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REPRESENTATION OF UNI-NULLNORMS AND NULL-UNINORMS ON BOUNDED LATTICES

Yi-Qun Zhang, Ya-Ming Wang and Hua-Wen Liu

In this paper, we present the representation for uni-nullnorms with disjunctive underlying uninorms on bounded lattices. It is shown that our method can cover the representation of nullnorms on bounded lattices and some of existing construction methods for uni-nullnorms on bounded lattices. Illustrative examples are presented simultaneously. In addition, the representation of null-uninorms with conjunctive underlying uninorms on bounded lattices is obtained dually.

Keywords: bounded lattice, uni-nullnorm, null-uninorm, uninorm, T-norm

Classification: 03E72, 03B52, 03G10, 18B35

1. INTRODUCTION

Serving as the union and intersection in fuzzy sets theory, t-norms and t-conorms on the real unit interval $[0,1]$ were introduced by Menger [10], and they have been employed in abundant areas. Unifying t-norms and t-conorms that always fix their neutral elements at one or zero, uninorms were introduced by Yager and Rybalov in [20] via allowing the neutral elements to lie anywhere in $[0,1]$. The fact that uninorms are combinations of t-norms and t-conorms enables uninorms to be important in both theory and applications including decision making [21], fuzzy system modeling [19] and so on. Another essential generalizations of t-norms and t-conorms are nullnorms [4], which let the zero elements to lie anywhere in $[0,1]$. Nullnorms also play important roles in applications, such as in expert systems, fuzzy sets theory, neural networks and so on [5, 7]. Uninorms and nullnorms can be brought together likewise, which are called 2-uninorms [1]. A particular case of 2-uninorms named uni-nullnorms was introduced by Sun et al. [12, 13]. And the dual functions of uni-nullnorms are called null-uninorms. Since then, many investigations about uni-nullnorms have arisen [11, 15, 24].

Recently, many researchers have focused on extending these aggregation functions from $[0,1]$ to bounded lattices. This kind of extensions, such as t-norms [2], uninorms [3], nullnorms [9] and uni-nullnorms [16] are significant research activities because of the wider applicability for practical applications of bounded lattices. The existence of uni-nullnorms on bounded lattices is shown by Wang et al. in [16]. Ertuğrul et al. [6] pro-

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posed two new methods to generate uni-nullnorms on bounded lattices. Null-uninorms on bounded lattices were studied simultaneously. Based on t-norms and disjunctive uninorms, Zhang et al. [22] presented a construction approach for uni-nullnorms on bounded lattices. Two concrete methods for constructing uni-nullnorms on bounded lattices were presented afterwards by employing several existing construction methods of disjunctive uninorms on bounded lattices, and similar methods for obtaining null-uninorms on bounded lattices were given. Wu et al. [18] constructed the largest and smallest uni-nullnorms on bounded lattices. And Wang et al. [17] provided an approach to construct uni-nullnorms on bounded lattices by applying uninorms and beam operations.

In this paper, we aim to continue the study of uni-nullnorms and null-uninorms on bounded lattices. From the aforementioned contributions, only some constructions have been given, which motivates us to study further on this topic. Thanks to the good structure of uni-nullnorms with disjunctive underlying uninorms on bounded lattices, we obtain the representation of them. We provide the representation in two steps: first we present the representation of uni-nullnorms with disjunctive underlying uninorms whose values are comparable with the zero element, and show that several existing construction methods are covered by this representation; then we present the complete representation for uni-nullnorms with disjunctive underlying uninorms. The representation obtained in this article extend the representation of nullnorms on bounded lattices proposed by Sun and Liu [14] and Zhang et al. [23] and several illustrative examples are provided. Dually, the representation of null-uninorms with conjunctive underlying uninorms can be obtained simultaneously.

This article contains five sections. In Section 2, we recall some basic notions of aggregation functions on bounded lattices, including t-norms, uninorms, nullnorms, uni-nullnorms and null-uninorms. In Section 3, we present the representation for uni-nullnorms with disjunctive underlying uninorms on L in two steps. And two examples are given. In Section 4, we obtain the representation for null-uninorms with conjunctive underlying uninorms on bounded lattices dually. In Section 5, we conclude the results in this article and provide some future perspectives.

2. PRELIMINARIES

In this section, we first recall some basic concepts including uni-nullnorms and null-uninorms on bounded lattices. In addition, relative properties are listed afterwards.

Throughout this paper, we always assume that $(L, \leq, \vee, \wedge, 1, 0)$ is a bounded lattice with the top element 1 and the bottom element 0. Let $u, v \in L$, the notation $u \parallel v$ denotes that u is *incomparable* with v. The notation $u \nparallel v$ means that u is *comparable* with v. And we denote $I_u = \{x \in L \mid x \parallel u\}$, $I_u^v = \{x \in L \mid x \parallel u \textit{ and } x \nparallel v\}$ and $I_{u,v} = \{x \in L \mid x \parallel u \textit{ and } x \parallel v\}$. A *subinterval* $[u, v]$ of L is defined as $[u, v] = \{x \in L \mid u \leq x \leq v\}$. Similarly, $]u, v] = \{x \in L \mid u < x \leq v\}$, $[u, v[= \{x \in L \mid u \leq x < v\}$ and $]u, v[= \{x \in L \mid u < x < v\}$. A subset $P \in L$ is called an upper set (resp. lower set) if, for all $u, v \in L$, $u \in P$ and $u \leq v$ (resp. $u \geq v$) imply $v \in P$.

Note that an order relation $\leq$ on a set P naturally induces an order relation on P^2, denoted by $\preceq$: $(x, y) \preceq (x_0, y_0)$ if and only if $x \leq x_0$ and $y \leq y_0$.

Definition 2.1. (Bedregal et al. [2]) If a binary function T (resp. S) : $L^2 \to L$ satisfies the associativity, commutativity, increasingness with respect to each variable and has a neutral element 1 (resp. 0), then it is called a *t-norm* (resp. *t-conorm*).

Definition 2.2. (Karaçal and Mesiar [9]) If a binary function $U : L^2 \to L$ satisfies the associativity, commutativity, increasingness with respect to each variable and has a neutral element $e \in L$, then it is called a *uninorm*.

For all $x \in L$, a uninorm $U : L^2 \to L$ is called *conjunctive* if $U(0, x) = 0$, while it is called *disjunctive* if $U(1, x) = 1$.

Definition 2.3. (Karaçal, Ince, and Mesiar [8]) If a binary function $V : L^2 \to L$ satisfies the associativity, commutativity, increasingness with respect to each variable and has a zero element $a \in L$ such that $V(0, x) = x$ for all $x \in [0, a]$ and $V(1, x) = x$ for all $x \in [a, 1]$, then it is called a *nullnorm*.

Definition 2.4. (Wang et al. [16]) If a binary function $K : L^2 \to L$ satisfies the associativity, commutativity, increasingness with respect to each variable and there exist two elements $e, a \in L$ with $e < a$ such that $K(e, x) = x$ for all $x \in [0, a]$ and $K(1, x) = x$ for all $x \in [a, 1]$, then it is called a *uni-nullnorm*.

We can easily observe that e is a neutral element of K on $[0, a]$ and a is a zero element of K on $[e, 1]$. For convenience, we call e the neutral element and a the zero element of K.

For $e, a \in L$ with $e < a$, let $\mathcal{UN}$ denote the family of all uni-nullnorms on L with neutral element e and zero element a. In addition, for a uni-nullnorm $K \in \mathcal{UN}$, if $e = 0$ then K is a nullnorm, and if $a = 1$ then K is a uninorm.

Dually, the definition of null-uninorms on bounded lattices are proposed by Zhang et al. [22].

Definition 2.5. (Zhang et al. [22]) If a binary function $R : L^2 \to L$ satisfies the associativity, commutativity, increasingness with respect to each variable and there exist two elements $a, e \in L$ with $a < e$ such that $R(0, x) = x$ for all $x \in [0, a]$ and $R(e, x) = x$ for all $x \in [a, 1]$, then is called a *null-uninorm*.

For $a, e \in L$ with $a < e$, let $\mathcal{NU}$ denote the family of all null-uninorms on L with neutral element e and zero element a.

Next, we list some useful properties of uni-nullnorms on bounded lattices.

Proposition 2.6. (Wang et al. [16]) Let $e, a \in L \setminus \{0, 1\}$ with $e < a$ and $K \in \mathcal{UN}$. Then

(1) $U^\star$ is a uninorm on $[0, a]$, where $U^\star = K|_{[0,a]^2}$.
(2) $V^\star$ is a nullnorm on $[e, 1]$, where $V^\star = K|_{[e,1]^2}$.
(3) $T_1^\star$ is a t-norm on $[0, e]$, where $T_1^\star = K|_{[0,e]^2}$.
(4) $S^\star$ is a t-conorm on $[e, a]$, where $S^\star = K|_{[e,a]^2}$.
(5) $T_2^\star$ is a t-norm on $[a, 1]$, where $T_2^\star = K|_{[a,1]^2}$.

In this paper, we call $U^\star = K|_{[0,a]^2} : [0,a]^2 \to [0,a]$ the *underlying uninorm* of auni-nullnorm K.

Proposition 2.7. (Wang et al. [16]) Let $e, a \in L \setminus \{0,1\}$ with $e < a$ and $K \in \mathcal{UN}$. Then

(1) $K(u,v) = a$ for $(u,v) \in [e,a] \times [a,1] \cup [a,1] \times [e,a]$.
(2) $K(u,v) \le a$ for $(u,v) \in [0,a] \times L \cup L \times [0,a]$.
(3) $K(u,v) \ge a$ for $(u,v) \in [a,1] \times [e,1] \cup [e,1] \times [a,1]$.

3. REPRESENTATION OF UNI-NULLNORMS WITH DISJUNCTIVE UNDERLYING UNINORMS

We denote all the uni-nullnorms in $\mathcal{UN}$ with disjunctive underlying uninorms by $\mathcal{UN}_\mathcal{D}$.

Proposition 3.1. Let $e, a \in L$ with $e < a < 1$ and $K \in \mathcal{UN}_\mathcal{D}$. Then

(1) $K(x,y) \ge a$ for $(x,y) \in [a,1] \times L \cup L \times [a,1]$.
(2) $K(x,y) \le a$ for $(x,y) \in [0,a] \times L \cup L \times [0,a]$.
(3) $K(x,y) \le (x \vee a) \wedge (y \vee a)$ for $(x,y) \in I_a^2$.

Moreover, $K(x,y) = a$ for $(x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a]$.

Proof.

(1) Since $K \in \mathcal{UN}_\mathcal{D}$, then $a = K(0,a) \le K(x,y) \le K(a,1) = a$ for all $(x,y) \in [0,a] \times [a,1]$. Besides, if $I_{e,a} \ne \emptyset$, then for $(x,y) \in I_{e,a} \times [a,1]$, $K(x,y) \ge K(0,a) = a$. Thus, together with the result in Proposition 2.7 (3), we have $K(x,y) \ge a$ for $(x,y) \in L \times [a,1]$. The commutativity of K ensures that $K(x,y) \ge a$ for $(x,y) \in [a,1] \times L$.

(2) Directly from Proposition 2.7 (2).

(3) For $(x,y) \in I_a^2$, $K(x,y) \le K(x \vee a, y \vee a) \le (x \vee a) \wedge (y \vee a)$.

Moreover, from (1) and (2), we have $K(x,y) = a$ for $(x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a]$ immediately. □

Notice that from Proposition 3.1, for a uni-nullnorm $K \in \mathcal{UN}_\mathcal{D}$, if one variable of K is comparable with the zero element a, then the value of K is comparable with a, in other words, only when $(x,y) \in I_a^2$ can $K(x,y) \parallel a$ happen. In the following, we will provide the representation of uni-nullnorms with disjunctive underlying uninorms on bounded lattices in two steps.

Denote that $\mathcal{UN}_{\mathcal{D}c} = \{K \in \mathcal{UN}_\mathcal{D} \mid K(x,y) \nparallel a, \forall x,y \in L\}$. In the following, we first present the representation for uni-nullnorms in $\mathcal{UN}_{\mathcal{D}c}$ in terms of two order-preserving maps, a disjunctive uninorm and a t-norm.

Theorem 3.2. Let $e, a \in L$ with $e < a < 1$ and K be a binary function on L. Then $K \in \mathcal{UN}_{\mathcal{D}c}$ if and only if there exist two order-preserving functions $f : L \to [a, 1]$ and $g : L \to [0, a]$, a disjunctive uninorm U on $[0, a]$ with neutral element e and a t-norm T on $[a, 1]$ such that

$$K(x,y) = \begin{cases} U(g(x), g(y)), & (x,y) \in N \\ T(f(x), f(y)), & (x,y) \in M, \end{cases} \tag{1}$$

where

$f(x) = x$ for all $x \in [a, 1]$;
$g(x) = x$ for all $x \in [0, a]$;
$M = ([a, 1] \times L) \cup (L \times [a, 1]) \cup A$ and $A = \{(x, y) \in I_a^2 \mid U(g(x), g(y)) = a\}$;
$N = ([0, a] \times L) \cup (L \times [0, a]) \cup B$ and $B = \{(x, y) \in I_a^2 \mid T(f(x), f(y)) = a\}$;
$A \cup B = I_a^2$.

Proof. *Necessity.* Let $K \in \mathcal{UN}_{\mathcal{D}c}$. Define f, g, U, T, A_0, B_0 as follows:

$f(x) = K(x, 1)$ for all $x \in L$;
$g(x) = K(x, e)$ for all $x \in L$;
$U = K|_{[0,a]^2}$, $T = K|_{[a,1]^2}$;
$A_0 = \{(x, y) \in I_a^2 \mid K(x, y) \geq a\}$;
$B_0 = \{(x, y) \in I_a^2 \mid K(x, y) \leq a\}$.

Since U is a disjunctive uninorm, i. e., $K(0, a) = U(0, a) = a$, then $f(x) = K(x, 1) \geq K(0, a) = a$ for all $x \in L$, T is a t-norm and $f(x) = K(x, 1) = T(x, 1) = x$ for all $x \in [a, 1]$. Besides, $K(x, e) \leq a$ for all $x \in L$ and $g(x) = x$ for all $x \in [0, a]$. From the increasingness of K, we know that f and g are both order-preserving. Since $K \in \mathcal{UN}_{\mathcal{D}c}$, then we have $A_0 \cup B_0 = I_a^2$.

If $K(x, y) \geq a$, i. e., for all $(x, y) \in ([a, 1] \times L) \cup (L \times [a, 1]) \cup A_0$, by the associativity we have

$$\begin{aligned} K(x,y) &= K(K(x,y),1) = K(x, K(y,1)) \\ &= K(x, f(y)) = K(x, K(f(y), 1)) = T(f(x), f(y)) \end{aligned} \tag{2}$$

and

$$a = K(K(x,y),e) = K(x,K(y,e)) = K(x,g(y)) = K(x,K(g(y),e)) = U(g(x),g(y)). \tag{3}$$

If $K(x, y) \leq a$, i. e., for all $(x, y) \in ([0, a] \times L) \cup (L \times [0, a]) \cup B_0$, similarly we have

$$\begin{aligned} K(x,y) &= K(K(x,y),e) = K(x, K(y,e)) \\ &= K(x, g(y)) = K(x, K(g(y), e)) = U(g(x), g(y)) \end{aligned} \tag{4}$$

and

$$a = K(K(x,y),1) = K(x,K(y,1)) = K(x,f(y)) = T(f(x),f(y)). \tag{5}$$

Next, we show that $A = A_0$ and $B = B_0$, where $A = \{(x, y) \in I_a^2 \mid U(g(x), g(y)) = a\}$ and $B = \{(x, y) \in I_a^2 \mid T(f(x), f(y)) = a\}$. Formula (3) implies that $A_0 \subseteq A$. If $A \setminus A_0 \neq \emptyset$, then there exist some $(x, y) \in A \setminus A_0$ such that $U(g(x), g(y)) = a$ and $K(x, y) < a$. However, if $K(x, y) < a$, then $K(x, y) = U(g(x), g(y)) < a$ from formula (4), which contradicts $U(g(x), g(y)) = a$. Therefore, we have $A \setminus A_0 = \emptyset$, i. e., $A = A_0$. And $B = B_0$ can be proven similarly.

To sum up, K can be represented by (1) and all the conditions are met.

Sufficiency. Let f and g be two order-preserving functions, U be a disjunctive uninorm on $[0,a]$ with neutral element e, T be a t-norm on $[a,1]$ such that K is given by (1) and the conditions are satisfied. First, the function K is well-defined since for all $(x,y) \in A \cap B$, $U(g(x),g(y)) = T(f(x),f(y)) = a$. For $(x,y) \in A$ and $(x_0,y_0) \in I_a^2$ with $(x,y) \preceq (x_0,y_0)$, then it follows that $U(g(x_0),g(y_0)) \geq U(g(x),g(y)) = a$ since g is order-preserving. Besides, since U is a disjunctive uninorm on $[0,a]$, then $U(g(x_0),g(y_0)) = a$, which implies that $(x_0,y_0) \in A$. So A is an upper set. And the fact that B is a lower set can be obtained similarly.

Now we prove that K is a uni-nullnorm. Obviously, K is commutative, e is the neutral element on $[0,a]$ and $K(x,1) = T(x,1) = x$ for all $x \in [a,1]$. Besides, we have $g(x) = a$ for all $x \in [a,1]$ and $f(x) = a$ for all $x \in [0,a]$.

Next, we verify the increasingness of K, i.e., $K(x,y) \leq K(x,z)$ for all $x,y,z \in L$ with $y \leq z$. If (x,y) and (x,z) belong to the identical set between M and N, the increasingness holds immediately. Hence we only need to consider the case when (x,y) and (x,z) belong to distinct sets. Since M is an upper set and N is a lower set, there is only one possible case: $(x,y) \in N$ and $(x,z) \in M$. In this case, we have $K(x,y) = U(g(x),g(y)) \leq a \leq T(f(x),f(z)) = K(x,z)$.

Finally, we consider the associativity. First, we show that $f(K(x,y)) = T(f(x),f(y))$ holds for all $x,y \in L$. Noticing that $f(x) = T(f(x),1) = T(f(x),f(1)) = K(x,1)$ for all $x \in L$, then for all $x,y \in L$, $f(K(x,y)) = K(K(x,y),1) = K(K(x,y),K(1,1)) = K(K(x,1),K(y,1)) = K(f(x),f(y)) = T(f(x),f(y))$. Similarly, we have $g(K(x,y)) = U(g(x),g(y))$ for all $x,y \in L$. Then we verify the associativity of K, i.e., we need to prove $K(K(x,y),z) = K(x,K(y,z))$ for all $x,y,z \in L$. If $(K(x,y),z)$ and $(x,K(y,z))$ belong to the identical set between M and N, the associativity holds clearly. Thus we only need to consider the case when $(K(x,y),z)$ and $(x,K(y,z))$ belong to different sets and the commutativity of K implies that there is only one feasible case: $(K(x,y),z) \in M$ and $(x,K(y,z)) \in N$. In this case, we have $K(K(x,y),z) = T(f(K(x,y)),f(z)) = T(T(f(x),f(y)),f(z)) = T(f(x),T(f(y),f(z))) = T(f(x),f(K(y,z))) = a$ and $K(x,K(y,z)) = U(g(x),g(K(y,z))) = U(g(x),U(g(y),g(z))) = U(U(g(x),g(y)),g(z)) = U(g(K(x,y)),z) = a$. Thus, it follows that $K(K(x,y),z) = K(x,K(y,z))$.

To sum up, $K \in \mathcal{UN}_{\mathcal{D}c}$. □

Example 3.3. Let $L = \{0,a,b,c,d,e,1\}$ be the lattice shown in Fig. 1, g and f be order-preserving functions given as Table 1 and 2, respectively.

x	0	e	a	b	c	d	1
$g(x)$	0	e	a	a	a	a	a

Tab. 1. The function g in Example 3.3.

x	0	e	a	b	c	d	1
$f(x)$	a	a	a	1	1	1	1

Tab. 2. The function f in Example 3.3.

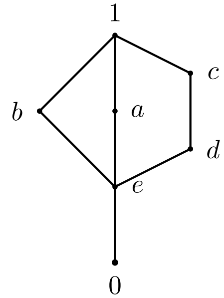

Fig. 1. The lattice L in Example 3.3.

Notice that it can be defined just one disjunctive uninorm U with the neutral element e on $[0, a]$ and also just one t-norm T on $[a, 1]$ considering the lattice L characterized by Hasse diagram in Fig. 1. We can construct a uni-nullnorm K by Theorem 3.2 and the structure of the constructed uni-nullnorm K is shown in Table 3.

$K(x,y)$	0	e	a	b	c	d	1
0	0	0	a	a	a	a	a
e	0	e	a	a	a	a	a
a	a	a	a	a	a	a	a
b	a	a	a	1	1	1	1
c	a	a	a	1	1	1	1
d	a	a	a	1	1	1	1
1	a	a	a	1	1	1	1

Tab. 3. The uni-nullnorm K in Example 3.3.

Remark 3.4.

(i) In the representation of a uni-nullnorm K as in formula (1), U, T, f and g are uniquely determined by K from the proof of Theorem 3.2.

(ii) In Theorem 3.2, if we take

$$g(x) = \begin{cases} x, & x \in [0, a] \\ a, & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} x, & x \in [a, 1] \\ a, & \text{otherwise,} \end{cases}$$

then the structure of uni-nullnorm K on L coincides with that in Theorem 3.1 [22] and in Corollary 2 [17]. And in this case, if we let the uninorm U be the disjunctive uninorm U_d constructed in [3], then the structure of uni-nullnorm K on L coincides with that in Theorem 3.3 [22]. Theorem 3.4 in [22] can be obtained by Theorem 3.2 similarly.

(*iii*) In Theorem 3.2, if we take

$$g(x) = \begin{cases} x, & x \in [0,a] \\ a, & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} x, & x \in [a,1] \\ x \vee a, & \text{otherwise,} \end{cases}$$

then $I_a^2 = A$ and the structure of uni-nullnorm K on L coincides with that in Theorem 4.1 [18] and in Corollary 1 [17].

(*iv*) If the neutral element of the underlying uninorm U is zero, i. e., $e = 0$, then Theroem 3.2 coincides with Theorem 3.2 in [14] and Theorem 3.8 in [23].

Based on the Theorem 3.2 and noticing that only when $x, y \in I_a$ can $K(x,y) \parallel a$ happen, we can provide the representation of uni-nullnorms in $\mathcal{UN}_\mathcal{D}$ by dividing I_a^2 into three subsets.

Theorem 3.5. Let $e, a \in L$ with $e < a < 1$ and K be a binary function on L. Then $K \in \mathcal{UN}_\mathcal{D}$ if and only if there exist three symmetric sets A, B, $C \in I_a^2$, two order-preserving functions $f : L \to [a,1]$ and $g : L \to [0,a]$, an increasing and commutative function $H : C \to I_a$, a disjunctive uninorm U on $[0,a]$ with neutral element e and a t-norm T on $[a,1]$ such that

$$K(x,y) = \begin{cases} U(g(x), g(y)), & (x,y) \in N \\ T(f(x), f(y)), & (x,y) \in M \\ H(x,y), & (x,y) \in C, \end{cases} \tag{6}$$

where
$f(x) = x$ for all $x \in [a,1]$;
$g(x) = x$ for all $x \in [0,a]$;
$M = ([a,1] \times L) \cup (L \times [a,1]) \cup A$ is an upper set, and $(x,y) \in A$ implies $U(g(x), g(y)) = a$;
$N = ([0,a] \times L) \cup (L \times [0,a]) \cup B$ is a lower set, and $(x,y) \in B$ implies $T(f(x), f(y)) = a$;
$C = I_a^2 \setminus (A \cup B)$;
$U(g(x), g(y)) \le H(x,z)$ for any $(x,y) \in ([0,a] \times L) \cup (L \times [0,a]) \cup B$, $(x,z) \in C$ with $y \le z$;
$T(f(x), f(z)) \ge H(x,y)$ for any $(x,y) \in C$, $(x,z) \in ([a,1] \times L) \cup (L \times [a,1]) \cup A$ with $y \le z$;
if $(x,y) \in C$, then $U(g(x), g(y)) = g(H(x,y))$ and $T(f(x), f(y)) = f(H(x,y))$;
if $(x,y) \in C$ and $(y,z) \in C$, then $(H(x,y), z) \in C \Leftrightarrow (x, H(y,z)) \in C$;
if $(x,y) \in C$, $(y,z) \in C$ and $(H(x,y), z) \in C$, then $H(H(x,y), z) = H(x, H(y,z))$.

Proof. *Necessity.* Let $K \in \mathcal{UN}_\mathcal{D}$. Define f, g, U, T, A, B, C as follows:
$f(x) = K(x,1)$ for all $x \in L$;
$g(x) = K(x,e)$ for all $x \in L$;
$U = K|_{[0,a]^2}$, $T = K|_{[a,1]^2}$, $H = K|_C$;
$A = \{(x,y) \in I_a^2 \mid K(x,y) \ge a\}$;
$B = \{(x,y) \in I_a^2 \mid K(x,y) \le a\}$;
$C = \{(x,y) \in I_a^2 \mid K(x,y) \parallel a\}$.

Obviously, $M = ([a,1] \times L) \cup (L \times [a,1]) \cup A$ is a symmetric upper set, $N = ([0,a] \times L) \cup (L \times [0,a]) \cup B$ is a symmetric lower set and $C = I_a^2 \setminus (A \cup B)$. In addition, based on the proof of Theorem 3.2, we only need to prove the conditions involving the function H on C. Obviously, $H : C \to I_a$ is commutative and increasing and $A \cup B \cup C = I_a^2$. For all $(x,y) \in C$, we have

$$\begin{aligned} U(g(x), g(y)) &= K(K(x,e), K(y,e)) = K(K(x,y), K(e,e)) \\ &= K(K(x,y), e) = g(K(x,y)) = g(H(x,y)), \end{aligned}$$

and similarly, we have $T(f(x), f(y)) = f(H(x,y))$. In addition, the increasingness of K requires that $U(g(x), g(y)) \le H(x,z)$ for any $(x,y) \in ([0,a] \times L) \cup (L \times [0,a]) \cup B$, $(x,z) \in C$ with $y \le z$, and $T(f(x), f(z)) \ge H(x,y)$ for any $(x,y) \in C$, $(x,z) \in ([a,1] \times L) \cup (L \times [a,1]) \cup A$ with $y \le z$.

The associativity of K and the fact that $K(x,y) \in I_a$ iff $(x,y) \in C$ imply that if $(x,y) \in C$ and $(y,z) \in C$, then $(H(x,y), z) \in C \Leftrightarrow (x, H(y,z)) \in C$, and if $(x,y) \in C$, $(y,z) \in C$ and $(H(x,y), z) \in C$, then $H(H(x,y), z) = H(x, H(y,z))$.

Hence, K can be represented by formula (6) and all the conditions are satisfied.

Sufficiency. Let f and g be two order-preserving functions, $H : C \to I_a$ be an increasing and commutative function, U be a disjunctive uninorm on $[0,a]$ with neutral element e, T be a t-norm on $[a,1]$ such that K is given by (6) and all the conditions are satisfied. As we have illustrated in the proof of Theorem 3.2, K is well defined and commutative, e is the neutral element on $[0,a]$ and $K(x,1) = T(x,1) = x$ for all $x \in [a,1]$. Then it remains to show that K is increasing and associative.

To show that $K(x,y) \le K(x,z)$ for all $x,y,z \in L$ with $y \le z$, we only need to verify the case when one of (x,y) and (x,z) belongs to C. If $(x,y) \in C$, then $(x,z) \in (I_a \times [a,1]) \cup ([a,1] \times I_a) \cup A$, from the conditions above, we know that $K(x,y) = H(x,y) \le T(f(x), f(z)) = K(x,z)$. Then case when $(x,z) \in C$ can be proved similarly.

To show that $K(K(x,y), z) = K(x, K(y,z))$ for all $x,y,z \in L$, we only need to consider the case when at least one of $(K(x,y), z)$ and $(x, K(y,z))$ belongs to C. If $(K(x,y), z) \in C$, then we have $(x,y) \in C$. In this case, if $(y,z) \in ([0,a] \times L) \cup (L \times [0,a]) \cup B$, then $H(H(x,y), z) = K(K(x,y), z) \le K(K(1,y), z) = K(f(y), z) = T(f(y), f(z)) = a$, which is a contradiction. Similarly, if $(y,z) \in ([a,1] \times L) \cup (L \times [a,1]) \cup A$, a contradiction can be obtained. Hence, $(y,z) \in C$ and thus $(x, K(y,z)) \in C$ from the given condition. If $(x, K(y,z)) \in C$, we can also obtain that $(K(x,y), z) \in C$, and this implies that $(K(x,y), z) \in C$ and $(x, K(y,z)) \in C$ coincides. Therefore, from the given condition, the associativity is true.

To sum up, $K \in \mathcal{UN}_{\mathcal{D}}$. □

Example 3.6. Let $L = \{0, a, b, c, d, e, 1\}$ be the lattice shown in Fig. 1 and the function K' shown in Table 4 be a uni-nullnorm on L.

Then, the uni-nullnorm K' can be represented by Theorem 3.5. In the representation of K', we have that $C = \{(b,b)\}$. In addition, the order-preserving functions g and f are shown in Table 3.6 and 6.

$K'(x,y)$	0	e	a	b	c	d	1
0	0	0	a	0	a	a	a
e	0	e	a	e	a	a	a
a	a	a	a	a	a	a	a
b	0	e	a	b	a	a	1
c	a	a	a	a	a	a	a
d	a	a	a	a	a	a	a
1	a	a	a	1	a	a	1

Tab. 4. The uni-nullnorm K' in Example 3.6.

x	0	e	a	b	c	d	1
$g(x)$	0	e	a	e	a	a	a

Tab. 5. The function f in Example 3.6

x	0	e	a	b	c	d	1
$f(x)$	a	a	a	1	a	a	1

Tab. 6. The function g in Example 3.6.

Remark 3.7.

(i) In the representation of a uni-nullnorm K as in formula (6), U, T, f and g are uniquely determined by K from the proof of Theorem 3.5.

(ii) If the neutral element of the underlying uninorm U is zero, i. e., $e = 0$, then Theorem 3.5 coincides with the representation of nullnorms on bounded lattices (Theorem 3.2 in [23]).

4. REPRESENTATION OF NULL-UNINORMS WITH CONJUNCTIVE UNDERLYING UNINORMS

In this section, we denote all the null-uninorms in $\mathcal{NU}$ with conjunctive underlying uninorms by $\mathcal{NU}_\mathcal{C}$. The following proposition can be obtained similarly.

Proposition 4.1. Let $e, a \in L \setminus \{0,1\}$ and $R \in \mathcal{UN}_\mathcal{C}$. Then
(i) $R(x,y) \geq a$ for $(x,y) \in [a,1] \times L \cup L \times [a,1]$.
(ii) $R(x,y) \leq a$ for $(x,y) \in [0,a] \times L \cup L \times [0,a]$.
(iii) $R(x,y) \geq (x \wedge a) \vee (y \wedge a)$ for $(x,y) \in I_a^2$.
Moreover, $R(x,y) = a$ for $(x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a]$.

We denote $\mathcal{NU}_{\mathcal{C}c} = \{R \in \mathcal{NU}_\mathcal{C} \mid R(x,y) \nparallel a, \forall x,y \in L\}$. The following theorems can be obtained similarly based on the results in Section 3 and thus we omit the proof.

Theorem 4.2. Let $a, e \in L$ with $0 < a < e$ and R be a binary function on L. Then $R \in \mathcal{NU}_{\mathcal{C}c}$ if and only if there exist two order-preserving functions $f : L \to [a, 1]$ and $g : L \to [0, a]$, a conjunctive uninorm U on $[a, 1]$ with neutral element e and a t-conorm S on $[0, a]$ such that

$$R(x,y) = \begin{cases} S(g(x), g(y)), & (x,y) \in N, \\ U(f(x), f(y)), & (x,y) \in M, \end{cases} \tag{7}$$

where
$f(x) = x$ for all $x \in [a, 1]$;
$g(x) = x$ for all $x \in [0, a]$;
$M = ([a, 1] \times L) \cup (L \times [a, 1]) \cup A$ and $A = \{(x, y) \in I_a^2 \mid S(g(x), g(y)) = a\}$;
$N = ([0, a] \times L) \cup (L \times [0, a]) \cup B$ and $B = \{(x, y) \in I_a^2 \mid U(f(x), f(y)) = a\}$;
$A \cup B = I_a^2$.

Remark 4.3. In Theorem 4.2, if we take

$$g(x) = \begin{cases} x, & x \in [0, a] \\ a, & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} x, & x \in [a, 1] \\ a, & \text{otherwise,} \end{cases}$$

then the structure of null-uninorm R on L coincides with that in Theorem 3.9 [22]. And in this case, if we let the uninorm U be the conjunctive uninorm U_c constructed in [3], then the structure of null-uninorm R on L coincides with that in Theorem 3.10 [22]. Theorem 3.11 in [22] can be obtained similarly.

Theorem 4.4. Let $a, e \in L$ with $0 < a < e$ and R be a binary function on L. Then $R \in \mathcal{NU}_{\mathcal{C}}$ if and only if there exist three symmetric sets A, B, $C \in I_a^2$, two order-preserving functions $f : L \to [a, 1]$ and $g : L \to [0, a]$, a commutative and increasing function $H : C \to I_a$, a conjunctive uninorm U on $[a, 1]$ with neutral element e and a t-norm S on $[0, a]$ such that

$$R(x,y) = \begin{cases} S(g(x), g(y)), & (x,y) \in N \\ U(f(x), f(y)), & (x,y) \in M \\ H(x,y), & (x,y) \in C, \end{cases} \tag{8}$$

where
$f(x) = x$ for all $x \in [a, 1]$;
$g(x) = x$ for all $x \in [0, a]$;
$M = ([a, 1] \times L) \cup (L \times [a, 1]) \cup A$ is an upper set, and $(x, y) \in A$ implies $S(g(x), g(y)) = a$;
$N = ([0, a] \times L) \cup (L \times [0, a]) \cup B$ is a lower set, and $(x, y) \in B$ implies $U(f(x), f(y)) = a$;
$C = I_a^2 \setminus (A \cup B)$;
$S(g(x), g(y)) \leq H(x, z)$ for any $(x, y) \in ([0, a] \times L) \cup (L \times [0, a]) \cup B$, $(x, z) \in C$ with $y \leq z$;
$U(f(x), f(z)) \geq H(x, y)$ for any $(x, y) \in C$, $(x, z) \in ([a, 1] \times L) \cup (L \times [a, 1]) \cup A$ with $y \leq z$;

if $(x, y) \in C$, then $S(g(x), g(y)) = g(H(x, y))$ and $U(f(x), f(y)) = f(H(x, y)$;
if $(x, y) \in C$ and $(y, z) \in C$, then $(H(x, y), z) \in C \Leftrightarrow (x, H(y, z)) \in C$;
if $(x, y) \in C$, $(y, z) \in C$ and $(H(x, y), z) \in C$, then $H(H(x, y), z) = H(x, H(y, z))$.

If the neutral element of the underlying uninorm U is one, i. e., $e = 1$, then Theroem 4.2 coincides with Theorem 3.2 in [14] and Theorem 3.8 in [23]. Besides, Theorem 4.4 coincides with Theorem 3.2 in [23] simultaneously in this case.

5. CONCLUSION

In this paper, we presented the representation for uni-nullnorms with disjunctive underlying uninorms on bounded lattices as well as the representation for null-uninorms with conjunctive underlying uninorms on bounded lattices. Some of existing construction methods of uni-nullnorms and null-uninorms on bounded lattices as well as the representation of nullnorms on bounded lattices were proved to be special cases of the representation in this paper and illustrating examples were provided.

In the future, we will investigate the representation of uni-nullnorms with conjunctive underlying uninorms on bounded lattices.

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