruckte Kritik des Buches: "Religionsbekenntnisse zweier Vernunftsfreunde usw." Sulzbach 1837.
[27] Beleuchtung der in den Göttingischen gelehrten Anzeigen befindlichen Recension des Buches: Lehrbuch der Religionswissenschaft etc. Stuttgart 1837.
[28] Leben Franz Joseph Ritters von Gerstner. Prague 1837.
[28a] 1838 — Ueber das Recht der Geistlichkeit, ihren Lebensunterhalt von Personen zu beziehen, welche nicht ihres Glaubens sind. Eine kirchenrechtliche Abhandlung nach B. Bolzano's Ansichten von einem seiner Schüler bearbeitet. Stuttgart 1838.
[29] 1839 — Dr. Bolzano und seine Gegner. Ein Beitrag zur neuesten Literaturgeschichte. Sulzbach 1839.
[30] 1840 — Prüfung der Philosophie des seligen Georg Hermes von einem Freunde der Ansichten Bolzano's. Sulzbach 1840.
[31] 1840 — Versuche einer objectiven Begründung der Lehre von der Zusammensetzung der Kräfte. Prague 1842.
[32] 1843 — Versuch einer objectiven Begründung der Lehre von den drei Dimensionen des Raumes. Prague 1843.
[33] Über den Begriff des Schönen. Eine philosophische Abhandlung. Prague 1843.
[34] Aufsatz, worin eine von Hern. Exner in seiner Abhandlung: "Über den Nominalismus und Realismus" angeregte logische Frage beantwortet wird. Prague 1843.
[35] Ein Paar Bemerkungen über die neue Theorie in Herrn Professor Ch. Doppler's Schrift: "Ueber das farbige Licht der Doppelsterne und einiger anderer Gestirne des Himmels". Leipzig 1843.
[36] 1843—44 — Über Haltung, Richtung, Krümmung und Schnörkelung bei Linien sowohl als Flächen samt einigen verwandten Begriffen, in: Spisy B. Bolzana, vol. 5, pp. 139—183. Prague 1948.
[37] Geometrische Begriffe, die Jeder kennt und nicht kennt, in: Bernard Bolzano — Gesamtausgabe, II.A.11.d. [J. Berg, E. Morscher, eds.], Stuttgart — Bad Cannstatt.
[38] 1844—45 — Paradoxien des Unendlichen [J. Příhonský, ed.], Leipzig 1851.
[39] 1845 — Ueber die Perfectibilität des Katholicismus. Streitschriften zweier katholischer Theologen; zugleich ein Beitrag zur Aufhellung einiger wichtiger Begriffe aus Bolzano's Religionswissenschaft. Leipzig 1845.
[40] Dr. Vincenz Julius Edler v. Krombholz nach seinem Leben und Wirken. Prague 1845.
[41] 1847 — Christ. Doppler's neueste Leistungen auf dem Gebiete der physikalischen Apparatenlehre, Akustik, Optik und optischen Astronomie. Leipzig 1847.
[42] Untersuchungen auf dem Gebicte der allgemeinen Sprachlehre. Prague 1847.

[43] Vorschläge zur Behebung des unter einem beträchtlichen Theile der Bewohner Prags dermal um sich greifenden Nothstandes. Prague 1847.
[44] Über die Wohlthätigkeit. Dem Wohle der leidenden Menschheit gewidmet von einem Menschenfreunde. Prague 1847.

B) *Works of other authors*

[45] Berg, J.: Bolzano's Logic. Stockholm 1962.
[46] Berka, K. - Prokešová, B.: Bolzano's struggles for the publication and recognition of "Wissenschaftslehre". Filozofický časopis XXVII (1979), 697—725 (Czech).
[47] Beth, E. W.: The philosophy of Mathematics from Parmenides to Bolzano. Antwerp 1944.
[48] Boole, G.: An investigation of the laws of thought on which are founded the mathematical theories of logic and probabilities. London 1854.
[49] Boole, G.: The mathematical analysis of logic. Cambridge 1847.
[50] Cantor, G.: Über unendliche lineare Punktmannigfaltigkeit. Math. Annalen 21 (1883), 51—58, 545—591.
[51] Carnap, R.: Abriss der Logistik. 1929.
[52] Cauchy, A. L.: Cours d'analyse de l'Ecole royale Polytechnique. I^{ère} partie: Analyse algébrique. Paris 1821.
[53] Dubislav, W.: Bolzano als Vorläufer mathematischer Logik, Philosophisches Jahrbuch 44, Fulda 1931, 448—456.
[54] Fels, H.: Bernard Bolzano. Leipzig 1929.
[55] Folta, J.: Bernard Bolzano and the Foundation of Geometry, Acta historiae rerum naturalium necnon technicarum, Spec. Issue 2, Prague 1966, 75—104.
[56] Frint, J.: Handbuch der Religionswissenschaft für die Candidaten der Philosophie I—III. Wien — Triest 1806—1813.
[57] Grattan-Guinness, I.: Had Cauchy read Bolzano before writing Cours d'analyse? Pokroky MFA XV (1970), 133—137 (Czech).
[58] Grattan-Guinness, I.: The development of the foundations of mathematical analysis from Euler to Riemann. MIT Press, Cambridge—London 1970.
[59] Grassmann, H.: Die Ausdehnungslehre aus 1844 oder Die lineale Ausdehnungslehre. Leipzig 1878.
[60] Hankel, H.: "Grenze", in: Allgemeine Enzyklopädie [Ersch, Gruber]. Leipzig 1871.
[61] Jarník, V.: On Bolzano's function, the present volume, 67—81.
[62] Jarník, V.: Bolzano's Functionenlehre, the present volume, 43—66.
[63] Jarník, V.: Bernard Bolzano and the foundations of mathematical analysis, the present volume, 33—42.
[64] Jarník, V.: Bernard Bolzano (October 5, 1781—December 18, 1848), the present volume, 82—86.
[65] Jašek, M.: Aus dem handschriftlichen Nachlass Bernard Bolzanos. Das erste historischnachweisbare Beispiel einer stetigen nirgends differenzierbaren Funktion. Enthalten in dem Manuskripte Bolzano's "Functionenlehre" — Nationalbibliothek in Wien. Věstník Královské české společnosti nauk, tř. mat. př. 1 (1921), 1—32.
[66] Jašek, M.: Bolzano's function. From his unpublished manuscripts in the former court library in Vienna. Časopis pro pěstování matematiky a fysiky 51 (1922), 69—76 (Czech).
[67] Jašek, M.: On functions with infinitely many oscillations in the manuscripts of Bernard Bolzano. An appendix to the report on Bolzano's function. Časopis pro pěstování matematiky a fysiky 53 (1924), 102—110 (Czech).

[68] Johnson, D.: Prelude to Dimension Theory: The Geometrical Investigations of Bernard Bolzano. Archive for History of Exact Sciences, Vol. 17, No. 3 (1977), 261—295.
[69] Kästner, A. G.: Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie und Perspective. I.Th., I.Abtl., 5.Aufl. Göttingen 1792.
[70] Křivský, P.: Jan Křtitel Bolzano. Literary legacy. Památník národního písemnictví. Praha 1972 (Czech).
[71] Laugwitz, D.: Bemerkungen zu Bolzano's Grössenlehre, Archive for History of Exact Sciences 2 (1965), 398—409.
[72] Loužil, J.: Bernard Bolzano. Prague 1978 (Czech).
[73] Menger, K.: Grundzüge einer Theorie der Kurven. Mathematische Annalen 95 (1925), 277—306.
[74] Morgan, A. de: Formal logic, or the calculus of inference necessary and probable. London 1847.
[75] Nový, L.: Bernard Bolzano (1781—1848), in: [Wussing, Arnold, eds.] Biographien bedeutender Mathematiker. Berlin 1975.
[76] Nový, L. et al.: History of Exact Sciences in Bohemia till the end of the 19th century. Prague 1961 (Czech).
[77] Petr, K.: Bernard Bolzano and his significance for Mathematics. Prague 1926 (Czech).
[78] Rootselaar, B. van: Bernard Bolzano, in: Dictionary of Scientific Biography, Vol. II, New York 1967, 273—279.
[79] Rootselaar, B. van: Bolzano's Corrections to his Functionenlehre, in: Janus 50 (1969), 1—21.
[80] Rootselaar, B. van: Bolzano's theory of real numbers. Archive for History of Exact Sciences 2 (1962—66), 168—180.
[81] Rychlík, K.: La théorie de fonctions de Bolzano, in: 6. Congresso internationale dei matematici. Bologna, 3.—10. settembre 1928. Argumenti delle communicazioni, 105.
[82] Rychlík K.: Theory of real numbers in Bernard Bolzano's manuscript legacy. Časopis pro pěstování matematiky 81 (1956), 391—395 (Czech); reprinted in Istoriko-matematičeskie issledovania 11 (1958), 515—532 and in Revue d'histoire des sciences et leurs applications 14 (1961), 313—327.
[83] Rychlík, K.: Considerations in logic in Bolzano's manuscript legacy. Časopis pro pěstování matematiky 83 (1958), 230—235; reprinted in Czechoslovak Mathematical Journal 8 (1958), 197—202.
[84] Slavík, B.: From Dobner to Dobrovský. Prague 1975 (Czech).
[85] Schwarz, H. A.: Zur Integration der partiellen Differentialgleichung. Journal für die reine und angewandte Mathematik 74 (1872), 218—253.
[86] Stolz, O.: Bolzano's Bedeutung in der Geschichte der Infinitesimalrechnung. Mathematische Annalen XVIII (1881), 255—279.
[87] Veselý, F.: Life of Bernard Bolzano and his works in mathematics and natural sciences. Pokroky MFA II (1957), 119—127, 234—243 (Czech).
[88] Wessel, C.: Om Directionens analytiske Betegning, in: Nye Samling af det Kongelige Danske Videnskarbernes Selskabs Skrifter V (1799).
[89] Winter, E.: Bernard Bolzano, Ein Lebensbild, in: Bernard Bolzano — Gesamtausgabe. Einleitungsband, I. T. Biographie. Stuttgart — Bad Cannstatt.
[90] Wussing, H.: Bernard Bolzano und die Grundlegung der Infinitesimalrechnung. Zeitschrift für Geschichte der Naturwissenschaft und Medizin 1 (1964), 57—73.