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Mbekezeli Nxumalo

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ON OPEN MAPS AND RELATED FUNCTIONS OVER THE SALBANY COMPACTIFICATION

Mbekezeli Nxumalo

Abstract. Given a topological space X, let $\mathcal{U}X$ and $\eta_X : X \to \mathcal{U}X$ denote, respectively, the Salbany compactification of X and the compactification map called the Salbany map of X. For every continuous function $f\colon X \to Y$, there is a continuous function $\mathcal{U}f\colon \mathcal{U}X \to \mathcal{U}Y$, called the Salbany lift of f, satisfying $(\mathcal{U}f) \circ \eta_X = \eta_Y \circ f$. If a continuous function $f\colon X \to Y$ has a stably compact codomain Y, then there is a Salbany extension $F\colon \mathcal{U}X \to Y$ of f, not necessarily unique, such that $F \circ \eta_X = f$. In this paper, we give a condition on a space such that its Salbany map is open. In particular, we prove that in a class of Hausdorff spaces, the spaces with open Salbany maps are precisely those that are almost discrete. We also investigate openness of the Salbany lift and a Salbany extension of a continuous function. Related to open continuous functions are initial maps as well as nearly open maps. It turns out that the Salbany map of every space is both initial and nearly open. We repeat the procedure done for openness of Salbany maps, Salbany lifts and Salbany extensions to their initiality and nearly openness.

Introduction

Salbany [11] in 2000, constructed a topological space $\mathcal{U}X$, called the *ultrafilter space*, using ultrafilters of a space X as points. He showed that the ultrafilter space $\mathcal{U}X$ is a compactification of a space X with the compactification map $\eta_X : X \to \mathcal{U}X$ taking each point of X to its induced principal ultrafilter. For every continuous function $f\colon X \to Y$, there is a continuous function $\mathcal{U}f\colon \mathcal{U}X \to \mathcal{U}Y$ such that $\eta_Y \circ f = (\mathcal{U}f) \circ \eta_X$. Another notable result from the cited Salbany's paper is that given a continuous function $f\colon X \to Y$ to a compact, locally compact and supersober space Y, there is a continuous function $F\colon \mathcal{U}X \to Y$, not necessarily unique, such that $F \circ \eta_X = f$. The ultrafilter space $\mathcal{U}X$ was later called the *Salbany compactification* of X in [5], a term we shall use in paper. The maps η_X, $\mathcal{U}f$ and F shall be referred to as the *Salbany map* of X, the *Salbany lift* of f and a *Salbany extension* of f, respectively. The Salbany compactification also appears in a number of articles such as [4] and [10]. To our knowledge, none of the work done on this compactification addresses the following questions:

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(1) Under which conditions does a space have an open Salbany map?

(2) When is the Salbany lift of a continuous map open?

(3) When does a continuous function have an open Salbany extension?

In this paper, we address the above questions. We do not only focus on openness, but also on initiality and nearly openness. The idea of giving conditions on a space such that its compactification map is open has appeared in articles such as [6] where the author showed that a non-compact space X is locally compact-small if and only if the Wallman compactification map $w_X \colon X \to WX$ is open. In [7], Dimov studied conditions on a continuous function such that its Banaschewski extension is open. In [1], Adjei and Dube characterized continuous maps such that their Banaschewski extensions are nearly open.

This article is organized as follows: Section one covers some terminologies that will be useful in this paper and also recalls the construction of the Salbany compactification given in [11]. In Section two, we discuss openness of (i) the Salbany map of a Hausdorff space, (ii) the Salbany lift of a surjective continuous function, and (iii) Salbany extension of a continuous function. We prove that in the class of Hausdorff spaces, spaces with open Salbany maps are precisely the almost discrete ones. Section three and Section four, respectively, consider initiality and nearly openness of the Salbany map of a space, Salbany lift and Salbany extension of a continuous function.

1. Preliminaries

For basic notations of topological spaces see [8] and refer to [11] for the construction of the Salbany compactification.

The terms topological space and space shall be used interchangeably and we shall only write X for a topological space if there is no possible danger of confusion. Throughout the paper, no separation axioms are assumed on spaces unless stated. We shall use $\mathcal{U}_x$ to denote the system of neighbourhoods (nhoods, in short) of a point x of a space X.

A space X is *supersober* if it is compact and for each ultrafilter $\mathcal{F}$ on X, there is $x \in X$ such that $\bigcap\{\overline{C} \colon C \in \mathcal{F}\} = \overline{\{x\}}$. It is called *stably compact* if it is compact, locally compact and supersober.

A space X is said to be *spectral space* provided that it is compact, sober and has a basis of compact-open sets closed under finite intersections. With the sober requirement replaced by supersober, X is called *quasi-spectral*.

Construction of the Salbany compactification: Let (X, τ) be a topological space. We denote the collection of all ultrafilters on X by $\mathcal{U}X$ and the points of $\mathcal{U}X$ by lowercase letters such as p, q, etc. For each $A \subseteq X$, define $A^* = \{p \in \mathcal{U}X : A \in p\}$. The collection $\mathcal{B} = \{G^* : G \in \tau\}$ forms a base for some topology on $\mathcal{U}X$ which is denoted by $\mathcal{U}\tau$. The *Salbany map* of X is the continuous function $\eta_X \colon (X, \tau) \to (\mathcal{U}X, \mathcal{U}\tau)$ defined by $x \mapsto \{A \subseteq X : x \in A\}$. The pair $((\mathcal{U}X, \mathcal{U}\tau), \eta_X)$ is referred to as the *Salbany compactification* of X, and we shall only write $\mathcal{U}X$ if there is no possible danger of confusion.

Here are some properties that we shall use:

(1) $A \cap B = \emptyset$ if and only if $A^* \cap B^*$.

(2) $A \subseteq B$ if and only if $A^* \subseteq B^*$.

(3) For each $U \subseteq X$, $U = \eta_X^{-1}(U^*)$.

Given a continuous function $f : X \to Y$, there is a continuous function $\mathcal{U}f : \mathcal{U}X \to \mathcal{U}Y$, called the *Salbany lift* of f, defined by $p \mapsto \{A \subseteq Y : f^{-1}(A) \in p\}$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ \mathcal{U}X & \xrightarrow[\mathcal{U}f]{} & \mathcal{U}Y \end{array} \tag{1.1}$$

commutes.

For each stably compact space X, there is a retraction map $r_X : \mathcal{U}X \to X$, defined by $r_X(p) = x$ for some x such that $\overline{\{x\}} = \bigcap\{\overline{A} : A \in p\}$, satisfying that $r_X \circ \eta_X = \mathrm{id}_X$. Hence, for a continuous function $f : X \to Y$ with a stably compact codomain Y, there is a continuous function (not necessarily unique) $F = r_Y \circ (\mathcal{U}f) : \mathcal{U}X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{U}X \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle F} \\ & & Y \end{array} \tag{1.2}$$

For such F, we have that $F(p) = y$ for some $y \in Y$ such that $\overline{\{y\}} = \bigcap\{\overline{A} : f^{-1}(A) \in p\}$, for all $p \in \mathcal{U}X$, and we call F a *Salbany extension* of f.

2. When Salbany maps, Salbany lifts and Salbany extensions are open

We begin this section with an investigation of spaces with open Salbany maps.

Recall from [9] that a topological space is *almost discrete* if every open subset is closed. It is clear that every Hausdorff almost discrete space is discrete. To see this, let $x \in X$. Since X is Hausdorff, $\{x\}$ is closed and hence open because X is almost discrete.

In the following theorem, we give conditions on a class of Hausdorff spaces such that their Salbany maps are open.

Theorem 2.1. *Let (X, τ) be a Hausdorff space. The following statements are equivalent.*

(1) *The Salbany map η_X of X is open.*

(2) *X is almost discrete.*

Proof. (1) $\Rightarrow$ (2): Let A be an open subset of X and assume that there is $x \in \overline{A} \smallsetminus A$. Then $A \cap N \neq \emptyset$ for every nhood N of x. Extend the collection $\{A \cap N : N \in \mathcal{U}_x\}$ to some ultrafilter p on X. Then $A \in p$ and p converges to x. Because $A \in p$ and $x \notin A$, we have that $p \neq \eta_X(x)$. We show that $\eta_X(x) \notin U$ for every open set $U\mathcal{U}X$ such that $U \subseteq \eta_X(X)$. This will contradict that $\eta_X(X)$ is open. Let $U \in \mathcal{U}\tau$ be such that $U \subseteq \eta_X(X)$. Then $\eta_X(x) \notin U$, otherwise $\eta_X(x) \in V^*$ for some $V \in \tau$ such that $V^* \subseteq U$. This makes V a nhood of x which implies that $V \in p$. Therefore $p \in V^* \subseteq \eta_X(X)$ so that $p = \eta_X(y)$ for some $y \in X$ different from x. It follows that p converges to y. This means that p has more than one limit which contradicts that X is Hausdorff. Thus $\eta_X(x) \notin U$ which is impossible. Hence A is closed.

(2) $\Rightarrow$ (1): Let U be open in X and choose $p \in \eta_X(U)$. Then $p = \eta_X(x)$ for some $x \in U$. Since X is discrete, $\{x\}$ is open in X. Therefore $p = \eta_X(x) \in (\{x\})^*$. Observe that $(\{x\})^* = \eta_X(\{x\})$. Indeed, if $q \in (\{x\})^*$, then $\{x\} \in q$. Therefore each $A \in \eta_X(x)$ belongs to q so that $\eta_X(x) \subseteq q$. Because $\eta_X(x)$ is an ultrafilter, $\eta_X(x) = q$. Thus $q \in \eta_X(\{x\})$. The other containment is straightforward.

Therefore $p \in (\{x\})^* \subseteq \eta_X(U)$, making $\eta_X(X)$ open. □

Denote by X_0 and e_{0X} the T_0-reflection of a space X and the T_0-reflection map, respectively. It is well-known that T_0-reflection maps are open.

In [11], Salbany proved that the T_0-reflection of the Salbany compactification $\mathcal{U}X$ of a T_0-space X is a *stable compactification* of X (see [12] for a formal definition of a stable compactification) with the compactification map denoted by $\beta_{0X} \colon X \to \beta_0 X$.

We have the following result following from Theorem 2.1.

Corollary 2.2. *If X is discrete, then β_{0X} is open.*

Proof. Follows since $\beta_{0X} = e_{0\mathcal{U}X} \circ \eta_X$ where both $e_{0\mathcal{U}X}$ and η_X are open. □

We give an example which shows that the statements of Theorem 2.1 are not equivalent to almost discrete $\mathcal{U}X$ unless X is a finite discrete space.

We shall need the following lemma.

Lemma 2.3. *Let (X, τ) be a topological space and $\mathcal{B} \subseteq \mathcal{P}(X)$. Then $\left(\overline{\bigcup_{V\in\mathcal{B}} V}\right)^* = \overline{\bigcup_{V\in\mathcal{B}} V^*}$.*

Proof. Let $q \in \left(\overline{\bigcup_{V\in\mathcal{B}} V}\right)^*$ and assume that $q \in X^* \smallsetminus \overline{\bigcup_{V\in\mathcal{B}} V^*}$. Then there is $G \in \tau$ such that $q \in G^* \subseteq X^* \smallsetminus \overline{\bigcup_{V\in\mathcal{B}} V^*}$. This makes $G^* \cap \left(\bigcup_{V\in\mathcal{B}} V^*\right) = \emptyset$. Therefore $G^* \cap V^* = \emptyset$ for each $V \in \mathcal{B}$ so that $G \cap V = \emptyset$ for each $V \in \mathcal{B}$. Therefore $G \cap \left(\bigcup_{V\in\mathcal{B}} V\right) = \emptyset$. Since G is open in X, $G \cap \overline{\bigcup_{V\in\mathcal{B}} V} = \emptyset$, which contradicts that both G and $\overline{\bigcup_{V\in\mathcal{B}} V}$ belong to q.

On the other hand, choose $q \in \overline{\bigcup_{V\in\mathcal{B}} V^*}$ and assume that $q \notin \left(\overline{\bigcup_{V\in\mathcal{B}} V}\right)^*$. Then $\overline{\bigcup_{V\in\mathcal{B}} V} \notin q$, making $X \smallsetminus \overline{\bigcup_{V\in\mathcal{B}} V} \in q$. Therefore $X^* \smallsetminus \left(\overline{\bigcup_{V\in\mathcal{B}} V}\right)^*$ is an open neighbourhood of q so that

$$\emptyset \neq \bigcup_{V\in\mathcal{B}} V^* \cap \left(X^* \smallsetminus \left(\overline{\bigcup_{V\in\mathcal{B}} V}\right)^*\right) = \bigcup_{V\in\mathcal{B}} V^* \cap \left(X^* \smallsetminus \left(\bigcup_{V\in\mathcal{B}} V\right)^*\right).$$

Since $\bigcup_{V\in\mathcal{B}} V^* \subseteq \left(\bigcup_{V\in\mathcal{B}} V\right)^*$, $(\bigcup_{V\in\mathcal{B}} V)^* \cap \left(X^* \smallsetminus \left(\bigcup_{V\in\mathcal{B}} V\right)^*\right) \neq \emptyset$, which is not possible.

Thus $\left(\overline{\bigcup_{V\in\mathcal{B}} V}\right)^* = \overline{\bigcup_{V\in\mathcal{B}} V^*}$. □

The previous lemma is a generalized result of the following.

Corollary 2.4. *Let (X,τ) be a topological space. Then $(\overline{A})^* = \overline{A^*}$ for every $A \subseteq X$.*

Example 2.5. A space X is finite and discrete if and only if $\mathcal{U}X$ is almost discrete. Indeed, if X is finite, then η_X is a homeomorphism. Because being discrete is a topological property, $\mathcal{U}X$ is discrete, making it almost discrete.

On the other hand, for each open $U \subseteq X$, we have that U^* is open in $\mathcal{U}X$ and hence closed because $\mathcal{U}X$ is almost discrete. Therefore $U = \eta_X^{-1}(U^*)$ is closed in X, making X almost discrete.

We show that X is compact. Let $\mathcal{C} = \{U_i : i \in I\}$ be a collection of open subsets of X such that $X = \bigcup_{i\in I} U_i$. Then $X^* = \left(\bigcup_{i\in I} U_i\right)^*$. Because $\bigcup_{i\in I} U_i$ is open in X and hence closed since X is now almost discrete, we get that $X^* = \left(\overline{\bigcup_{i\in I} U_i}\right)^*$. Therefore

$$X^* = \overline{\bigcup_{i\in I} U_i^*} = \bigcup_{i\in I} U_i^*$$

where the former equality follows from Lemma 2.3 and the latter equality follows since $\bigcup_{i\in I} U_i^*$ is open in $\mathcal{U}X$ and hence closed because $\mathcal{U}X$ is almost discrete. Since X^* is compact, there is a finite $J \subseteq I$ such that $X^* = \bigcup_{i\in J} U_i^*$. Therefore

$$X = \eta_X^{-1}(X^*) = \bigcup_{i\in J} \eta_X^{-1}(U_i^*) = \bigcup_{i\in J} U_i\,.$$

Thus X is compact.

Now, X is Hausdorff, compact and almost discrete making it compact and discrete. Therefore X is finite since every compact discrete space is finite.

Therefore, an infinite almost discrete and Hausdorff space X does not imply that $\mathcal{U}X$ is almost discrete. This tells us that we cannot replace X in the second statement of Theorem 2.1 with $\mathcal{U}X$ unless X is finite.

Our next step is to find conditions on some continuous functions such that their Salbany lifts are open. We give the following result some part of which will be used below.

Proposition 2.6. *Let $f : X \to Y$ be a function. Then f is surjective iff $\mathcal{U}f$ is surjective.*

Proof. ($\Longrightarrow$): Let $p \in \mathcal{U}Y$. Consider the filter $q = \{B \subseteq X : f^{-1}(D) \subseteq B$ for some $D \in p\}$ on X.

Extend q to an ultrafilter r on X. Then $(\mathcal{U}f)(r) = p$. To see this, choose $D \in p$. Then $f^{-1}(D) \in q \subseteq r$. By definition of $(\mathcal{U}f)(r)$, $D \in (\mathcal{U}f)(r)$ so that $p \subseteq (\mathcal{U}f)(r)$. Since p is an ultrafilter on Y, $p = (\mathcal{U}f)(r)$.

($\Longleftarrow$): Let $y \in Y$. Then $\eta_Y(y) \in \mathcal{U}Y$. Since $\mathcal{U}f$ is surjective, there is $p \in \mathcal{U}X$ such that $\eta_Y(y) = (\mathcal{U}f)(p)$. Choose a non-empty subset A of X such that $A \in p$.

Then $f^{-1}(f(A)) \in p$ so that $f(A) \in (\mathcal{U}f)(p) = \eta_Y(y)$. Therefore $y \in f(A)$ which implies the existence of $x \in A$ such that $f(x) = y$. Thus f is surjective. □

We show that openness of a surjective continuous function is equivalent to openness of its Salbany lift.

Theorem 2.7. *Let $f\colon (X, \tau) \to (Y, \rho)$ be a surjective continuous function. Then f is open iff $\mathcal{U}f$ is open.*

Proof. ($\Longrightarrow$): Let U be open in $\mathcal{U}X$. Then $U = \bigcup\{A^* : A \in \mathcal{B}\}$ for some $\mathcal{B} \subseteq \tau$. Therefore $(\mathcal{U}f)(U) = \bigcup\{(\mathcal{U}f)(A^*) : A \in \mathcal{B}\}$. Observe that $(\mathcal{U}f)(A^*) = (f(A))^*$. We only verify the containment $(\mathcal{U}f)(A^*) \supseteq (f(A))^*$. Choose $p \in (f(A))^*$. Then $f(A) \in p$. But $\mathcal{U}f$ is surjective by Lemma 2.6, so there is $q \in \mathcal{U}X$ such that $(\mathcal{U}f)(q) = p$. Therefore $A \in q$ so that $q \in A^*$. As a result, $p = (\mathcal{U}f)(q) \in (\mathcal{U}f)(A^*)$, as required.

Therefore $(\mathcal{U}f)(U) = \bigcup\{(f(A))^* : A \in \mathcal{B}\}$ and each $f(A)$ is open in Y. Thus $(\mathcal{U}f)(U)$ is open, making $\mathcal{U}f$ an open map.

($\Longleftarrow$): Let U be an open subset of X and choose $y \in f(U)$. Then $y = f(a)$ for some $a \in U$. Therefore $\eta_X(a) \in U^*$, where U^* is open in $\mathcal{U}X$. We then get that

$$\eta_Y(y) = \eta_Y(f(a)) = (\mathcal{U}f)(\eta_X(a)) \in (\mathcal{U}f)(U^*)\,,$$

with $(\mathcal{U}f)(U^*)$ being an open subset of $\mathcal{U}Y$ because of openness of $\mathcal{U}f$. Therefore $y \in \eta_Y^{-1}((\mathcal{U}f)(U^*))$. By continuity of η_Y, we have that $\eta_Y^{-1}((\mathcal{U}f)(U^*))$ is open in Y and since $(\mathcal{U}f)(U^*) \subseteq (f(U))^*$ which implies $\eta_Y^{-1}((\mathcal{U}f)(U^*)) \subseteq \eta_Y^{-1}((f(U))^*) = f(U)$, we get that $\eta_Y^{-1}((\mathcal{U}f)(U^*)) \subseteq f(U)$. Thus $y \in \operatorname{int}(f(U))$, making f open. (We did not need surjectivity). □

We close this section with a discussion of openness of Salbany extensions.

Proposition 2.8. *Let $f\colon X \to Y$ be a continuous function with a Salbany extension F. Then F is open only if f is open.*

Proof. Let U be open in $\mathcal{U}X$. Then $U = \bigcup_{V \in \mathcal{B}} V^*$ for some collection $\mathcal{B}$ of open subsets of X. Therefore $\eta_X^{-1}(U)$ is open in X so that $f(\eta_X^{-1}(U))$ is open in Y by openness of f. Let $y \in f(\eta_X^{-1})(U)$. Then $y = f(x)$ for some $x \in \eta_X^{-1}(U)$. Therefore $\eta_X(x) \in U$. We get that $y = f(x) = F(\eta_X(x)) \in F(U)$. Thus $F(U)$ is open in Y. □

The converse of Proposition 2.8 is not always true, as shown below.

Example 2.9. Consider a Hausdorff space X which is not almost discrete. The identity map $\mathrm{id}_{\mathcal{U}X}$ is open but η_X is not open by Theorem 2.1.

With some conditions on both the domain and codomain of a continuous function, we are able to improve Theorem 2.8.

Recall from [5] that every continuous function $f\colon X \to Y$ with a spectral codomain Y has a unique Salbany extension F which satisfies the condition that $F^{-1}(U) = (f^{-1}(U))^*$ for every compact-open $U \subseteq Y$.

Proposition 2.10. *Let $f\colon X \to Y$ be a continuous function from a space X with compact-open basis to a spectral space Y. Then f is open iff its Salbany extension F is open.*

Proof. The existence of the Salbany extension F of f follows since Y is spectral.

Proof for the necessary condition follows from Proposition 2.8.

For the sufficient condition, let U be compact-open in X. Then U^* is compact-open in $\mathcal{U}X$. Since images of compact sets are compact and F is open, $F(U^*)$ is compact-open in Y. Now, choose $y \in F(U^*)$. Then $y = F(p)$ for some $p \in U^*$. We get that $U \in p$ and $p \in F^{-1}(F(U^*))$. Since $F(U^*)$ is compact-open, $F^{-1}(F(U^*)) = (f^{-1}(F(U^*)))^*$.

Therefore $f^{-1}(F(U^*)) \in p$, making $f^{-1}(F(U^*)) \cap U \in p$. We have that

$$f^{-1}(F(U^*) \cap f(U)) = f^{-1}(F(U^*)) \cap f^{-1}(f(U)) \in p\,.$$

Since $F(U^*) \cap f(U)$ is compact, $\bigcap\{\overline{C} \cap F(U^*) \cap f(U) : f^{-1}(C) \in p\}$ is nonempty and

$$\begin{aligned}\bigcap\{\overline{C} \cap F(U^*) \cap f(U) : f^{-1}(C) \in p\} &= \overline{\{F(p)\}} \cap F(U^*) \cap f(U)\\ &= \{F(p)\} \cap F(U^*) \cap f(U)\,,\end{aligned}$$

where the first equality follows since $\bigcap\{\overline{C} : f^{-1}(C) \in p\} = \overline{\{F(p)\}}$ and intersections distribute over arbitrary intersections. Therefore $y = F(p) \in f(U)$ so that $f(U) = F(U^*)$.

Now, for open $V \subseteq X$, we have that $V = \bigcup_{U\in\mathcal{B}} U$ for some collection $\mathcal{B}$ of compact-open subsets of X. Therefore

$$f(V) = f\left(\bigcup_{U\in\mathcal{B}} U\right) = \bigcup_{U\in\mathcal{B}} f(U) = \bigcup_{U\in\mathcal{B}} F(U^*)\,,$$

where each $F(U^*)$ is open (and compact), making $f(V)$ open. Thus f is open. □

Example 2.11. I proved in [10, Proposition 3.5.] that a space $\beta_0 X$ is sober and hence spectral. Therefore the map $\beta_{0X} : X \to \beta_0 X$ from an infinite discrete space X is an example of an open continuous map (see Corollary 2.2) which is not a homeomorphism. For, if β_{0X} is a homeomorphism, then $\beta_0 X$ is discrete, making $\mathcal{U}X$ discrete and hence almost discrete. By Example 2.5, X is finite which is impossible.

3. Initiality of Salbany maps, Salbany lifts and Salbany extensions

In [3], the authors call a continuous function $f\colon X \to Y$ *initial* in case $A = f^{-1}(\overline{f(A)})$ for all closed $A \subseteq X$. This is equivalent to saying that for each open $A \subseteq X$, $A = f^{-1}(U)$ for some open $U \subseteq Y$. To see this, assume that f is initial and let $U \subseteq X$ be open. Then $X \smallsetminus U = f^{-1}(\overline{f(X \smallsetminus U)})$. Therefore

$$U = X \smallsetminus f^{-1}(\overline{f(X \smallsetminus U)}) = f^{-1}(Y \smallsetminus \overline{f(X \smallsetminus U)})\,.$$

On the other hand, we always have that $F \subseteq f^{-1}(\overline{f(F)})$ for all $F \subseteq X$. For closed $F \subseteq X$, choose $x \in f^{-1}(\overline{f(F)})$ such that $x \notin F$. Then $X \smallsetminus F = f^{-1}(V)$ for some open $V \subseteq Y$. Therefore $f(x) \in \overline{f(F)} \cap V$ so that $f(F) \cap V \neq \emptyset$ since V is open. It is clear that

$$\emptyset \neq F \cap f^{-1}(V) = F \cap (X \smallsetminus F)$$

which is impossible. Thus $x \in F$ and hence $F = f^{-1}(\overline{f(F)})$, making f initial.

Although initiality is not directly a type of open maps, but every surjective initial continuous map is open. Indeed, if $U \subseteq X$ is open, then $U = f^{-1}(V)$ for some open $V \subseteq Y$. By surjectivity of f, $f(U) = f(f^{-1}(V)) = V$, making $f(U)$ open and hence f open.

The Salbany map is initial, as we show below.

Proposition 3.1. *For any space X, the Salbany map η_X is initial.*

Proof. Follows since $U = \eta_X^{-1}(U^*)$ for every $U \subseteq X$. □

Observation 3.2. Since the Salbany map is seldomly open, Proposition 3.1 tells us that not every initial map is open.

We recall the following result from [10] which we shall use below.

Lemma 3.3. *Let $f\colon X \to Y$ be a function. Then $(f^{-1}(A))^* = (\mathcal{U}f)^{-1}(A^*)$ for every $A \subseteq Y$.*

In the following result, we characterize initiality of the Salbany lift of any continuous function.

Theorem 3.4. *Let $f\colon (X, \tau) \to (Y, \rho)$ be a continuous function. Then f is initial iff $\mathcal{U}f$ is initial.*

Proof. ($\Longrightarrow$): Let U be an open subset of $\mathcal{U}X$. Then there is $\mathcal{B} \subseteq \tau$ such that

$$U = \bigcup\{B^* : B \in \mathcal{B}\}.$$

Since f is initial, for each $B \in \mathcal{B}$, there is $A_B \in \rho$ such that $B = f^{-1}(A_B)$. Therefore $B^* = (f^{-1}(A_B))^* = (\mathcal{U}f)^{-1}(A_B^*)$ where the latter equality follows from Lemma 3.3. Therefore

$$U = \bigcup\{(\mathcal{U}f)^{-1}(A_B^*) : B \in \mathcal{B}\} = (\mathcal{U}f)^{-1}\left(\bigcup\{A_B^* : B \in \mathcal{B}\}\right).$$

Thus $\mathcal{U}f$ is initial.

($\Longleftarrow$): Let V be open in X. Then V^* is open in $\mathcal{U}X$. Since f is initial, there is $W \in \mathcal{U}\rho$ such that $V^* = (\mathcal{U}f)^{-1}(W)$. Therefore

$$V = \eta_X^{-1}(V^*) = \eta_X^{-1}((\mathcal{U}f)^{-1}(W)) = f^{-1}(W).$$

Thus f is initial. □

We consider initiality of Salbany extensions.

Theorem 3.5. *Let $f\colon (X, \tau) \to (Y, \rho)$ be a continuous function from a hereditary compact space X onto a spectral space Y and let F be the Salbany extension of f. The following statements are equivalent.*

(1) *f is initial.*

(2) *F is initial.*

Proof. (1) $\Longrightarrow$ (2): Let U be open in $\mathcal{U}X$. Then $U = \bigcup_{V\in\mathcal{B}} V^*$ for some collection $\mathcal{B} \subseteq \tau$. Since X is hereditary compact, each $V \in \mathcal{B}$ is compact. By initiality of f, for each $V \in \mathcal{B}$, there is an open set $W_V \subseteq Y$ such that $V = f^{-1}(W_V)$. Because f is surjective, $f(V) = W_V$, making W_V compact-open in Y. Since Y is spectral, $F^{-1}(W_V) = (f^{-1}(W_V))^*$ so that $F^{-1}(W_V) = V^*$. Therefore

$$U = \bigcup_{V\in\mathcal{B}} F^{-1}(W_V) = F^{-1}\left(\bigcup_{V\in\mathcal{B}} W_V\right).$$

Thus F is initial.

(2) $\Longrightarrow$ (1): Let U be open in X. Then U^* is open in $\mathcal{U}X$ so that $U^* = F^{-1}(W)$ for some open $W \subseteq Y$. We get that

$$U = \eta_X^{-1}(U^*) = \eta_X^{-1}(F^{-1}(W)) = f^{-1}(W).$$

Thus f is initial. (We did not use the assumptions that X is hereditary compact and f is surjective). □

We consider an example of a continuous map with properties hypothesized above that is not a homeomorphism.

Example 3.6. It is clear that every sober and finite space is spectral. For instance, the space (Y, ρ), where $Y = \{0, 1\}$ and $\rho = \{\emptyset, Y, \{1\}\}$, is spectral (see [10] for verification of sobriety). Define a map $f\colon (X, \tau) \to (Y, \rho)$, where where $X = \{0, 1, 2\}$ and $\tau = \{\emptyset, X\}$, by $f(0) = f(1) = f(2) = 1$. Then f is an initial map which is not a homeomorphism.

4. Nearly openness of Salbany maps, Salbany lifts and Salbany extensions

Recall from [1] that a continuous function $f : X \to Y$ is *nearly open* if for each open $V \subseteq Y$, $f^{-1}(\overline{V}) = \overline{f^{-1}(V)}$. This is equivalent to saying that for every open set $V \subseteq Y$, $f^{-1}(\overline{V}) \subseteq \overline{f^{-1}(V)}$ since every continuous function $g : Z \to Q$ satisfies the condition that $\overline{g^{-1}(A)} \subseteq g^{-1}(\overline{A})$ for every $A \subseteq Q$, [8]. We shall freely use the fact that the composition of two nearly open maps is nearly open.

We start by showing that Salbany maps are nearly open. This result will follow from the following proposition.

Proposition 4.1. *Let $f\colon X \to Y$ be an initial map such that $f(X)$ is dense in Y, then f is nearly open.*

Proof. Let $U \subseteq Y$ be open. If $U = \emptyset$, then $f^{-1}(\overline{U}) = \overline{f^{-1}(U)}$. For $U \neq \emptyset$, choose $x \in f^{-1}(\overline{U})$ and assume that there is an open nhood V of x missing $f^{-1}(U)$. Since f is initial, there is an open $W \subseteq Y$ such that $V = f^{-1}(W)$. Therefore W is an open nhood of $f(x)$ and hence meets U. Since $f(X)$ is dense, $f(X) \cap W \cap U \neq \emptyset$. Therefore $V \cap f^{-1}(U) = f^{-1}(W) \cap f^{-1}(U) \neq \emptyset$ which is a contradiction. Thus $f^{-1}(\overline{U}) \subseteq \overline{f^{-1}(U)}$ as required. □

Corollary 4.2. *For each space X, the Salbany map η_X is nearly open.*

In the next theorem, we characterize nearly openness of the Salbany lift of a continuous function.

Theorem 4.3. *A continuous function* $f\colon (X,\tau)\to(Y,\rho)$ *is nearly open iff* $\mathcal{U}f$ *is nearly open.*

Proof. ($\Longrightarrow$): Let $Q\subseteq \mathcal{U}Y$ be open and assume that there is $p\in(\mathcal{U}f)^{-1}(\overline{Q})$ such that $p\notin \overline{(\mathcal{U}f)^{-1}(Q)}$. There is a collection $\mathcal{B}\subseteq\rho$ such that $Q=\bigcup_{V\in\mathcal{B}}V^*$ so that

$$\begin{aligned}
p\in(\mathcal{U}f)^{-1}(\overline{Q}) &= (\mathcal{U}f)^{-1}\left(\overline{\bigcup_{V\in\mathcal{B}}V^*}\right)\\
&= (\mathcal{U}f)^{-1}\left(\left(\overline{\bigcup_{V\in\mathcal{B}}V}\right)^*\right) \quad \text{by Lemma 2.3}\\
&= \left(f^{-1}\left(\overline{\bigcup_{V\in\mathcal{B}}V}\right)\right)^* \quad \text{by Lemma 3.3}\\
&= \left(\overline{f^{-1}\left(\bigcup_{V\in\mathcal{B}}V\right)}\right)^* \quad \text{since } f \text{is nearly open}\\
&= \left(\overline{\bigcup_{V\in\mathcal{B}}f^{-1}(V)}\right)^*\\
&= \overline{\bigcup_{V\in\mathcal{B}}(f^{-1}(V))^*} \quad \text{by Lemma 2.3.}
\end{aligned}$$

We also have that $p\in X^*\smallsetminus\overline{(\mathcal{U}f)^{-1}(Q)}$ so that $\bigcup_{V\in\mathcal{B}}(f^{-1}(V))^*\cap\Big(X^*\smallsetminus\overline{(\mathcal{U}f)^{-1}(Q)}\Big)\neq\emptyset$. Therefore

$$\begin{aligned}
&\bigcup_{V\in\mathcal{B}}((\mathcal{U}f)^{-1}(V^*))\cap\big(X^*\smallsetminus(\mathcal{U}f)^{-1}(Q)\big)\neq\emptyset\\
&\qquad\Longrightarrow (\mathcal{U}f)^{-1}\left(\bigcup_{V\in\mathcal{B}}V^*\right)\cap\big(X^*\smallsetminus(\mathcal{U}f)^{-1}(Q)\big)\neq\emptyset\\
&\qquad\Longrightarrow (\mathcal{U}f)^{-1}(Q)\cap\big(X^*\smallsetminus(\mathcal{U}f)^{-1}(Q)\big)\neq\emptyset
\end{aligned}$$

which is impossible. Thus $p\in\overline{(\mathcal{U}f)^{-1}(Q)}$ and hence $\mathcal{U}f$ is nearly open.

($\Longleftarrow$): Let $U\subseteq Y$ be open. Then U^* is open in $\mathcal{U}X$. Since $\mathcal{U}f$ is nearly open, $(\mathcal{U}f)^{-1}(\overline{U^*})=\overline{(\mathcal{U}f)^{-1}(U^*)}$, i.e., $(\mathcal{U}f)^{-1}(\overline{U^*})=\overline{(f^{-1}(U))^*}$. Therefore $\eta_X^{-1}((\mathcal{U}f)^{-1}(\overline{U^*}))=\eta_X^{-1}(\overline{(f^{-1}(U))^*})$ so that $f^{-1}(\eta_Y^{-1}(\overline{U^*}))=\overline{\eta_X^{-1}((\mathcal{U}f)^{-1}(U^*))}$ because $\eta_Y\circ f=(\mathcal{U}f)\circ\eta_X$ and η_X is nearly open. We further get that

$$f^{-1}(\overline{U})=f^{-1}\left(\overline{\eta_Y^{-1}(U^*)}\right)=\overline{f^{-1}(\eta_Y^{-1}(U^*))}=\overline{f^{-1}(U)}.$$

Thus f is nearly open. $\square$

We consider nearly openness of Salbany extensions.

We start by showing that continuous functions with nearly open Salbany extensions are nearly open.

Proposition 4.4. *Let $f : X \to Y$ be a continuous function. If f has a nearly open Salbany extension, then f is nearly open.*

Proof. Let F be a nearly open Salbany extension of f and choose an open $U \subseteq Y$. Then $F^{-1}(\overline{U}) = \overline{F^{-1}(U)}$ since F is nearly open. Because $F \circ \eta_X = f$ and η_X is nearly open, we have that

$$f^{-1}(\overline{U}) = \eta_X^{-1}(F^{-1}(\overline{U})) = \eta_X^{-1}(\overline{F^{-1}(U)}) = \overline{\eta_X^{-1}(F^{-1}(U))} = \overline{f^{-1}(U)}$$

which proves the result. □

Recall that a space X is *Stonean* if X is compact, Hausdorff and *extremally disconnected* in the sense that for each open $U \subseteq X$ and each $x \in \overline{U}$, there is a clopen set $C \subseteq X$ such that $x \in C \subseteq \overline{U}$. Extremally disconnection is equivalent to a condition that every regular-closed subset is open.

Lemma 4.5. *If X is Stonean, then the retraction r_X map of the Salbany map η_X is nearly open.*

Proof. Let $U \subseteq X$ be open and assume that there is $p \in r_X^{-1}(\overline{U})$ such that $p \notin \overline{r_X^{-1}(U)}$. Then $r_X(p) \in \overline{U}$ and there is open $V \subseteq X$ such that $p \in V^* \subseteq X^* \smallsetminus \overline{r_X^{-1}(U)}$. Therefore $V^* \cap r_X^{-1}(U) = \emptyset$, making

$$\emptyset = \overline{V^*} \cap r_X^{-1}(U) = (\overline{V})^* \cap r_X^{-1}(U)$$

because $r_X^{-1}(U)$ is open. We also have that $V \in p$ which, by definition of r_X, implies that $r_X(p) \in \overline{V}$. Since X is Stonean, there is a clopen $C \subseteq X$ such that $r_X(p) \in C \subseteq \overline{V}$. Therefore $C \cap U \neq \emptyset$.

We claim that $r_X^{-1}(C) \subseteq C^*$. To see this, let $q \in r_X^{-1}(C)$ be such that $q \notin C^*$. Then $r_X(q) \in C$ and $X \smallsetminus C \in q$. Therefore $r_X(p) \in \overline{X \smallsetminus C} = X \smallsetminus C$ which is impossible.

Now, since r_X is surjective, $r_X^{-1}(C) \cap r_X^{-1}(U) \neq \emptyset$ so that

$$\emptyset \neq C^* \cap r_X^{-1}(U) \subseteq (\overline{V})^* \cap r_X^{-1}(U)$$

which is impossible. Thus $p \in \overline{r_X^{-1}(U)}$. □

Remark 4.6. The claim that $r_X^{-1}(C) \subseteq C^*$ for clopen $C \subseteq X$ actually holds for all open subset C of a space X in which η_X has a retraction map r_X.

Theorem 4.7. *Let $f\colon X \to Y$ be a continuous map to a Stonean space Y. Then f is nearly open iff it has a nearly open Salbany extension.*

Proof. ($\Longrightarrow$): It is clear that $r_X \circ \mathcal{U}f$ is the Salbany extension of f. Now, since f is nearly open, it follows from Theorem 4.3 that $\mathcal{U}f$ is nearly open. Because r_X is also nearly open, we have that $r_X \circ \mathcal{U}f$ is nearly open.

($\Longleftarrow$): Follows from Proposition 4.4. □

Remark 4.8. The map $r_X \circ \mathcal{U}f$ in Theorem 4.7 is unique. This follows from [10] where I proved that every continuous function with a compact Hausdorff codomain has a unique Salbany extension. A similar result can also be found in [5] where the authors showed that every continuous function with a spectral codomain has a unique Salbany extension, a result we recalled and used in Section 3. Since every Stonean space is spectral, this makes Y spectral.

In the next example we give an example of a non-homeomorphism nearly open map with a Stonean codomain. We shall need the following result.

Lemma 4.9. *Let (X, τ) be a topological space. If X is a clopen subset of $\mathcal{U}$, then $C = V^*$ for some clopen $V \subseteq X$.*

Proof. See [2]. □

Example 4.10. Let (X, τ) be a topological space. Then X is discrete if and only if $\mathcal{U}X$ is Stonean. To verify this, we start by showing that X is extremally disconnected if and only if $\mathcal{U}X$ is extremally disconnected. Suppose that X is extremally disconnected, let U be an open subset of $\mathcal{U}X$ and choose $p \in \overline{U}$. Then $p \in \overline{U} = \overline{\bigcup_{V \in \mathcal{B}} V^*}$ for some $\mathcal{B} \subseteq \tau$. By Lemma 2.3, $\overline{U} = \left(\overline{\bigcup_{V \in \mathcal{B}} V}\right)^*$ so that $\overline{\bigcup_{V \in \mathcal{B}} V} \in p$. Because $\overline{\bigcup_{V \in \mathcal{B}} V}$ is a regular-closed subset of X, it follows from that it is open and hence clopen. Therefore $\left(\overline{\bigcup_{V \in \mathcal{B}} V}\right)^*$ is clopen in $\mathcal{U}X$, making $\overline{U}$ the required clopen set.

Conversely, assume that $\mathcal{U}X$ is extremally disconnected, let $U \in \tau$ and choose $x \in \overline{U}$. Then $\eta_X(x) \in (\overline{U})^* = \overline{U^*}$, where the latter equality follows from Corollary 2.4. Therefore there is a clopen $C \subseteq \mathcal{U}X$ such that $\eta_X(x) \in C \subseteq \overline{U^*}$. By Lemma 4.9, such $C = V^*$ for some clopen $V \subseteq X$. Therefore $x \in V \subseteq \overline{U}$ so that $x \in V \subseteq \int(\overline{U})$. Thus $\overline{U} = \int(\overline{U})$, as required.

Now, if X is discrete, then X is extremally disconnected, making $\mathcal{U}X$ extremally disconnected. Since $\mathcal{U}X$ is compact and X being discrete also implies that $\mathcal{U}X$ is Hausdorff, we have that $\mathcal{U}X$ is a Stonean space.

On the other hand, if $\mathcal{U}X$ is Stonean, then it is Hausdorff, making X discrete.

As a result, the Salbany map of an infinite discrete space is nearly open, not a homeomorphism and has a Stonean codomain.

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Department of Mathematics, Rhodes University,
P.O. Box 94, Grahamstown 6140,
South Africa
E-mail: sibahlezwide@gmail.com