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COMBINATION OF T-NORMS AND THEIR CONORMS

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Non-negative linear combinations of $t_{\min}$-norms and their conorms are used to formulate some decision making problems using systems of max-separable equations and inequalities and optimization problems under constraints described by such systems. The systems have the left hand sides equal to the maximum of increasing functions of one variable and on the right hand sides are constants. Properties of the systems are studied as well as optimization problems with constraints given by the systems and appropriate solution methods are proposed. Motivation of this research are decision making investment situations both in deterministic and uncertain environment. Possibilities of further research are briefly discussed in the concluding remarks of the paper.

Keywords: combining triangular norms and conorms, nonlinear optimization, decision making, operations research

Classification: 90C30, 94D05, 90B50, 90C08

1. INTRODUCTION

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Expressions $a \wedge x \equiv \min(a, x)$, $a \vee x \equiv \max(a, x)$, where a, x are real numbers are known in the fuzzy sets theory as so called triangular norm ($t_{\min} - norm$) and its conorm, and are used as membership functions of intersection or union of fuzzy sets to model various decision making situations in the fuzzy environment. In this paper we will show how non-negative linear combinations of $t_{\min} - norms$ and their conorms can be used to formulate a class of constrained optimization problems, which are motivated by some deterministic or stochastic decisions. The following motivating example describes such simple decision making model for a single decision maker and two options.

Example 1.1. Decision maker wants to invest certain amount $x \in [0, \overline{x}]$ in two investment options.

Option 1 is to invest αx money units with profit (rate of return) $\alpha r(x)$, where $\alpha \in [0, 1]$ and we assume that $r : R_+ \to R_+$, $R_+ = [0, \infty)$ is a strictly increasing and continuous function of x. The rate of return increases if x increases, but it may be maximally equal to a given upper bound αa and further remains unchanged if x increases, i. e. guaranteed rate of return is $\alpha a \wedge \alpha r(x) = \alpha(a \wedge r(x))$.

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If the decision maker decides to invest $\beta x, \beta \in [0,1], \alpha+\beta \leq 1$ in Option 2, his guaranteed rate of return is $\beta b \vee \beta s(x) = \beta(b \vee s(x))$, where we assume that $s : R_+ \to R_+$ is a strictly increasing continuous function. The total guaranteed rate of return is therefore equal to $\alpha(a \wedge r(x)) + \beta(b \vee s(x))$. Let $\hat{x}$ be a recommended (secure) investment level. The decision maker has to decide which amount x he is going to invest if he has at his disposal $x \in [0, \overline{x}]$ and wants to minimize the distance $|x - \hat{x}|$ from a given recommended (secure) level of investment $\hat{x}$. In other words he has to solve the minimization problem

$$|x - \hat{x}| \to \min$$

subject to

$$\tilde{a} \wedge \tilde{r}(x) + \tilde{b} \vee \tilde{s}(x) \leq c,$$

where c is given upper bound on the total rate of return, $\tilde{a} = \alpha a,\ \tilde{b} = \beta b,\ \tilde{r} = \alpha r,\ \tilde{s} = \beta s.$

Note that we could consider similar problems with equality or inequality $\geq$ in the constraint restriction. Coefficients $\alpha,\ \beta$ may be interpreted as an intensity of investment in Option 1 and 2.

Similar problems arise e. g. if the utility of applied incentives (e. g. vitamins, combinations of medicaments) increases only up to a certain level and then remains unchanged or increases only if applied on a sufficiently high level being ineffective below this level and we want to be as close as possible to a recommended level of application.

The next example shows a more complex problem with n decision makers and m places for decisions.

Example 1.2. Let us have n decision makers $j,\ j \in J = \{1, \ldots, n\}$. Each decision maker $j \in J$ decides how to invest a part of the amount $x_j \in [0, \overline{x}_j]$. Otherwise the situation is the same as in the preceding example, i. e. he invests under the same restrictions at place $i \in I = \{1, \ldots, m\}$ the amounts $r_{ij}(x_j)$ in Option 1 and $s_{ij}(x_j)$ in Option 2, so that the total guaranteed rate of return will be $a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j))$, where $r_{ij},\ s_{ij}$ are given continuous strictly increasing functions. Let $\hat{x_j}$ be a given level of recommended (secure) investment. We want to solve the problem

$$\max_{j \in J} |x_j - \hat{x}_j| \to \min \ \textit{subject to}\ x_j \in [0, \overline{x}_j],\ \max_{j \in J}(a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j)) *_i c_i,\ i \in I,$$

where $*_i$ is one of the relations $\leq, =, \geq$.

In the next section, we will propose a solution method for solving problems like in Example 1.2. The problems are in general non-convex and non-smooth and we will solve them by methods, which take advantage of the special structure of the problems. Let us note further that the number of decision options may be greater than two according to the type of the given decision problem.

The following example shows a case of a stochastic decision making.

Example 1.3. Let us have again the two Options of Example 1.2. and we invest each time either with probability p_{ij} in Option 1 or with probability q_{ij} in Option 2 for all $i \in I, j \in J$. The sum $p_{ij}(a_{ij} \wedge r_{ij}(x_j)) + q_{ij}(b_{ij} \vee s_{ij}(x_j)),\ p_{ij} + q_{ij} = 1$ is the expected value of rate of return in place $i \in I$, which can the decision maker $j \in J$ expect. This maximum of the expected level of rate of return is bounded by c_i at each place $i \in I$. The objective function is $\max_{j\in J} f_j(x_j)$, where $f_j : R \to R,\ j \in J$ are continuous functions. Functions $f_j(x_j),\ f \in J$ may express e. g. the risk connected with the investment x_j or a distance $|x_j - \hat{x}_j|$ from a recommended (secure) level of investment $\hat{x}$. The objective function is therefore minimized under the constraints

$$x_j \in [0, \overline{x}_j],\ \max_{j\in J} t_{ij}(x_j) *_i c_i,\ i \in I,$$

where $*_i$ stands for one of the relations $\leq, =, \geq$ and $t_{ij}(x_j) = p_{ij}(a_{ij} \wedge r_{ij}(x_j) + q_{ij}(b_{ij} \vee s_{ij}(x_j)$.

Other problems can be formulated in frame of the fuzzy set theory, where expression $\alpha(a \wedge x) + \beta(b \vee x),\ \alpha, \beta \in R_+$ may be interpreted e. g. as a linear combination of two membership functions: one belonging to the (fuzzy) intersection of fuzzy sets A and X having a finite support N $= \{l, \ldots, n\}$ with membership functions $a : N \to [0, 1], x : N \to [0, 1]$ and the other one to the (fuzzy) union of fuzzy set B (with membership function $b : N \to [0, 1]$) and the fuzzy set X.

2. GENERAL FORMULATION OF THE MINIMIZATION PROBLEMS

The following notation will be introduced throughout the paper:

$\alpha \wedge \beta \equiv \min(\alpha, \beta)$, $\alpha \vee \beta \equiv \max(\alpha, \beta)$ for all $\alpha,\ \beta \in R_+ = (0, \infty)$. Let $r : R_+ \to R_+$ be a strictly increasing continuous function. Then $r^{-1} : R_+ \to R_+$ will denote the inverse function to r. Under the combination of norms and/or conorms we will consider in the sequel their non-negative linear combinations. Since for any $\alpha \geq 0, \beta \geq 0, a, b \in R_+, r : R_+ \to R_+, s : R_+ \to R_+$ hold the equalities

$$\alpha(a \wedge r(x)) + \beta(b \vee s(x)) = (\alpha a) \wedge (\alpha r(x)) + (\beta b) \vee (\beta s(x)) = \tilde{a} \wedge \tilde{r}(x) + \tilde{b} \vee \tilde{s}(x),$$

where $\tilde{a} = \alpha a,\ \tilde{r}(x) = \alpha r(x), \tilde{b} = \beta b,\ \tilde{s}(x) = \beta s(x)$, we can further consider only the sum of t-norms and their conorms instead of their explicit linear combination.

Let us consider the following systems of inequalities and equations:

$$\max_{j\in J}(a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j)) \leq b_i,\ i \in I, \tag{1}$$

$$\max_{j\in J}(a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j)) = b_i,\ i \in I, \tag{2}$$

$$\max_{j\in J}(a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j)) \geq b_i,\ i \in I, \tag{3}$$

where $r_{ij} : R_+ \to R_+,\ s_{ij} : R_+ \to R_+$ are for all $i \in I,\ j \in J$ strictly increasing continuous functions of x_j, and $a_{ij}, b_{ij}, b_i \in R_+$ are given numbers. The set of all $x = (x_1, x_2, \ldots, x_n)^T$ satisfying (1), (2), (3) will be denoted $M_1(b)$, $M_2(b)$,$M_3(b)$ respectively. Let $M \subset R^n,\ M \neq \emptyset$, let $\overline{x} \in M$ with the property $x \leq \overline{x}$ for all $x \in M$. Then $\overline{x}$ will be called the maximum element of M.

In the next section, we will study properties of the sets $M_i(b),\ i = 1, 2, 3$ and propose methods for solving the following three optimization problems successively for $i = 1, 2, 3$:

$$f(x) = \max_{j\in J}(f_j(x_j)) \ \rightarrow \ \min, \ \textit{subject to } x \in M_i(b), \ x \geq \underline{x}, \tag{4}$$

where $f_j(x_j)$ are continuous monotone or convex functions and $\underline{x}_j$ are given lower bounds. The methods generalize some results of [1] – [4], [8]. Equations and inequalities describing sets $M_i, i = 1, 2, 3$ can be considered as a generalization of fuzzy relation equations or fuzzy relational inequalities (see e. g. [6, 7]).

3. SOLUTION METHODS

First we will study properties of sets $M_i(b),\ i = 1, 2, 3,$ which will serve as sets of feasible solutions for the considered optimization problems. Our first aim is to find maximum elements of sets $M_1(b),\ M_2(b),\ M_3(b)$ and derive conditions, under which the sets $M_i(b) \neq \emptyset,\ i = 1, 2, 3.$ The knowledge of the maximum elements of sets $M_i,\ i = 1,\ 2,\ 3$ is substantial for solving the optimization problems considered later.

Let

$$V_{ij} \equiv \{x_j \mid a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j) \leq b_i\}. \tag{5}$$

Let $r_{ij}^{-1}(a_{ij}) < s_{ij}^{-1}(b_{ij})$. Then the following three cases can take place:

(i) If $r_{ij}(x_j) < a_{ij}$, then $x_j < r_{ij}^{-1}(a_{ij}) < s_{ij}^{-1}(b_{ij})$ and we have further $x_j < s_{ij}^{-1}(b_{ij})$, which implies $s_{ij}(x_j) < b_{ij}$ so that $a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j) = r_{ij}(x_j) + b_{ij}$.

(ii) If $r_{ij}^{-1}(a_{ij}) \leq x_j \leq s_{ij}^{-1}(b_{ij})$, then $a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j) = a_{ij} + b_{ij}$.

(iii) If $x_j > s_{ij}^{-1}(b_{ij})$, then $a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j) = a_{ij} + s_{ij}(x_j)$.

Let $\overline{x}_j(b_i)$ denote the maximum element of set $V_{ij}(b_i)$. Then we have:

in case (i) $\overline{x}_j(b_i) = r_{ij}^{-1}(b_i - b_{ij})$,
in case (ii) it is $\overline{x}_j(b_i) = s_{ij}^{-1}(b_i - a_{ij})$, and
in case (iii) we have $\overline{x}_j(b_i) = s_{ij}^{-1}(b_i - a_{ij})$.

Let us assume now that $r_{ij}^{-1}(a_{ij}) \geq s_{ij}^{-1}(b_{ij})$. Then the following three cases can occur:

(iv) If $x_j < s_{ij}^{-1}(b_{ij}) \leq r_{ij}^{-1}(a_{ij})$, then $s_{ij}(x_j) < b_{ij}, r_{ij}(x_j) < a_{ij}$ and we have $a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j) = r_{ij}(x_j) + b_{ij}$.

(v) If $s_{ij}^{-1}(b_{ij}) \leq x_j \leq r_{ij}^{-1}(a_{ij})$, we have $a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j) = r_{ij}(x_j) + s_{ij}(x_j)$.

(vi) If $x_j > r_{ij}^{-1}(a_{ij})$, we have $a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j) = a_{ij} + s_{ij}(x_j)$.

Therefore we have:

in case (iv) $\overline{x}_j(b_i) = r_{ij}^{-1}(b_i - b_{ij})$,
in case (v) it is $\overline{x}_j(b_i) = (r_{ij} + s_{ij})^{-1}(b_i)$, where $(r_{ij} + s_{ij})(x_j) = r_{ij}(x_j) + s_{ij}(x_j)$,
in case (vi) $\overline{x}_j(b_i) = s_{ij}^{-1}(b_i - a_{ij})$.

We derived therefore a procedure, which makes possible to compute in all cases the maximum element $\overline{x}_j(b_i)$ for each $i \in I,\ j \in J,\ b_i \in R_+$.

The following algorithm summarizes the computation of $\overline{x}_j(b_i)$ for a fixed $i \in I,\ j \in J$.

Algorithm 1

$\boxed{1}$ Input $i \in I,\ j \in J,\ a_{ij},\ b_{ij},\ b_i, r_{ij}(x_j), s_{ij}(x_j)$;
$\boxed{2}$ If $r_{ij}^{-1}(a_{ij}) \geq s_{ij}^{-1}(b_{ij})$, go to $\boxed{5}$;
$\boxed{3}$ If $r_{ij}^{-1}(b_i - b_{ij}) < r_{ij}^{-1}(a_{ij})$, set $\overline{x}_j(b_i) = r_{ij}^{-1}(b_i - b_{ij})$;
$\boxed{4}$ If $s_{ij}^{-1}(b_i - b_{ij}) \geq s_{ij}^{-1}(b_{ij})$, set $\overline{x}_j(b_i) = s_{ij}^{-1}(b_i - a_{ij})$;
$\boxed{5}$ If $r_{ij}^{-1}(b_i - b_{ij}) < b_{ij}$, set $\overline{x}_j(b_i) = r_{ij}^{-1}(b_i - b_{ij})$;
$\boxed{6}$ If $b_{ij} \leq (r_{ij} + s_{ij})^{-1}(b_i) \leq a_{ij}$, set $\overline{x}_j(b_i) = (r_{ij} + s_{ij})^{-1}(b_i)$;
$\boxed{7}$ If $r_{ij}^{-1}(a_{ij}) < s_{ij}^{-1}(b_i - a_{ij})$, set $\overline{x}_j(b_i) = s_{ij}^{-1}(b_i - a_{ij})$ STOP.

The following numerical examples illustrate the procedure.

Example 3.1.

(a) Let $a_{ij} = 5,\ b_{ij} = 7,\ b_i = 9,\ r_{ij}(x_j) = x_j = s_{ij}(x_j)$ so that $a_{ij} < b_{ij}$(Steps $\boxed{1}$,$\boxed{2}$). Since $b_i - b_{ij} = 9 - 7 = 2 \leq a_{ij} = 5$, we have $\overline{x}_j(b_i) = b_i - b_{ij} = 2$ (Step $\boxed{3}$).

(b) Let $b_{ij} = 5,\ a_{ij} = 7,\ b_i = 9,\ r_{ij}(x_j) = x_j = s_{ij}(x_j)$ so that $a_{ij} \geq b_{ij}$ (Steps $\boxed{1}$,$\boxed{2}$). Since $b_i - a_{ij} = 9 - 7 = 2 < b_{ij} = 5$, we have $\overline{x}_j(b_i) = b_i - a_{ij} = 9 - 7 = 2$ (Step $\boxed{5}$).

(c) Let $a_{ij} = 7,\ b_{ij} = 5,\ b_i = 14,\ r_{ij}(x_j) = x_j = s_{ij}(x_j)$ so that $a_{ij} \geq b_{ij}$(Steps $\boxed{1}$,$\boxed{2}$). Since $b_i - b_{ij} = 14 - 5 = 9 \geq a_{ij} = 7$ and $b_i - a_{ij} = 14 - 7 = 7 \geq b_{ij} = 5$, we proceed according to Step $\boxed{7}$ and obtain: $\overline{x}_j(b_i) = s_{ij}^{-1}(b_i - a_{ij}) = 14 - 7 = 7$.

(d) Let $a_{ij} = 6,\ b_{ij} = 6,\ b_i = 12,\ r_{ij}(x_j) = x_j = s_{ij}(x_j)$ so that $a_{ij} \geq b_{ij}$ (Steps $\boxed{1}$,$\boxed{2}$).Since $b_i - b_{ij} = 12 - 5.5 = 6.5 > a_{ij} = 6$ and $b_i - a_{ij} = 12 - 6 = 6 > b_{ij} = 5.5$, we proceed according to Step $\boxed{7}$ and obtain: $\overline{x}_j(b_i) = b_i = 12$.

We will summarize the obtained results in the following theorems.

Theorem 3.1. Let $M_1(b) \neq \emptyset$. Let us set

$$\overline{x}_j(b) = \min_{i \in I} \overline{x}_j(b_i),\ j \in J,\ b = (b_1, \dots, b_m)^T.$$

Then $\overline{x}(b) = (\overline{x}_1(b), \dots,\ \overline{x}_n(b))^T$ is the maximum element of $M_1(b)$, i.e. it holds $\overline{x}(b) \in M_1(b)$ and $x \leq \overline{x}(b)$ for all $x \in M_1(b)$. The maximum elements $\overline{x}_j(b_i)$ for $i \in I,\ j \in J$ are computed according to formulas (i) – (vi) above.

Proof. If $x = (x_1, \dots, x_n) \in M_1$, then it must for each $j \in J, i \in I$ hold $x_j \in V_{ij}(b_i)$ for all $i \in I$. Therefore it must be $x_j \le \overline{x}_j(b_i)\ \forall i \in I$. It follows that $\overline{x}_j(b) = \min_{i \in I} \overline{x}_j(b_i)$ for each $j \in J$. □

It follows from the results above that set $M_1(b) = \{x \in R^n \mid x \le \overline{x}(b)\}$, i. e. $M_1(b)$ is always a closed convex set with the upper bound $\overline{x}(b)$.

Let us consider now set $M_2(b)$. Since $M_2(b) \subseteq M_1(b)$, then if $x \in M_2(b)$, then $x \le \overline{x}(b)$. It follows that if $\overline{x}(b) \in M_2(b)$, then $\overline{x}(b)$ is the maximum element of $M_2(b)$.

Theorem 3.2. Let $\overline{x}(b)$ be the maximum element of $M_1(b)$. Then $M_2(b) \ne \emptyset$ if and only if $\overline{x}(b) \in M_2(b)$.

Proof. If $\overline{x}(b) \in M_2(b)$, the assertion of the theorem is evident.

Let us set

$$g_i(x) \equiv \max_{j \in J}(a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j)).$$

Let us assume that $M_2(b) \ne \emptyset$, $\tilde{x} \in M_2(b)$ and $\overline{x}(b) \notin M_2(b)$. Since $M_2(b) \subset M_1(b)$, it is $x \le \overline{x}(b)$ for all $x \in M_2(b)$. Since $\overline{x}(b) \notin M_2(b)$, there exists an index $i \in I$ such that $g_i(\overline{x}(b)) < b_i$. Then we have for the index $i \in I$:

$$b_i = g_i(\tilde{x}) \le g_i(\overline{x}(b)) < b_i,$$

which is a contradiction. □

Let us now study properties of set $M_3(b)$. For this purpose, we will first investigate set

$$W_{ij}(b_i) \equiv \{x_j \mid a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j) \ge b_i\}. \tag{6}$$

We will find the minimum element of the set $W_{ij}(b_i)$, i. e. element $\underline{x}_j(b_i) \in M_3$ satisfying the condition $\underline{x}_j(b_i) \le x_j,\ j \in J$.

It follows from the construction of the element $\overline{x}_j(b_i)$ using (i) - (vi) that it holds except the case (ii) the equality $\overline{x}_j(b_i) = \underline{x}_j(b_i)$. In case (ii), we have

$$\underline{x}_j(b_i) = r_{ij}^{-1}(b_i - b_{ij}) \le s_{ij}^{-1}(b_i - a_{ij}) = \overline{x}_j(b_i).$$

The following theorem characterizes the elements of the set $M_3(b)$.

Theorem 3.3. $x \in M_3(b)$ if and only if for each $i \in I$ there exists an index $j(i) \in J$ such that

$$a_{ij(i)} \wedge r_{ij(i)}(x_{j(i)}) + b_{ij(i)} \vee s_{ij(i)}(x_{j(i)}) \ge b_i \ i.e. W_{ij(i)}(b_i) \ne \emptyset. \tag{7}$$

Proof. If (7) is fulfilled, then evidently $x \in M_3(b)$. Assume further that (7) is not fulfilled and there exists an element $x \in M_3(b)$. Since we assume that (7) is not fulfilled, there exists an index $i_0 \in I$ such that

$$\max_{j \in J}(a_{i_0 j} \wedge r_{i_0 j}(x_j) + b_{i_0 j} \vee s_{i_0 j}(x_j)) < b_{i_0}.$$

It follows that $x \notin M_3(b)$. This contradiction proves the theorem. □

We will turn our attention to optimization problems, in which the set of feasible solutions is one of the sets $M_i(b),\ i = 1, 2, 3$. Let us note that set $M_1(b)$ is a closed convex set and as such can be incorporated in any convex optimization problem. We will concentrate further on optimization problems, in which in general non-convex sets $M_2(b),\ M_3(b)$ take part in generating the set of feasible solutions. Proposed solution procedures are modifications of methods in [1, 3].

Let us consider the following optimization problem:

$$f(x) \equiv \max_{j\in J} f_j(x_j) \ \rightarrow \ \min \text{ subject to } x \in M_2(b),\ x \geq \underline{x}, \tag{8}$$

where $f_j : R_+ \ \rightarrow \ R_+,\ j \in J$ are strictly increasing continuous functions. To avoid discussions of trivial cases, we will assume that $\underline{x} \leq \overline{x}(b), 0 \leq \ \underline{x} < \overline{x}(b)$ and $\overline{x}(b) \in M_2(b)$ so that $M_2(b) \neq \emptyset$. The following algorithm is a steepest descent algorithm solving problem (8). We begin with the maximum element $\overline{x}(b)$ of set $M_2(b)$, find the set of indices $F(\overline{x}(b))$ of components $\overline{x}_k(b)$, which determine the value of the objective function at the point $\overline{x}(b)$, i.e. set of indices $F(\overline{x}(b)) = \{\ k \mid f_k(\overline{x}(b)) = f(\overline{x}(b))\}$. We set $\tilde{f} = \max_{j\in J\setminus F(\overline{x}(b)} f_j(\overline{x}_j(b))$, so that $\tilde{f}$ is the next threshold value, to which the objective function may be decreased without checking the feasibility. We set further $\tilde{x}_k = \overline{x}_k(b)\ \forall k \in J \setminus F(\overline{x}(b))$. If $\widetilde{x(b)}$ is not feasible i.e. $\tilde{x}(b) \notin M_2(b)$, then $\overline{x}(b)$ is the optimal solution of the problem (8), otherwise we set $\tilde{x}(b) =: \overline{x}(b)$ and repeat the cycle with this new upper bound. We will summarize the procedure in the following algorithm.

Algorithm 2

$\boxed{1}$ Compute $\overline{x}(b),\ f(\overline{x}(b))$.

$\boxed{2}$ Find the set of active indices in $f(\overline{x}(b))$, i.e. set

$$F(\overline{x}(b)) = \{k \in J \mid f(\overline{x}(b)) = f_k(\overline{x}_k(b))\}.$$

$\boxed{3}$ Set

$$\tilde{f} = \max_{j\in J\setminus F(\overline{x}(b))} f_j(\overline{x}_j(b))$$

$\boxed{4}$ Set

$$\tilde{x}_k = f_k^{-1}(\tilde{f}) \vee \underline{x}_k \ \forall k \in F(\overline{x}(b)),\ \tilde{x}_k = \overline{x}_k(b)\ \forall k \in (J \setminus F(\overline{x}(b)))$$

$\boxed{5}$ If $\tilde{x} \notin M_2(b)$, then $\overline{x}(b)$ is the optimal solution $\boxed{\text{STOP}}$.

$\boxed{6}$ Set $\tilde{x} =: \overline{x}(b)$, goto $\boxed{2}$.

Note that the number of cycles of Algorithm 2 is at most equal to n. Each cycle requires $O(mn)$ operations.

Remark 3.1. Let us note that the Algorithm 2 can be adjusted also for problems, in which functions $f_j(x_j)$ in the objective function are convex. In such a case, it is necessary replace Step $\boxed{4}$ by a step, in which we set $\tilde{x_k}$ for all $k \in F(\overline{x}(b))$ equal to the point, where the minimum of function $f_k(x_k)$ on interval $[\underline{x}_k, \overline{x}_k]$ is attained. After such adjustment, the algorithm can be further extended to problems with continuous functions f_j, which can be effectively minimized on a closed interval (e. g. unimodal or quasiconvex functions).

Let us consider now the following optimization problem:

$$f(x) \equiv \max_{j \in J} f_j(x_j) \;\; \rightarrow \;\; \min \;\; \textit{subject to } x \in M_3(b), \tag{9}$$

where $f_j : R_+ \;\rightarrow\; R_+,\; j \in J$ are increasing continuous functions. Note that under our assumptions the set of feasible solutions of this problem is always nonempty.

Let us set $c_{ij}(x_j) \equiv a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j)$ to simplify further notation. Then we have according to (6) and the definition of $\underline{x}_j(b_i)$ above:

$$W_{ij}(b_i) = \{x_j \mid \underline{x}_j(b_i) \le x_j < \infty\} = \{x_j \mid c_{ij}(x_j) \ge b_i\}, i \in I,\; j \in J. \tag{10}$$

Since functions f_j are assumed to be increasing, we have for all $i \in I,\; j \in J$ such that $W_{ij}(b_i) \neq \emptyset$:

$$\min_{x_j \in W_{ij}(b_i)} f_j(x_j) = f_j(\underline{x}_j(b_i)),\; i \in I,\; j \in J. \tag{11}$$

Let us introduce the following notations:

$$\min_{j \in J} f_j(\underline{x}_j(b_i)) = f_{j(i)}(\underline{x}_{j(i)}(b_i)),\; i \in I. \tag{12}$$

Let us define further sets $I_j,\; W_j$ for all $j \in J$ as follows:

$$I_j = \{i \in I \mid j(i) = j\},\; W_j = \bigcap_{i \in I_j} W_{ij}(b_i), \tag{13}$$

so that $W_j = \{x_j \mid x_j \ge \max_{i \in I_j} \underline{x}_j(b_i)\}$.

Theorem 3.4. Let notations (10)−(13) be introduced. Let for all $j \in J$

$$\hat{x}_j(b) = \max_{i \in I_j} \underline{x}_j(b_i),\; if\; I_j \neq \emptyset,\; \hat{x}_j(b) = 0\; if\; I_j = \emptyset.$$

Then $\hat{x}(b) = (\hat{x}_1(b), \dots, \hat{x}_n(b))$ is the optimal solution of problem (9).

P r o o f. Let us assume that $\hat{x}(b)$ is not optimal solution of problem (9). Then there exists a feasible solution $\tilde{x}$ of problem (9) such that

$$f(\tilde{x}) < f(\hat{x}(b)) = f_k(\hat{x}_k) = f_k(\underline{x}_k(b_{i_0}))$$

for some indexes $k \in J$ and $i_0 \in I_k$. Since $f(\tilde{x}) < f(\hat{x}(b))$ and $f_k(\hat{x}_k) = f_k(\underline{x}_k(b_{i_0})) = \min_{x_k \in W_{i_0 k}} f_k(x_k)$, it must be $\tilde{x}_k \notin W_{i_0 k}$ and there exists an index $k_0 \in J,\; k_0 \neq k$ such that $\tilde{x}_{k_0} \in W_{i_0 k_0}(b_{i_0})$ and it holds

$$f(\tilde{x}) \ge f_{k_0}(\tilde{x}_{k_0}) \ge f_{k_0}(\underline{x}_{k_0}(b_{i_0})) \ge f_k(\underline{x}_k(b_{i_0})) = f_k(\hat{x}_k(b) = f(\hat{x}(b)).$$

This contradiction proves the theorem. $\square$

We can summarize the procedure computing the optimal solution of problem (9) as follows:

Algorithm 3

$\boxed{1}$ Find $\underline{x}_j(b_i)$ for $i \in I,\ j \in J$;
$\boxed{2}$ set $\min_{j\in J} f_j(\underline{x}_j(b_i)) = f_{j(i)}(\underline{x}_{j(i)}(b_i))\ \forall i \in I$;
$\boxed{3}$ Set $I_j = \{i \in I \mid j = j(i)\ \forall j \in J\}$;
$\boxed{4}$ $\hat{x}_j = \max_{i\in I_j} \underline{x}_j(b_i)\ if\ I_j \neq \emptyset$, $\hat{x}_j = \underline{x}_j\ if\ I_j = \emptyset$.
$\boxed{5}$ STOP

Remark 3.2. Algorithm 3 has under the given assumptions computational complexity $O(mn)$. Its methodology may be extended to a wider class of functions. In that case the computations contain mn-times computing the minimum of a continuous function of one variable on a closed interval. The final complexity is then dependent on complexity of these minimizations.

Remark 3.3. The aim of the paper was to show how linear combination of the norm $\wedge$ and its co-norm $\vee$ can be applied to selected decision making problems both in crisp and in uncertain environment. Let us note that the methodology presented in the paper can be applied to a wider class of similar problems. We will mention some of them here.

1. We can develop a "dual" or "symmetric" theory, in which the left hand sides of the equations and inequalities will have the form

$$\min_{j\in J}(a_{ij} \wedge r_{ij}(x_j) + b_{ij} \vee s_{ij}(x_j)).$$

2. We can consider combinations of other triangular norms and/or conorms or more norms resp. conorms than two. For example we can consider the left-hand sides of the form

$$\max_{j\in J}(a_{ij} \wedge r_{ij}(x_j) + b_{ij} \wedge s_{ij}(x_j) + d_{ij} \vee u_{ij}(x_j)),$$

   where $a_{ij},\ b_{ij},\ d_{ij} \in R_+$ and $r_{ij} : R_+ \to R_+,\ s_{ij} : R_+ \to R_+,\ u_{ij} : R_+ \to R_+$ are strictly increasing functions.

3. The explicitly visible dependence of maximal and minimal elements on the input data a_{ij}, b_{ij}, b_i makes possible to develop postoptimal, parametric or interval investigations of the problems.

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