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ON GENERALIZED BIHYPERBOLIC MERSENNE NUMBERS

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Abstract. In this paper, a new generalization of Mersenne bihyperbolic numbers is introduced. Some of the properties of presented numbers are given. A general bilinear index-reduction formula for the generalized bihyperbolic Mersenne numbers is obtained. This result implies the Catalan, Cassini, Vajda, d'Ocagne and Halton identities. Moreover, generating function and matrix generators for these numbers are presented.

Keywords: Mersenne number; hyperbolic number; bihyperbolic number; recurrence relation

MSC 2020: 11B37, 11B39

1. Introduction

Let $\mathbf{h}$ be the unipotent element such that $\mathbf{h} \neq \pm 1$ and $\mathbf{h}^2 = 1$. A hyperbolic number z is defined as $z = x + y\mathbf{h}$, where $x, y \in \mathbb{R}$. Denote by $\mathbb{H}$ the set of hyperbolic numbers. The hyperbolic numbers were introduced by Cockle, see [5]–[8].

The addition and subtraction of hyperbolic numbers is done by adding and subtracting the appropriate terms and thus their coefficients. The hyperbolic numbers multiplication can be made analogously as multiplication of algebraic expressions using the rule $\mathbf{h}^2 = 1$. The real numbers x and y are called the real and unipotent parts of the hyperbolic number z, respectively. For others details concerning hyperbolic numbers see for example [12], [13].

The extension of complex numbers to a higher dimension is of interest not only to mathematics but also to modern physics and engineering. Quaternions are one of the well-known sets, however they form a non-commutative algebra.

In [11], Olariu introduced commutative hypercomplex numbers in different dimensions. One of 4-dimensional commutative hypercomplex numbers is called the hyperbolic fourcomplex number. In [12], the authors used the name bihyperbolic numbers.

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Note that bihyperbolic numbers are a special case of generalized Segre's quaternions, being a 4-dimensional commutative number system, and they are named as canonical hyperbolic quaternions, see [3]. In this paper, we use the name bihyperbolic numbers. Analogously as bicomplex numbers are an extension of complex numbers, bihyperbolic numbers are a natural extension of hyperbolic numbers to 4-dimension.

Let $\mathbb{H}_2$ be the set of bihyperbolic numbers ζ of the form

$$\zeta = x_0 + j_1 x_1 + j_2 x_2 + j_3 x_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $j_1, j_2, j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1, \quad j_1 j_2 = j_2 j_1 = j_3, \quad j_1 j_3 = j_3 j_1 = j_2, \quad j_2 j_3 = j_3 j_2 = j_1. \tag{1.1}$$

The addition and multiplication on $\mathbb{H}_2$ are commutative and associative. Moreover, $(\mathbb{H}_2, +, \cdot)$ is a commutative ring. For the algebraic properties of bihyperbolic numbers, see [1].

In this paper, we study some generalization of bihyperbolic Mersenne numbers. The Mersenne sequence $\{M_n\}$ is defined by the recurrence

$$M_n = 2M_{n-1} + 1 \quad \text{for } n \geqslant 1 \tag{1.2}$$

with the initial condition $M_0 = 0$ or

$$M_n = 3M_{n-1} - 2M_{n-2} \quad \text{for } n \geqslant 2 \tag{1.3}$$

with $M_0 = 0$, $M_1 = 1$. The Binet formula for the Mersenne numbers has the form

$$M_n = 2^n - 1. \tag{1.4}$$

Some interesting properties of the Mersenne numbers can be found in [2], [14].

In the literature there are some generalizations of the Mersenne numbers, see [4], [10], [15]. In [10], a one parameter generalization of the Mersenne numbers was investigated. We recall this generalization.

Let $n \geqslant 0$, $k \geqslant 3$ be integers, the generalized Mersenne numbers $M(k,n)$ are defined by the recurrence relation

$$M(k,n) = kM(r,n-1) - (k-1)M(k,n-2) \quad \text{for } n \geqslant 2 \tag{1.5}$$

with the initial conditions $M(k,0) = 0$, $M(k,1) = 1$. It is easily seen that $M(3,n) = M_n$.

In [9], the generalized Mersenne quaternion $\mathcal{M}_n^{(p,r,s)}$ was introduced. For an integer n and any integers p, r, s the generalized Mersenne quaternion is defined by

$$\mathcal{M}_n^{(p,r,s)} = M_n + iM_{n+p} + jM_{n+r} + kM_{n+s},$$

where $\{i, j, k\}$ is the standard basis of quaternions. Motivated by the mentioned concept, in this paper, we introduce and study generalized bihyperbolic Mersenne numbers.

2. Generalized bihyperbolic Mersenne numbers

Let $p \geqslant 1$, $r \geqslant 1$, $s \geqslant 1$, $n \geqslant 0$ be integers, the nth generalized bihyperbolic Mersenne number $BhM_n^{(p,r,s)}$ is defined as

$$BhM_n^{(p,r,s)} = M_n + j_1 M_{n+p} + j_2 M_{n+r} + j_3 M_{n+s}, \tag{2.1}$$

where M_n is the nth Mersenne number and operators j_1, j_2, j_3 satisfy (1.1).

By (2.1) we obtain

$$\begin{aligned} BhM_0^{(p,r,s)} &= M_0 + j_1 M_p + j_2 M_r + j_3 M_s, \\ BhM_1^{(p,r,s)} &= M_1 + j_1 M_{1+p} + j_2 M_{1+r} + j_3 M_{1+s}, \\ BhM_2^{(p,r,s)} &= M_2 + j_1 M_{2+p} + j_2 M_{2+r} + j_3 M_{2+s}. \end{aligned} \tag{2.2}$$

For $p = 1$, $r = 2$, $s = 3$ we obtain the definition of the nth bihyperbolic Mersenne number $BhM_n^{(1,2,3)}$, i.e., $BhM_n^{(1,2,3)} = BhM_n$.

By the definition of the generalized bihyperbolic Mersenne numbers we get the following recurrence relations.

Theorem 2.1. *Let $n \geqslant 2$, $p \geqslant 1$, $r \geqslant 1$, $s \geqslant 1$ be integers. Then*

$$BhM_n^{(p,r,s)} = 3BhM_{n-1}^{(p,r,s)} - 2BhM_{n-2}^{(p,r,s)}, \tag{2.3}$$

where $BhM_0^{(p,r,s)}$, $BhM_1^{(p,r,s)}$ are given by (2.2).

Proof. Using (2.1) and (1.3), we have

$$\begin{aligned} 3BhM_{n-1}^{(p,r,s)} - 2BhM_{n-2}^{(p,r,s)} &= 3(M_{n-1} + j_1 M_{n-1+p} + j_2 M_{n-1+r} + j_3 M_{n-1+s}) \\ &\quad - 2(M_{n-2} + j_1 M_{n-2+p} + j_2 M_{n-2+r} + j_3 M_{n-2+s}) \\ &= M_n + j_1 M_{n+p} + j_2 M_{n+r} + j_3 M_{n+s} = BhM_n^{(p,r,s)}. \end{aligned}$$

□

Theorem 2.2. *Let $n \geqslant 1$, $p \geqslant 1$, $r \geqslant 1$, $s \geqslant 1$ be integers. Then*

$$BhM_n^{(p,r,s)} = 2BhM_{n-1}^{(p,r,s)} + 1 + j_1 + j_2 + j_3, \tag{2.4}$$

where $BhM_0^{(p,r,s)}$ is given by (2.2).

P r o o f. Using (2.1) and (1.2), we get

$$\begin{aligned} 2BhM_{n-1}^{(p,r,s)} &= 2(M_{n-1} + j_1 M_{n-1+p} + j_2 M_{n-1+r} + j_3 M_{n-1+s}) \\ &= M_n - 1 + j_1(M_{n+p} - 1) + j_2(M_{n+r} - 1) + j_3(M_{n-1+s} - 1) \\ &= M_n + j_1 M_{n+p} + j_2 M_{n+r} + j_3 M_{n+s} - (1 + j_1 + j_2 + j_3) \\ &= BhM_n^{(p,r,s)} - (1 + j_1 + j_2 + j_3), \end{aligned}$$

which ends the proof. □

In the proof of the next theorem we will use the following result.

Theorem 2.3 ([10])**.** *Let $n \geqslant 1$, $t \geqslant 1$ be integers. Then*

$$M_{n+1} - M_n = 2^n.$$

Theorem 2.4. *Let $n \geqslant 0$, $p \geqslant 1$, $r \geqslant 1$, $s \geqslant 1$, $t \geqslant 1$ be integers. Then*

$$BhM_{n+1}^{(p,r,s)} - BhM_n^{(p,r,s)} = 2^n(1 + 2^p j_1 + 2^r j_2 + 2^s j_3).$$

P r o o f. By the equality (2.1) and Theorem 2.3, we have

$$\begin{aligned} BhM_{n+1}^{(p,r,s)} - BhM_n^{(p,r,s)} &= M_{n+1} + j_1 M_{n+1+p} + j_2 M_{n+1+r} + j_3 M_{n+1+s} \\ &\quad - (M_n + j_1 M_{n+p} + j_2 M_{n+r} + j_3 M_{n+s}) \\ &= M_{n+1} - M_n + j_1(M_{n+1+p} - M_{n+p}) \\ &\quad + j_2(M_{n+1+r} - M_{n+r}) + j_3(M_{n+1+s} - M_{n+s}) \\ &= 2^n(1 + 2^p j_1 + 2^r j_2 + 2^s j_3). \end{aligned}$$

□

Now, we give the Binet formula for the generalized bihyperbolic Mersenne numbers.

Theorem 2.5 (Binet formula)**.** *Let $n \geqslant 0$, $p \geqslant 1$, $r \geqslant 1$, $s \geqslant 1$ be integers. Then*

$$BhM_n^{(p,r,s)} = 2^n(1 + 2^p j_1 + 2^r j_2 + 2^s j_3) - (1 + j_1 + j_2 + j_3). \tag{2.5}$$

P r o o f. By the formulas (2.1) and (1.4) we get

$$\begin{aligned} BhM_n^{(p,r,s)} &= M_n + j_1 M_{n+p} + j_2 M_{n+r} + j_3 M_{n+s} \\ &= 2^n - 1 + j_1(2^{n+p} - 1) + j_2(2^{n+r} - 1) + j_3(2^{n+s} - 1) \\ &= 2^n(1 + 2^p j_1 + 2^r j_2 + 2^s j_3) - (1 + j_1 + j_2 + j_3). \end{aligned}$$

□

By Theorem 2.5, we get the Binet formula for the bihyperbolic Mersenne numbers.

Corollary 2.1. *Let $n \geqslant 0$ be an integer. Then*

$$BhM_n = 2^n(1 + 2j_1 + 4j_2 + 8j_3) - (1 + j_1 + j_2 + j_3).$$

Assume that $p \geqslant 1$, $r \geqslant 1$, $s \geqslant 1$ are integers.

Theorem 2.6 (General bilinear index-reduction formula for the generalized bihyperbolic Mersenne numbers)**.** *Let $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$, $d \geqslant 0$ be integers such that $a + b = c + d$. Then*

$$BhM_a^{(p,r,s)} \cdot BhM_b^{(p,r,s)} - BhM_c^{(p,r,s)} \cdot BhM_d^{(p,r,s)} = (2^c + 2^d - 2^a - 2^b)AB,$$

where $A = 1 + 2^p j_1 + 2^r j_2 + 2^s j_3$, $B = 1 + j_1 + j_2 + j_3$.

P r o o f. By formula (2.5), we get

$$\begin{aligned} &BhM_a^{(p,r,s)} \cdot BhM_b^{(p,r,s)} - BhM_c^{(p,r,s)} \cdot BhM_d^{(p,r,s)} \\ &\quad= (2^a A - (1 + j_1 + j_2 + j_3))(2^b A - (1 + j_1 + j_2 + j_3)) \\ &\qquad - (2^c A - (1 + j_1 + j_2 + j_3))(2^d A - (1 + j_1 + j_2 + j_3)) \\ &\quad= -2^a A(1 + j_1 + j_2 + j_3) - 2^b A(1 + j_1 + j_2 + j_3) \\ &\qquad + 2^c A(1 + j_1 + j_2 + j_3) + 2^d A(1 + j_1 + j_2 + j_3) \\ &\quad= (2^c + 2^d - 2^a - 2^b)AB, \end{aligned}$$

where $A = 1 + 2^p j_1 + 2^r j_2 + 2^s j_3$, $B = 1 + j_1 + j_2 + j_3$. □

It is easily seen that for special values of a, b, c, d, by Theorem 2.6, we get new identities for generalized bihyperbolic Mersenne numbers:

- ▷ Catalan identity (for $a = n - m$, $b = n + m$ and $c = d = n$),
- ▷ Cassini identity (for $a = n - 1$, $b = n + 1$ and $c = d = n$),
- ▷ d'Ocagne identity (for $a = n$, $b = m + 1$, $c = n + 1$ and $d = m$),
- ▷ Vajda identity (for $a = m + k$, $b = n - k$, $c = m$ and $d = n$),
- ▷ Halton identity (for $a = m + k$, $b = n$, $c = k$ and $d = m + n$).

Corollary 2.2 (Catalan identity for generalized bihyperbolic Mersenne numbers)**.** *Let $n \geqslant 0$, $m \geqslant 0$ be integers such that $n \geqslant m$. Then*

$$BhM_{n-m}^{(p,r,s)} \cdot BhM_{n+m}^{(p,r,s)} - (BhM_{n}^{(p,r,s)})^2 = -2^{n-m}(1-2^m)^2 AB.$$

Corollary 2.3 (Cassini identity for generalized bihyperbolic Mersenne numbers)**.** *Let $n \geqslant 1$ be an integer. Then*

$$BhM_{n-1}^{(p,r,s)} \cdot BhM_{n+1}^{(p,r,s)} - (BhM_{n}^{(p,r,s)})^2 = -2^{n-1} AB.$$

Corollary 2.4 (d'Ocagne identity for generalized bihyperbolic Mersenne numbers)**.** *Let $n \geqslant 0$, $m \geqslant 0$ be integers such that $n \geqslant m$. Then*

$$BhM_{n}^{(p,r,s)} \cdot BhM_{m+1}^{(p,r,s)} - BhM_{n+1}^{(p,r,s)} \cdot BhM_{m}^{(p,r,s)} = (2^n - 2^m) AB.$$

Corollary 2.5 (Vajda identity for generalized bihyperbolic Mersenne numbers)**.** *Let $n \geqslant 0$, $m \geqslant 0$, $k \geqslant 0$ be integers such that $n \geqslant k$. Then*

$$BhM_{m+k}^{(p,r,s)} \cdot BhM_{n-k}^{(p,r,s)} - BhM_{m}^{(p,r,s)} \cdot BhM_{n}^{(p,r,s)} = \left(2^m(1-2^k) + 2^n\left(1 - \frac{1}{2^k}\right)\right) AB.$$

Corollary 2.6 (Halton identity for generalized bihyperbolic Mersenne numbers)**.** *Let $n \geqslant 0$, $m \geqslant 0$, $k \geqslant 0$ be integers such that $n \geqslant k$. Then*

$$BhM_{m+k}^{(p,r,s)} \cdot BhM_{n}^{(p,r,s)} - BhM_{k}^{(p,r,s)} \cdot BhM_{m+n}^{(p,r,s)} = (2^m - 1)(2^n - 2^k) AB.$$

In the proof of the next theorem we will use the following result.

Theorem 2.7 ([15])**.** *Let $m \geqslant 1$, $n \geqslant 1$ be integers. Then*

$$M_{n+m} = M_n M_{m+1} - 2M_{n-1} M_m. \tag{2.6}$$

Theorem 2.8. *Let $m \geqslant 1$, $n \geqslant 1$ be integers. Then*

$$\begin{aligned} &BhM_{n}^{(p,r,s)} \cdot BhM_{m+1}^{(p,r,s)} - 2BhM_{n-1}^{(p,r,s)} \cdot BhM_{m}^{(p,r,s)} \\ &\quad = BhM_{n+m}^{(p,r,s)} + j_1 BhM_{n+m+p}^{(p,r,s)} + j_2 BhM_{n+m+r}^{(p,r,s)} + j_3 BhM_{n+m+s}^{(p,r,s)}. \end{aligned}$$

Proof. By simple calculations we have

$$
\begin{aligned}
& BhM_n^{(p,r,s)} \cdot BhM_{m+1}^{(p,r,s)} - 2BhM_{n-1}^{(p,r,s)} \cdot BhM_m^{(p,r,s)} \\
&= (M_n + j_1 M_{n+p} + j_2 M_{n+r} + j_3 M_{n+s}) \\
&\quad \cdot (M_{m+1} + j_1 M_{m+1+p} + j_2 M_{m+1+r} + j_3 M_{m+1+s}) \\
&\quad - 2(M_{n-1} + j_1 M_{n-1+p} + j_2 M_{n-1+r} + j_3 M_{n-1+s}) \\
&\quad \cdot (M_m + j_1 M_{m+p} + j_2 M_{m+r} + j_3 M_{m+s}) \\
&= M_n M_{m+1} - 2M_{n-1} M_m \\
&\quad + j_1 (M_n M_{m+1+p} - 2M_{n-1} M_{m+p} + M_{n+p} M_{m+1} - 2M_{n+p-1} M_m) \\
&\quad + j_2 (M_n M_{m+1+r} - 2M_{n-1} M_{m+r} + M_{n+r} M_{m+1} - 2M_{n+r-1} M_m) \\
&\quad + j_3 (M_n M_{m+1+s} - 2M_{n-1} M_{m+s} + M_{n+s} M_{m+1} - 2M_{n+s-1} M_m) \\
&\quad + M_{n+p} M_{m+1+p} - 2M_{n+p-1} M_{m+p} + M_{n+r} M_{m+1+r} \\
&\quad - 2M_{n+r-1} M_{m+r} + M_{n+s} M_{m+1+s} - 2M_{n+s-1} M_{m+s} \\
&\quad + j_1 (M_{n+r} M_{m+1+s} - 2M_{n+r-1} M_{m+s} + M_{n+s} M_{m+1+r} - 2M_{n+s-1} M_{m+r}) \\
&\quad + j_2 (M_{n+p} M_{m+1+s} - 2M_{n+p-1} M_{m+s} + M_{n+s} M_{m+1+p} - 2M_{n+s-1} M_{m+p}) \\
&\quad + j_3 (M_{n+r} M_{m+1+p} - 2M_{n+r-1} M_{m+p} + M_{n+p} M_{m+1+r} - 2M_{n+p-1} M_{m+r}).
\end{aligned}
$$

By the formula (2.6) we get

$$
\begin{aligned}
BhM_n^{(p,r,s)} \cdot BhM_{m-1}^{(p,r,s)} - 2BhM_{n-1}^{(p,r,s)} \cdot BhM_m^{(p,r,s)} & \\
= M_{n+m} + j_1 M_{n+m+p} + j_1 M_{n+m+p} + j_2 M_{n+m+r} & \\
\quad + j_2 M_{n+m+r} + j_3 M_{n+m+s} + j_3 M_{n+m+s} & \\
\quad + M_{n+m+p+p} + M_{n+m+r+r} + M_{n+m+s+s} & \\
\quad + j_1 M_{n+m+r+s} + j_1 M_{n+m+s+r} + j_2 M_{n+m+p+s} & \\
\quad + j_2 M_{n+m+s+p} + j_3 M_{n+m+p+r} + j_3 M_{n+m+r+p} & \\
= M_{n+m} + j_1 M_{n+m+p} + j_2 M_{n+m+r} + j_3 M_{n+m+s} & \\
\quad + j_1 (M_{n+m+p} + j_1 M_{n+m+p+p} + j_2 M_{n+m+p+r} + j_3 M_{n+m+p+s}) & \\
\quad + j_2 (M_{n+m+r} + j_1 M_{n+m+r+p} + j_2 M_{n+m+r+r} + j_3 M_{n+m+r+s}) & \\
\quad + j_3 (M_{n+m+s} + j_1 M_{n+m+s+p} + j_2 M_{n+m+s+r} + j_3 M_{n+m+s+s}) & \\
= BhM_{n+m}^{(p,r,s)} + j_1 BhM_{n+m+p}^{(p,r,s)} + j_2 BhM_{n+m+r}^{(p,r,s)} + j_3 BhM_{n+m+s}^{(p,r,s)}, &
\end{aligned}
$$

which ends the proof. □

Now, we give the ordinary generating function for the generalized bihyperbolic Mersenne numbers.

Theorem 2.9. *The generating function for the generalized bihyperbolic Mersenne sequence* $\{BhM_n^{(p,r,s)}\}$ *is*

$$g(x) = \frac{BhM_0^{(p,r,s)} + (BhM_1^{(p,r,s)} - 3BhM_0^{(p,r,s)})x}{1 - 3x + 2x^2}.$$

P r o o f. Let

$$g(x) = BhM_0^{(p,r,s)} + BhM_1^{(p,r,s)}x + BhM_2^{(p,r,s)}x^2 + \ldots + BhM_n^{(p,r,s)}x^n + \ldots$$

be the generating function of the generalized bihyperbolic Mersenne numbers. Hence we have

$$\begin{aligned} -3xg(x) = & -3BhM_0^{(p,r,s)}x - 3BhM_1^{(p,r,s)}x^2 - 3BhM_2^{(p,r,s)}x^3 \\ & - \ldots - 3BhM_{n-1}^{(p,r,s)}x^n - \ldots, \\ 2x^2g(x) = & \, 2BhM_0^{(p,r,s)}x^2 + 2BhM_1^{(p,r,s)}x^3 + 2BhM_2^{(p,r,s)}x^4 \\ & + \ldots + 2BhM_{n-2}^{(p,r,s)}x^n + \ldots \end{aligned}$$

Using the recurrence (2.3), we get

$$g(x)(1 - 3x + 2x^2) = BhM_0^{(p,r,s)} + (BhM_1^{(p,r,s)} - 3BhM_0^{(p,r,s)})x.$$

Thus

$$g(x) = \frac{BhM_0^{(p,r,s)} + (BhM_1^{(p,r,s)} - 3BhM_0^{(p,r,s)})x}{1 - 3x + 2x^2}.$$

□

In the next theorem we give a summation formula for the generalized bihyperbolic Mersenne numbers. In the proof we will use the following result.

Theorem 2.10 ([2]). *If* M_i *is the* i*th term of the Mersenne sequence then*

$$\sum_{i=0}^{n} M_i = 2M_n - n. \tag{2.7}$$

Theorem 2.11. *Let* $n \geqslant 0$, $p \geqslant 1$, $r \geqslant 1$, $s \geqslant 1$ *be integers. Then*

$$\sum_{i=0}^{n} BhM_i^{(p,r,s)} = 2BhM_n^{(p,r,s)} - BhM_0^{(p,r,s)} - n(1 + j_1 + j_2 + j_3).$$

Proof. By the definition of the generalized bihyperbolic Mersenne numbers we get

$$\begin{aligned}\sum_{i=0}^{n} BhM_i^{(p,r,s)} &= M_0 + j_1 M_p + j_2 M_r + j_3 M_s \\ &\quad + M_1 + j_1 M_{1+p} + j_2 M_{1+r} + j_3 M_{1+s} + \ldots \\ &\quad + M_n + j_1 M_{n+p} + j_2 M_{n+r} + j_3 M_{n+s} \\ &= \sum_{i=0}^{n} M_i + j_1 \sum_{i=0}^{n} M_{i+p} + j_2 \sum_{i=0}^{n} M_{i+r} + j_3 \sum_{i=0}^{n} M_{i+s} \\ &= \sum_{i=0}^{n} M_i + j_1 \left(\sum_{i=0}^{n+p} M_i - \sum_{i=0}^{p-1} M_i \right) \\ &\quad + j_2 \left(\sum_{i=0}^{n+r} M_i - \sum_{i=0}^{r-1} M_i \right) + j_3 \left(\sum_{i=0}^{n+s} M_i - \sum_{i=0}^{s-1} M_i \right).\end{aligned}$$

By the formula (2.7) we obtain

$$\begin{aligned}\sum_{i=0}^{n} BhM_n^{(p,r,s)} &= 2M_n - n + j_1[2M_{n+p} - (n+p) - (2M_{p-1} - (p-1))] \\ &\quad + j_2[2M_{n+r} - (n+r) - (2M_{r-1} - (r-1))] \\ &\quad + j_3[2M_{n+s} - (n+s) - (2M_{s-1} - (s-1))] \\ &= 2M_n - n + 2j_1 M_{n+p} - 2j_1 M_{p-1} + j_1(-n-1) \\ &\quad + 2j_2 M_{n+r} - 2j_2 M_{r-1} + j_2(-n-1) \\ &\quad + 2j_3 M_{n+s} - 2j_3 M_{s-1} + j_3(-n-1).\end{aligned}$$

Using the formula (1.2), we have

$$\begin{aligned}\sum_{i=0}^{n} BhM_i^{(p,r,s)} &= 2M_n + 2j_1 M_{n+p} + 2j_2 M_{n+r} + 2j_3 M_{n+s} \\ &\quad - 2j_1 M_{p-1} - 2j_2 M_{r-1} - 2j_3 M_{s-1} \\ &\quad - n + j_1(-n-1) + j_2(-n-1) + j_3(-n-1) \\ &= 2BhM_n^{(p,r,s)} - j_1(M_p - 1) - j_2(M_r - 1) - j_3(M_s - 1) \\ &\quad - n + j_1(-n-1) + j_2(-n-1) + j_3(-n-1) \\ &= 2BhM_n^{(p,r,s)} - j_1 M_p + j_1 - j_2 M_r + j_2 - j_3 M_s + j_3 - M_0 \\ &\quad - n + j_1(-n-1) + j_2(-n-1) + j_3(-n-1) \\ &= 2BhM_n^{(p,r,s)} - M_0 - j_1 M_p - j_2 M_r - j_3 M_s \\ &\quad - n(1 + j_1 + j_2 + j_3) \\ &= 2BhM_n^{(p,r,s)} - BhM_0^{(p,r,s)} - n(1 + j_1 + j_2 + j_3).\end{aligned}$$

□

At the end, we give matrix representations of the numbers $BhM_n^{(p,r,s)}$. By the equality (2.3) we get the following result.

Theorem 2.12. *Let $n \geqslant 1$ be an integer. Then*

$$\begin{bmatrix} BhM_{n+1}^{(p,r,s)} \\ BhM_n^{(p,r,s)} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} BhM_n^{(p,r,s)} \\ BhM_{n-1}^{(p,r,s)} \end{bmatrix}.$$

Theorem 2.13. *Let $n \geqslant 0$ be an integer. Then*

$$\begin{bmatrix} BhM_{n+2}^{(p,r,s)} & BhM_{n+1}^{(p,r,s)} \\ BhM_{n+1}^{(p,r,s)} & BhM_n^{(p,r,s)} \end{bmatrix} = \begin{bmatrix} BhM_2^{(p,r,s)} & BhM_1^{(p,r,s)} \\ BhM_1^{(p,r,s)} & BhM_0^{(p,r,s)} \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}^n. \tag{2.8}$$

P r o o f. We use induction on n. If $n = 0$ then the result is obvious. Assuming the formula (2.8) holds for $n \geqslant 0$, we shall prove it for $n+1$. Using the induction's hypothesis and formula (2.3), we have

$$\begin{aligned} &\begin{bmatrix} BhM_2^{(p,r,s)} & BhM_1^{(p,r,s)} \\ BhM_1^{(p,r,s)} & BhM_0^{(p,r,s)} \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} BhM_{n+2}^{(p,r,s)} & BhM_{n+1}^{(p,r,s)} \\ BhM_{n+1}^{(p,r,s)} & BhM_n^{(p,r,s)} \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3BhM_{n+2}^{(p,r,s)} - 2BhM_{n+1}^{(p,r,s)} & BhM_{n+2}^{(p,r,s)} \\ 3BhM_{n+1}^{(p,r,s)} - 2BhM_n^{(p,r,s)} & BhM_{n+1}^{(p,r,s)} \end{bmatrix} \\ &= \begin{bmatrix} BhM_{n+3}^{(p,r,s)} & BhM_{n+2}^{(p,r,s)} \\ BhM_{n+2}^{(p,r,s)} & BhM_{n+1}^{(p,r,s)} \end{bmatrix}, \end{aligned}$$

which ends the proof. □

Calculating determinants in the formula (2.8), we obtain the Cassini identity.

Corollary 2.7. *For $n \in \mathbb{N} \cup \{0\}$ we have*

$$BhM_{n+2}^{(p,r,s)} \cdot BhM_n^{(p,r,s)} - (BhM_{n+1}^{(p,r,s)})^2 = 2^n (BhM_2^{(p,r,s)} \cdot BhM_0^{(p,r,s)} - (BhM_1^{(p,r,s)})^2).$$

Corollary 2.8. *For $n \in \mathbb{N} \cup \{0\}$ we have*

$$BhM_{n+2} \cdot BhM_n - (BhM_{n+1})^2 = -15 \cdot 2^n (1 + j_1 + j_2 + j_3).$$

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