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EXISTENCE OF WEAK SOLUTIONS FOR ELLIPTIC DIRICHLET PROBLEMS WITH VARIABLE EXPONENT

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Abstract. This paper presents several sufficient conditions for the existence of weak solutions to general nonlinear elliptic problems of the type

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) + b(x,u,\nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$. In particular, we do not require strict monotonicity of the principal part $a(x,z,\cdot)$, while the approach is based on the variational method and results of the variable exponent function spaces.

Keywords: variable exponent; existence; variational methods; Dirichlet problem

MSC 2020: 35J20, 35J25, 35J70

1. Introduction and main results

We deal with the existence of weak solutions to general elliptic Dirichlet problems of the type

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) + b(x,u,\nabla u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$. Hereafter, $a(x,z,\xi)$ and $b(x,z,\xi)$ are always supposed to satisfy the assumption

(H1) $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $b\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfy the Carathéodory conditions and

$$|a(x,z,\xi)| \leqslant a_0|\xi|^{p(x)-1} + a_1|z|^{q(x)/p'(x)} + f_1(x), \tag{1.2}$$

$$|b(x,z,\xi)| \leqslant b_0|\xi|^{p(x)/q'(x)} + b_1|z|^{q(x)-1} + f_2(x) \tag{1.3}$$

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for a.e. $x \in \Omega$ and all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$, where p and q are functions such that

$$p, q \in C(\overline{\Omega}), \tag{1.4}$$

$$1 < p^- := \inf_\Omega p(x) \leqslant p^+ := \sup_\Omega p(x) < \infty, \tag{1.5}$$

$$p(x) \leqslant q(x) < p^*(x) := \begin{cases} \dfrac{np(x)}{n - p(x)} & \text{if } p(x) < n, \\ \infty & \text{if } p(x) \geqslant n, \end{cases} \tag{1.6}$$

and a_0, a_1, b_0 and b_1 are given nonnegative constants, $f_1 \in L^{p'(\cdot)}(\Omega)$, $f_2 \in L^{q'(\cdot)}(\Omega)$ and $p'(x) := p(x)/(p(x) - 1)$.

The study of partial differential equations and variational problems involving operators with variable exponent growth conditions has received more and more interest in the last few years. It was found that these problems with the $p(x)$-growth are related to modeling of electrorheological fluids, nonlinear elasticity and image restoration; see, e.g., [4], [27], [31].

Elliptic Dirichlet problems with variable exponent have been studied by several authors. In [3], [9], [7], [11], [30] the existence of nontrivial weak solutions for the $p(x)$-Laplacian Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = b(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has been established assuming, among others, that the lower order term b satisfies $|b(x, z)| \leqslant b_1|z|^{q(x)-1} + b_2$. Similar results have been proved for the problem with $b(x, u)$ replaced by $k(x)|u|^{q(x)-2}u + b(x, u)$; see, e.g., [8], [18], [22].

The principal part of the above equation, the $p(x)$-Laplacian, was generalized in some papers. Galewski (see [15]) and Zhou (seen [32]) studied the cases that $a(x, z, \xi) \equiv \alpha(x)|\xi|^{p(x)-2}\xi$ and $a(x, z, \xi) \equiv \bigl(1 + |\xi|^{p(x)}/\sqrt{1 + |\xi|^{2p(x)}}\bigr)|\xi|^{p(x)-2}\xi$, respectively, where $0 < \alpha_0 \leqslant \alpha(x) \leqslant \alpha_1 < \infty$. In particular, Pucci and Zhang (see [24]), and Rădulescu (see [25]) considered the equation with $a(x, z, \xi) \equiv a(x, \xi)$, provided that $a(x, \xi)$ is strictly monotone in ξ, i.e.,

$$(a(x, \xi) - a(x, \eta))(\xi - \eta) > 0 \quad \text{when } \xi \neq \eta,$$

among other requirements, while $b(x, z, \xi)$ is independent of ξ. Mihăilescu and Repovš (see [23]) studied the existence of weak solutions to the Dirichlet problem for a special equation

$$-\operatorname{div}(\alpha(x, u)\langle A\nabla u, \nabla u\rangle^{(p(x)-2)/2} A\nabla u) = f(x), \quad x \in \Omega,$$

where $A(x) = (a_{ij}(x))_{n\times n}$, $a_{ij} = a_{ji}$, $a_{ij} \in L^\infty(\Omega) \cap C^1(\Omega)$,

$$\langle A\xi, \xi\rangle = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant |\xi|^2 \quad \forall\, x \in \Omega\ \forall\, \xi \in \mathbb{R}^n$$

and $\alpha\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function for which there exist constants $0 < \alpha_0 \leqslant \alpha_1$ such that

$$\alpha_0 \leqslant \alpha(x,z) \leqslant \alpha_1 \quad \text{for a.e. } x \in \Omega\ \forall\, z \in \mathbb{R}.$$

Moreover, they assumed that a continuous function $p(x)$ is in $(2,\infty)$ and satisfies the inequality

$$\lambda_1 := \inf_{u\in C_0^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x}{\int_\Omega |u|^{p(x)}\,\mathrm{d}x} > 0,$$

which condition is very restrictive. Indeed, a sufficient condition on $p(\cdot)$ in order to satisfy $\lambda_1 > 0$ is that there exists a vector $l \in \mathbb{R}^n \setminus \{0\}$ such that, for any $x \in \Omega$, the function $f(t) := p(x + tl)$ is monotone in $t \in I_x := \{t\colon\ x + tl \in \Omega\}$, see [12].

In our opinion the paper which dealt with the existence of weak solutions for the most general elliptic Dirichlet problem with variable exponents seems to be that of Fu and Yu (see [14]) up to date. They considered the Dirichlet problem for the higher order elliptic equation

$$\begin{cases} \displaystyle\sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha a_\alpha(x, \xi_m(u)) = G + f(x,u), & x \in \Omega,\\ D^\beta u = 0, & x \in \partial\Omega\ \forall\, |\beta| \leqslant m-1, \end{cases}$$

where $\xi_m = \{\xi_\alpha\colon\ |\alpha| \leqslant m\}$, G is a bounded linear functional on $W_0^{m,p(\cdot)}(\Omega)$, and $f(x,z)$ is a Carathéodory function and satisfies some structural conditions including the critical growth. Moreover, they assumed that the principal part $(a_\alpha(x,\xi_{m-1},\zeta_m))_{|\alpha|=m}$ is strictly monotone in ζ_m, i.e.,

$$\sum_{|\alpha|=m} (a_\alpha(x,\xi_{m-1},\zeta_m) - a_\alpha(x,\xi_{m-1},\zeta_m'))(\xi_\alpha - \xi_\alpha') > 0$$

when $\zeta_m \neq \zeta_m'$, where $\zeta_m = \{\xi_\alpha\colon\ |\alpha| = m\}$, and

$$\sum_{|\alpha|\leqslant m} a_\alpha(x,\xi_m)\xi_\alpha \geqslant C|\zeta_m|^{p(x)} - h(x), \quad h \in L^1(\Omega),\ C > 0,$$

$$|a_\alpha(x,\xi_m)| \leqslant C|\xi_m|^{p(x)-1} + h_0(x), \quad h_0 \in L^{p'(\cdot)}(\Omega),$$

and, in particular, $p(x)$ is Lipschitz continuous and satisfies $1 < p^- \leqslant p^+ < n/m$.

To our knowledge, [13] is the only paper concerning the existence of weak solutions for elliptic Dirichlet problems with variable exponent of the type (1.1) under the condition that $a(x,z,\xi)$ is monotone in ξ, but not strictly monotone. In that paper, the authors have considered the Dirichlet problem for the quasilinear elliptic system

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in (W_0^{1,p(\cdot)}(\Omega,\mathbb{R}^m))^*$, $p(x)$ is Lipschitz continuous and $1 < p^- \leqslant p^+ < n$, and $a\colon \Omega \times \mathbb{R}^m \times M^{m\times n} \to M^{m\times n}$, where $M^{m\times n}$ denotes the real vector space of $m \times n$ matrices equipped with the inner product $M : N = M_{ij}N_{ij}$ (with the usual summation convention), satisfies the growth condition similar to (1.2) and coercivity $a(x,z,\xi) : \xi \geqslant a_0|\xi|^{p(x)} - a_1(x)$, $a_1 \in L^1(\Omega)$, $a_0 > 0$, and for any $x \in \Omega$ and $z \in \mathbb{R}^m$, $\xi \to a(x,z,\xi)$ is a C^1-function and monotone, i.e.,

$$(a(x,z,\xi) - a(x,z,\eta)) : (\xi - \eta) \geqslant 0 \quad \text{for any } \xi,\eta \in M^{m\times n}.$$

For an overview of elliptic equations with variable growth, we refer to [17] and [25].

The aim of the present paper is to improve the existence results of [13] and [14]. As seen above, the conditions on $p(\cdot)$ assumed in [13], [14] to obtain the existence of weak solutions are too restrictive, while such strong conditions on $p(\cdot)$ need for using the embedding results from [10]. In contrast to [13] and [14], we assume that p is only continuous in $\overline{\Omega}$ and satisfies (1.5), and we employ the embedding results from [20] (see Propositions 2.2 and 2.3 below).

The following assumptions on a and b will be used.

(H2) For a.e. $x \in \Omega$, all $z \in \mathbb{R}$ and all ξ, $\eta \in \mathbb{R}^n$, the inequality

$$(a(x,z,\xi) - a(x,z,\eta))(\xi - \eta) \geqslant 0 \tag{1.7}$$

holds.

(H3) For a.e. $x \in \Omega$, $a_i(x,\cdot,\cdot)$, $i = 1,\ldots,n$, and $b(x,\cdot,\cdot)$ belong to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^n)$ and impose the symmetry conditions

$$\frac{\partial a_i(x,z,\xi)}{\partial \xi_j} = \frac{\partial a_j(x,z,\xi)}{\partial \xi_i}, \qquad \frac{\partial a_i(x,z,\xi)}{\partial z} = \frac{\partial b(x,z,\xi)}{\partial \xi_i} \tag{1.8}$$

for all $1 \leqslant i,j \leqslant n$ and for a.e. $(x,z,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^n$.

(H4) There exist $c_0 > 0$, $f_3 \in L^{p(\cdot)/(p(\cdot)-r_1(\cdot))}(\Omega)$, $f_4 \in L^{q(\cdot)/(q(\cdot)-r_2(\cdot))}(\Omega)$ and functions $r_1, r_2 \in C(\overline{\Omega})$ with $r_1(x) < p(x)$ for any $x \in \overline{\Omega}$ and $1 \leqslant r_2^- \leqslant r_2^+ < p^-$ such that

$$a(x,z,\xi)\xi + b(x,z,\xi)z \geqslant c_0|\xi|^{p(x)} - f_3(x)|\xi|^{r_1(x)} - f_4(x)|z|^{r_2(x)}$$

for a.e. $x \in \Omega$ and all $(z,\xi) \in \mathbb{R}\times\mathbb{R}^n$.

Our first main result is the following.

Theorem 1.1. *Let* (H1)–(H4) *be satisfied. Then the Dirichlet problem* (1.1) *has a weak solution.*

In the present paper, we also show that there exists a weak solution of (1.1) without requiring the assumption (H3). Namely, in Theorem 4.4, we prove the existence of weak solutions of (1.1), assuming the lower order term $b(x,z,\xi)$ is independent of ξ or linearly dependent on ξ and some growth conditions are satisfied, where we do not require (H3). Moreover, the following theorem shows the existence result under the further assumption that $a(x,z,\cdot)$ is strictly monotone and $b(x,z,\xi)$ is nonlinearly dependent on not only z but also ξ. The second result of this paper is the following.

Theorem 1.2. *Let* (H1) *be satisfied. Assume that the following conditions hold:*

(H5) *For a.e.* $x \in \Omega$ *and all* $z \in \mathbb{R}$, *the inequality*

$$(a(x,z,\xi) - a(x,z,\eta))(\xi - \eta) > 0 \quad \text{when } \xi \neq \eta \tag{1.9}$$

holds.

(H6) *There exist* $c_0 > 0$, $f_4 \in L^{q(\cdot)/(q(\cdot)-r_2(\cdot))}(\Omega)$ *and* $f_5 \in L^1(\Omega)$ *such that*

$$a(x,z,\xi)\xi + b(x,z,\xi)z \geqslant c_0|\xi|^{p(x)} - f_4(x)|z|^{r_2(x)} - f_5(x), \tag{1.10}$$

where r_2 *is continuous on* $\overline{\Omega}$ *and satisfies* $1 \leqslant r_2^- \leqslant r_2^+ < p^-$. *Then the problem* (1.1) *has a weak solution.*

Compared with the known existence results for the elliptic Dirichlet problems with variable exponent, our results are more general in the following sense. As mentioned above, in [13] and [14] the authors assumed that p is Lipschitz continuous and $1 < p^- \leqslant p^+ < n$, but in this paper we only require that p is continuous on $\overline{\Omega}$ and $1 < p^- \leqslant p^+ < \infty$. Fu and Yang in [13] studied the general elliptic systems with variable growth in which the principal part $a(x,z,\xi)$ is monotone in ξ, but not strictly monotone, and the lower order term b is independent of u and $\nabla u : b(x,z,\xi) = b(x)$, while in Theorems 1.1 and 4.4 we generalized the condition on b to the case when b is dependent on u and ∇u. Fu and Yu (see [14]) studied the elliptic equation in which $a(x,z,\xi)$ is strictly monotone and b satisfies a critical growth condition that is more general than (1.3), but the conditions in [14] on the power exponents of $|z|$ in (1.2) and $|\xi|$ in (1.3) are stronger than those of Theorem 1.2 due to the assumption (1.6).

The proofs of Theorems 1.1, 1.2 and 4.4 are more difficult in comparison with [13], [14] for the reason of general and sharp conditions on the coefficients $a(x,z,\xi)$ and $b(x,z,\xi)$. The proof of Theorem 1.1 is based on the variational method and several properties of variable exponent spaces. Here, we adapt some ideas explored in [26] for the constant exponent case.

Theorems 1.2 and 4.4 are proved by using the theory of pseudomonotone operators. For more details on this topic see, e.g., [1], [16], [26]. For the proofs of Theorems 1.2 and 4.4 we need Lemmas 4.2 and 4.3, respectively, such lemmas are proved by using Leray-Lions' technique (see [21]) under suitable variable exponent growth conditions on the lower order term b.

The paper is divided into four sections. In Section 2 we present some preliminary knowledge of the variable exponent spaces and some results which we use in the next sections. In Section 3, we prove Theorem 1.1. In Section 4, by using the theory of pseudomonotone operators, we prove some existence results including Theorem 1.2.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, and $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p\colon \Omega \to [1,\infty]$. We put $\Omega_\infty := \{x \in \Omega\colon p(x) = \infty\}$. Given $p \in \mathcal{P}(\Omega)$, let

$$p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x), \quad p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$$

and let the conjugate exponent function p' be defined by the formula

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \Omega,$$

with the convention that $1/\infty = 0$.

Given $p \in \mathcal{P}(\Omega)$ and a measurable function u, define the modular associated with p by

$$\varrho_{p(\cdot)}(u) := \int_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)}\,\mathrm{d}x + \operatorname*{ess\,sup}_{x\in\Omega_\infty} |u(x)|.$$

We put $L^{p(\cdot)}(\Omega)$ to be the set of measurable functions u such that $\varrho_{p(\cdot)}(u/\lambda) < \infty$ for some $\lambda > 0$. Define the Luxemburg norm on $L^{p(\cdot)}(\Omega)$ by

$$\|u\|_{p(\cdot)} := \inf\{\lambda > 0\colon\ \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}.$$

Put

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega)\colon\ |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

with the norm

$$\|u\|_{1;p(\cdot)} := \|\nabla u\|_{p(\cdot)} + \|u\|_{p(\cdot)}, \quad \text{where } \|\nabla u\|_{p(\cdot)} := \||\nabla u|\|_{p(\cdot)}.$$

Denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

Given $p \in \mathcal{P}(\Omega)$, for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, then $uv \in L^1(\Omega)$ and the Hölder inequality

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \leqslant C_{p(\cdot)}\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}$$

holds, where

$$C_{p(\cdot)} := \Big(\frac{1}{p^-} - \frac{1}{p^+} + 1\Big)\|\chi_{\Omega_0}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty$$

and $\Omega_1 := \{x \in \Omega\colon\ p(x) = 1\}$, $\Omega_0 := \Omega \setminus (\Omega_1 \cup \Omega_\infty)$, and χ_E is the characteristic function of E (see Theorem 2.26 in [5]).

For every $f \in (L^{p(\cdot)}(\Omega))^*$ there exists a unique function $v \in L^{p'(\cdot)}(\Omega)$ such that

$$f(u) = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \forall\, u \in L^{p(\cdot)}(\Omega)$$

and

$$\frac{1}{2}\|v\|_{p'(\cdot)} \leqslant \|f\| \leqslant 3\|v\|_{p'(\cdot)},$$

provided $p \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$. The dual space to $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ if and only if $p \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$.

The variable exponent spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are complete and, moreover, separable if $p^+ < \infty$ or reflexive if $1 < p^- \leqslant p^+ < \infty$.

For a measurable set E in $\mathbb{R}^n$ we denote by $|E|$ the n-Lebesgue measure of E. If $p, q \in \mathcal{P}(\Omega)$ and $p(x) \leqslant q(x)$ for a.e. $x \in \Omega$, then $L^{q(\cdot)}(\Omega)$ is continuously embedded in $L^{p(\cdot)}(\Omega)$ and

$$\|u\|_{p(\cdot)} \leqslant (1 + |\Omega|)\|u\|_{q(\cdot)} \quad \forall\, u \in L^{q(\cdot)}(\Omega).$$

Given $p \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$, for every $u \in L^{p(\cdot)}(\Omega)$ there holds the inequality

$$\min\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\} \leqslant \int_\Omega |u|^{p(x)}\,\mathrm{d}x \leqslant \max\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\}.$$

Given $p \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$, and $r \in L^\infty(\Omega)$ with $0 < r^- \leqslant r^+ < \infty$ and $1 \leqslant r(x)p(x)$ for a.e. $x \in \Omega$, then for every $u \in L^{r(\cdot)p(\cdot)}(\Omega)$ it follows that

$$\min\{\|u\|_{r(\cdot)p(\cdot)}^{r^-}, \|u\|_{r(\cdot)p(\cdot)}^{r^+}\} \leqslant \||u|^{r(\cdot)}\|_{p(\cdot)} \leqslant \max\{\|u\|_{r(\cdot)p(\cdot)}^{r^-}, \|u\|_{r(\cdot)p(\cdot)}^{r^+}\}.$$

Proposition 2.1 ([20], Theorems 4.1 and 4.2). *Let $h\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function, where m is a given natural number. If for given p_i, $r \in \mathcal{P}(\Omega) \cap L^\infty(\Omega)$, $i = 1, \ldots, m$, there exist a function $g \in L^{r(\cdot)}(\Omega)$ and a constant $C > 0$ such that*

$$|h(x, \xi)| \leqslant C\sum_{i=1}^m |\xi_i|^{p_i(x)/r(x)} + g(x)$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^m$, then the Nemyckii operator $\mathcal{N}_h$ defined by

$$\mathcal{N}_h(u_1, \ldots, u_m)(x) := h(x, u_1(x), \ldots, u_m(x)), \quad x \in \Omega,$$

maps $L^{p_1(\cdot)}(\Omega) \times \ldots \times L^{p_m(\cdot)}(\Omega)$ in $L^{r(\cdot)}(\Omega)$, and is continuous and bounded.

Proposition 2.2 ([20], Theorem 3.9). *Assume that $p \in C(\overline{\Omega})$ with $p(x) \geqslant 1$ for all $x \in \overline{\Omega}$. If $q \in C(\overline{\Omega})$ and $1 \leqslant q(x) < p^*(x)$ for every $x \in \overline{\Omega}$, then there exists a continuous and compact embedding*

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega).$$

Proposition 2.3 ([20], Theorem 3.10). *Let $p \in \mathcal{P}(\Omega) \cap C(\overline{\Omega})$. Then*

$$\|u\| := \|\nabla u\|_{p(\cdot)}$$

is an equivalent norm in $W_0^{1,p(\cdot)}(\Omega)$.

For more results on the variable exponent spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, we refer to [6], [20].

Now we list several well known results which will be used in the next sections.

Proposition 2.4 ([2], Corollary 2.8.7). *Let $\varphi\colon \Omega \times (a,b) \to \mathbb{R}$ be a function such that for every $t \in (a,b)$ the function $\varphi(\cdot,t)$ is integrable on Ω and for a.e. $x \in \Omega$ the function $\varphi(x,\cdot)$ is differentiable, and there exists an integrable function g such that for a.e. $x \in \Omega$ we have $|\partial\varphi(x,t)/\partial t| \leqslant g(x)$ for all t simultaneously. Then the function*

$$F\colon\ t \mapsto \int_\Omega \varphi(x,t)\,\mathrm{d}x$$

is differentiable and

$$F'(t) = \int_\Omega \frac{\partial\varphi(x,t)}{\partial t}\,\mathrm{d}x.$$

Proposition 2.5 ([29], Proposition 42.6). *Let $\varphi\colon X \to \mathbb{R}$ be a functional on the real Banach space X. Suppose the Gâteaux derivative $\varphi'\colon X \to X^*$ exists on X. Then φ' is monotone on X if and only if*

$$\varphi(v) - \varphi(u) \geqslant \langle \varphi'(u), v-u\rangle \quad \forall\, u,v \in X,$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between X and its dual X^*.

A functional $\varphi\colon X \to \mathbb{R} \cup \{\pm\infty\}$ is called weakly lower semicontinuous if for any $u \in X$ the inequality

$$\varphi(u) \leqslant \liminf_{k\to\infty} \varphi(u_k)$$

holds for every sequence $\{u_k\}$ in X such that $u_k \rightharpoonup u$ as $k \to \infty$, where $u_k \rightharpoonup u$ means the weak convergence of the sequence $\{u_k\}$ to u.

A functional $\varphi\colon X \to \mathbb{R}$ on the Banach space X is called coercive if

$$\frac{\varphi(u)}{\|u\|} \to \infty \quad \text{as } \|u\| \to \infty.$$

Similarly an operator $A\colon X \to X^*$ is called coercive if

$$\frac{\langle A(u), u\rangle}{\|u\|} \to \infty \quad \text{as } \|u\| \to \infty.$$

An operator $A\colon X \to X^*$ is called pseudomonotone if A is bounded and

$$\begin{aligned} u_k \rightharpoonup u,\ \limsup_{k\to\infty} \langle A(u_k), u_k - u\rangle \leqslant 0 \\ \Rightarrow \forall\, v \in X,\ \langle A(u), u - v\rangle \leqslant \liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle. \end{aligned}$$

Proposition 2.6 ([26], Theorem 4.2). *Let $J\colon X \to \mathbb{R}$ be Gâteaux differentiable and weakly lower semicontinuous and $A = J'$. Then, if J is coercive, the equation $Au = f$ has a solution for any $f \in X^*$.*

Proposition 2.7 ([26], Theorem 2.6). *Let $A\colon X \to X^*$ be pseudomonotone and coercive. Then, for any $f \in X^*$, the equation $A(u) = f$ has solution.*

We say that a function $u \in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution to the problem (1.1) if

$$\int_\Omega (a(x, u, \nabla u)\nabla v + b(x, u, \nabla u)v)\,\mathrm{d}x = 0 \quad \text{for every } v \in W_0^{1,p(\cdot)}(\Omega). \tag{2.1}$$

3. Proof of Theorem 1.1

In what follows, for brevity, we write X instead of $W_0^{1,p(\cdot)}(\Omega)$. From Proposition 2.3 we are able to define the norm on X by $\|u\| := \|\nabla u\|_{p(\cdot)}$. The assumption (H1) and the results on $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ mentioned in Section 2 are sufficient for us to define the operator $A\colon X \to X^*$ by

$$\langle A(u), v\rangle := \int_\Omega (a(x, u, \nabla u)\nabla v + b(x, u, \nabla u)v)\,\mathrm{d}x, \quad u, v \in X. \tag{3.1}$$

It is easy to see that A is bounded. Obviously, $u \in X$ is a weak solution of (1.1) if and only if $A(u) = 0$. Put

$$\varphi(x, z, \xi) := \int_0^1 (a(x, tz, t\xi)\xi + b(x, tz, t\xi)z)\,\mathrm{d}t. \tag{3.2}$$

It is known from the assumption (H3) that the equalities

$$\frac{\partial \varphi(x, z, \xi)}{\partial \xi_i} = a_i(x, z, \xi), \quad i = 1, \ldots, n, \tag{3.3}$$

$$\frac{\partial \varphi(x, z, \xi)}{\partial z} = b(x, z, \xi) \tag{3.4}$$

hold for a.e. $x \in \Omega$ and all $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$; see Roubíček [26], page 117.

It follows that the functional $J\colon X \to \mathbb{R}$ defined by

$$J(u) := \int_\Omega \varphi(x, u, \nabla u)\,\mathrm{d}x, \quad u \in X, \tag{3.5}$$

is continuous by using the assumption (H1), Propositions 2.1 and 2.2.

Lemma 3.1. *Let* (H1) *and* (H3) *be satisfied. Then J is Gâteaux differentiable and $J' = A$ with A given by* (3.1).

P r o o f. Fix $u, v \in X$. Define the function

$$t \mapsto J(u + tv) = \int_\Omega \varphi(x, u + tv, \nabla u + t\nabla v)\,\mathrm{d}x. \tag{3.6}$$

Putting

$$\varphi'_\xi = \Bigl(\frac{\partial \varphi}{\partial \xi_1}, \ldots, \frac{\partial \varphi}{\partial \xi_n}\Bigr),$$

it follows from (1.2), (1.3), (3.3) and (3.4) that for any $t \in [0, 1]$,

$$\begin{aligned}
&|\varphi'_\xi(x, u + tv, \nabla u + t\nabla v)\nabla v| \\
&\qquad \leqslant (a_0|\nabla u + t\nabla v|^{p(x)-1} + a_1|u + tv|^{q(x)/p'(x)} + f_1(x))|\nabla v| \\
&\qquad \leqslant C_{p,q}(|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1} + |u|^{q(x)/p'(x)} + |v|^{q(x)/p'(x)} + f_1(x))|\nabla v|, \\
&|\varphi'_z(x, u + tv, \nabla u + t\nabla v)v| \\
&\qquad \leqslant C_{p,q}(|\nabla u|^{p(x)/q'(x)} + |\nabla v|^{p(x)/q'(x)} + |u|^{q(x)-1} + |v|^{q(x)-1} + f_2(x))|v|,
\end{aligned}$$

where the constant $C_{p,q}$ depends on a_0, a_1, b_0, b_1, $p(\cdot)$ and $q(\cdot)$. It is obvious that the right-hand sides of the above inequalities are integrable on Ω. Therefore, by Proposition 2.4, $J(u + tv)$ defined by (3.6) is differentiable with respect to $t \in [0, 1]$ and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} J(u + tv) &= \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\varphi(x, u + tv, \nabla u + t\nabla v)\,\mathrm{d}x \\
&= \int_\Omega (a(x, u + tv, \nabla u + t\nabla v)\nabla v + b(x, u + tv, \nabla u + t\nabla v)v)\,\mathrm{d}x
\end{aligned}$$

by (3.3) and (3.4). Thus, the directional derivative $DJ(u, v)$ of J at u in the direction v is

$$\begin{aligned}
DJ(u, v) &:= \lim_{t \to 0} \frac{J(u + tv) - J(u)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} J(u + tv)|_{t=0} \\
&= \int_\Omega (a(x, u, \nabla u)\nabla v + b(x, u, \nabla u)v)\,\mathrm{d}x = \langle A(u), v\rangle.
\end{aligned} \tag{3.7}$$

We use (1.2), (1.3) and the Young inequality to get

$$
\begin{aligned}
&|a(x,u,\nabla u)\nabla v + b(x,u,\nabla u)v| \\
&\qquad \leqslant C(|\nabla u|^{p(x)} + |\nabla v|^{p(x)} + |u|^{q(x)} + |v|^{q(x)} + |f_1(x)|^{p'(x)} + |f_2(x)|^{q'(x)}),
\end{aligned}
$$

where the constant C depends on a_0, a_1, b_0, b_1, $p(\cdot)$ and $q(\cdot)$.

Finally, it follows from Propositions 2.1 and 2.2 that $DJ(u,\cdot)\colon X \to \mathbb{R}$ is linear and continuous, and therefore J is Gâteaux differentiable in X and $J' = A$ by (3.7). Lemma 3.1 is proved. □

Lemma 3.2. *Let* (H1), (H2), (H3) *be satisfied. Then J is weakly lower semicontinuous.*

P r o o f. Suppose that a sequence of functions u_k weakly converges to u in X. Hence, by Proposition 2.2, $u_k \to u$ strongly in $L^{q(\cdot)}(\Omega)$. We have

$$|\varphi(x,z,\xi)| \leqslant C(|\xi|^{p(x)} + |z|^{q(x)}) + g(x)$$

by using (1.2), (1.3) and (3.2), where C is an constant and g is an integrable function on Ω. So, by Proposition 2.1, $\varphi(x,u_k,\nabla u) \to \varphi(x,u,\nabla u)$ strongly in $L^1(\Omega)$ and by (1.2) and (3.3) we have $\varphi'_\xi(x,u_k,\nabla u) \to \varphi'_\xi(x,u,\nabla u)$ strongly in $L^{p'(\cdot)}(\Omega)$. By (3.3), (H2), the monotonicity of $a(x,z,\cdot)$ is just the monotonicity of $\varphi'_\xi(x,z,\cdot)$, hence from Proposition 2.5 we have

$$\varphi(x,u_k,\nabla u_k) - \varphi(x,u_k,\nabla u) \geqslant \varphi'_\xi(x,u_k,\nabla u)\cdot(\nabla u_k - \nabla u).$$

Therefore, by using the above limits we get

$$
\begin{aligned}
&\liminf_{k\to\infty} \int_\Omega \varphi(x,u_k,\nabla u_k)\,\mathrm{d}x \\
&\qquad = \liminf_{k\to\infty} \int_\Omega (\varphi(x,u_k,\nabla u) + (\varphi(x,u_k,\nabla u_k) - \varphi(x,u_k,\nabla u)))\,\mathrm{d}x \\
&\qquad \geqslant \int_\Omega \varphi(x,u,\nabla u)\,\mathrm{d}x + \lim_{k\to\infty} \int_\Omega \varphi'_\xi(x,u_k,\nabla u)(\nabla u_k - \nabla u)\,\mathrm{d}x \\
&\qquad = \int_\Omega \varphi(x,u,\nabla u)\,\mathrm{d}x.
\end{aligned}
$$

Lemma 3.2 is proved. □

Lemma 3.3. *Let* (H1) *and* (H4) *be satisfied. Then J is coercive.*

P r o o f. By (H4) and (3.2), we have

$$\begin{aligned}\varphi(x,z,\xi) &= \int_0^1 \frac{t\xi a(x,tz,t\xi)+tzb(x,tz,t\xi)}{t}\,\mathrm{d}t\\ &\geqslant \int_0^1 \frac{c_0|t\xi|^{p(x)}-f_3(x)|t\xi|^{r_1(x)}-f_4(x)|tz|^{r_2(x)}}{t}\,\mathrm{d}t\\ &= \frac{c_0}{p(x)}|\xi|^{p(x)}-\frac{f_3(x)}{r_1(x)}|\xi|^{r_1(x)}-\frac{f_4(x)}{r_2(x)}|z|^{r_2(x)}.\end{aligned}$$

Suppose that $\|u\|$ is large enough. Using the Young inequality with $\varepsilon = c_0/(2p^+)$ and the results on $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ and (3.5), we get

$$\begin{aligned}J(u) &\geqslant \frac{c_0}{p^+}\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega (|f_3(x)||\nabla u|^{r_1(x)} + |f_4(x)||u|^{r_2(x)})\,\mathrm{d}x\\ &\geqslant \frac{c_0}{2p^+}\|u\|^{p^-} - C\int_\Omega |f_3(x)|^{p(x)/(p(x)-r_1(x))}\,\mathrm{d}x\\ &\quad - 2\|f_4\|_{q(\cdot)/(q(\cdot)-r_2(\cdot))}\max\{\|u\|_{q(\cdot)}^{r_2^-}, \|u\|_{q(\cdot)}^{r_2^+}\}\\ &\geqslant \biggl(\frac{c_0}{2p^+}\|u\|^{p^- - r_2^+} - C\int_\Omega |f_3(x)|^{p(x)/(p(x)-r_1(x))}\,\mathrm{d}x - C\|f_4\|_{q(\cdot)/(q(\cdot)-r_2(\cdot))}\biggr)\|u\|.\end{aligned}$$

Since $p^- - r_2^+ > 0$, thus J is coercive. $\square$

P r o o f of Theorem 1.1. By combining Lemmas 3.1–3.3 and Proposition 2.6 it follows that (1.1) has a weak solution. $\square$

R e m a r k 3.4. We emphasize that (H3) is not necessary for A to have a potential. Indeed, for example, if $n = 1$ and $a \equiv a(x,\xi)$ and $b \equiv b(x,z)$, then the Carathéodory condition is obviously sufficient; $\varphi(x,\cdot,\cdot)$ is just the sum of the primitive functions of $a(x,\cdot)$ and $b(x,\cdot)$.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is based on the theory of pseudomonotone operators.

Lemma 4.1. *Suppose that a sequence of functions $\{u_k\}$ weakly converges to u in $L^{p(\cdot)}(\Omega)$ and converges to v almost everywhere in Ω. Then $u = v$.*

P r o o f. From the assumptions it follows that $\{u_k\}$ is bounded in $L^{p(\cdot)}(\Omega)$ and

$$\lim_{k\to\infty} |u_k(x)|^{p(x)} = |v(x)|^{p(x)} \quad \text{for a.e. } x \in \Omega.$$

Thus, by the Fatou lemma, we have

$$\int_\Omega |v(x)|^{p(x)}\,\mathrm{d}x = \int_\Omega \liminf_{k\to\infty} |u_k(x)|^{p(x)}\,\mathrm{d}x \leqslant \liminf_{k\to\infty}\int_\Omega |u_k(x)|^{p(x)}\,\mathrm{d}x \leqslant M < \infty.$$

So $v \in L^{p(\cdot)}(\Omega)$ and v is finite almost everywhere in Ω. Put

$$\Omega_l := \Big\{x \in \Omega\colon \sup_{k \geqslant l} |u_k(x)| \geqslant l\Big\}, \quad l = 1, 2, \dots$$

Obviously, then Ω_l is a measurable set and

$$\bigcap_{l=1}^{\infty} \Omega_l = \Big\{x \in \Omega\colon \limsup_{k \to \infty} |u_k(x)| = \infty\Big\}.$$

It follows that $\lim\limits_{l \to \infty} |\Omega_l| = 0$ since $\Big|\bigcap\limits_{l=1}^{\infty} \Omega_l\Big| = 0$ and $\Omega_l \supset \Omega_{l+1}$. Let $w \in L^{p'(\cdot)}(\Omega)$. Then

$$|u_k(x)w(x)| \leqslant l|w(x)| \quad \forall\, k \geqslant l$$

for a.e. $x \in \Omega \setminus \Omega_l$. Therefore, by the dominated convergence theorem we get

$$\lim_{k \to \infty} \int_{\Omega \setminus \Omega_l} u_k(x)w(x)\,\mathrm{d}x = \int_{\Omega \setminus \Omega_l} v(x)w(x)\,\mathrm{d}x. \tag{4.1}$$

On the other hand, by the weak convergence of $\{u_k\}$ to u in $L^{p(\cdot)}(\Omega)$, we have

$$\lim_{k \to \infty} \int_{\Omega \setminus \Omega_l} u_k(x)w(x)\,\mathrm{d}x = \lim_{k \to \infty} \int_{\Omega} u_k(x)(\chi_{\Omega \setminus \Omega_l} w)(x)\,\mathrm{d}x = \int_{\Omega} u(x)(\chi_{\Omega \setminus \Omega_l} w)(x)\,\mathrm{d}x.$$

It follows from this result and (4.1) that

$$\int_{\Omega} u\chi_{\Omega \setminus \Omega_l} w\,\mathrm{d}x = \int_{\Omega} v\chi_{\Omega \setminus \Omega_l} w\,\mathrm{d}x \quad \forall\, w \in L^{p'(\cdot)}(\Omega)$$

and so $u(x) = v(x)$ for a.e. $x \in \Omega \setminus \Omega_l$. Therefore, $u(x) = v(x)$ for a.e. $x \in \Omega$ since $|\Omega_l| \to 0$ as $l \to \infty$. Lemma is proved. $\square$

Lemma 4.2. *Let* (H1), (H5) *and* (H6) *be satisfied. Then the operator A defined by* (3.1) *is pseudomonotone.*

P r o o f. Assume that $u_k \rightharpoonup u$ in X and

$$\limsup_{k \to \infty} \langle A(u_k), u_k - u\rangle \leqslant 0. \tag{4.2}$$

We have to prove that

$$\langle A(u), u - v\rangle \leqslant \liminf_{k \to \infty} \langle A(u_k), u_k - v\rangle \quad \text{for any } v \in X. \tag{4.3}$$

Let u, $w \in X$. Putting

$$\langle B(w, u), v\rangle := \int_{\Omega} (a(x, w, \nabla u)\nabla v + b(x, w, \nabla w)v)\,\mathrm{d}x, \quad v \in X,$$

it follows from (H1) that $B(w,u) \in X^*$. Obviously, $A(u) = B(u,u)$. Let us put $u_\varepsilon := (1-\varepsilon)u + \varepsilon v$, $\varepsilon \in (0,1]$. Since

$$\langle B(u_k,u_k) - B(u_k,u_\varepsilon), u_k - u_\varepsilon\rangle \geqslant 0$$

by (1.9), it is easy to see that

$$\varepsilon\langle A(u_k), u-v\rangle \geqslant -\langle A(u_k), u_k - u\rangle + \langle B(u_k,u_\varepsilon), u_k - u\rangle + \varepsilon\langle B(u_k,u_\varepsilon), u-v\rangle. \tag{4.4}$$

It follows from Proposition 2.2 that $u_k \to u$ in $L^{q(\cdot)}(\Omega)$ and by (1.3) the sequence $\{b(x,u_k,\nabla u_k)\}$ is bounded in $L^{q'(\cdot)}(\Omega)$. Therefore

$$\int_\Omega b(x,u_k,\nabla u_k)(u_k - u)\,\mathrm{d}x \to 0 \quad \text{as } k \to \infty. \tag{4.5}$$

Moreover, it follows from (1.2) and Proposition 2.1 that

$$a_i(x,u_k,\nabla u_\varepsilon) \to a_i(x,u,\nabla u_\varepsilon) \quad \text{as } k \to \infty \text{ in } L^{p'(\cdot)}(\Omega). \tag{4.6}$$

Since $\nabla u_k \rightharpoonup \nabla u$ in $(L^{p(\cdot)}(\Omega))^n$, from (4.5) and (4.6) we have

$$\begin{aligned} \lim_{k\to\infty} &\langle B(u_k,u_\varepsilon), u_k - u\rangle \\ &= \lim_{k\to\infty} \int_\Omega (a(x,u_k,\nabla u_\varepsilon)(\nabla u_k - \nabla u) + b(x,u_k,\nabla u_k)(u_k - u))\,\mathrm{d}x = 0. \end{aligned} \tag{4.7}$$

Now we prove that

$$\lim_{k\to\infty} \langle B(u_k,u_\varepsilon), u-v\rangle = \langle B(u,u_\varepsilon), u-v\rangle. \tag{4.8}$$

Since

$$\langle B(u_k,u_\varepsilon), u-v\rangle = \int_\Omega (a(x,u_k,\nabla u_\varepsilon)(\nabla u - \nabla v) + b(x,u_k,\nabla u_k)(u-v))\,\mathrm{d}x$$

and (4.6) hold, in order to show (4.8), it suffices to prove that

$$\lim_{k\to\infty} \int_\Omega b(x,u_k,\nabla u_k)(u-v)\,\mathrm{d}x = \int_\Omega b(x,u,\nabla u)(u-v)\,\mathrm{d}x. \tag{4.9}$$

Since $u - v \in L^{q(\cdot)}(\Omega)$, (4.9) follows if we prove that

$$b(x,u_k,\nabla u_k) \rightharpoonup b(x,u,\nabla u) \quad \text{as } k \to \infty \text{ in } L^{q'(\cdot)}(\Omega). \tag{4.10}$$

Since $\{b(x,u_k,\nabla u_k)\}$ is bounded in $L^{q'(\cdot)}(\Omega)$ and $L^{q'(\cdot)}(\Omega)$ is reflexive, there exist a subsequence $\{b(x,u_{k_j},\nabla u_{k_j})\}$ and $g \in L^{q'(\cdot)}(\Omega)$ such that $b(x,u_{k_j},\nabla u_{k_j}) \rightharpoonup g$ in $L^{q'(\cdot)}(\Omega)$. Putting

$$\alpha_{k_j}(x) := (a(x,u_{k_j}(x),\nabla u_{k_j}(x)) - a(x,u_{k_j}(x),\nabla u(x)))(\nabla u_{k_j}(x) - \nabla u(x)),$$

by (1.9), (4.2) and (4.7) we have

$$
\begin{aligned}
0 \leqslant \limsup_{j\to\infty} \int_\Omega \alpha_{k_j}(x)\,\mathrm{d}x &= \limsup_{j\to\infty} \langle B(u_{k_j}, u_{k_j}) - B(u_{k_j}, u), u_{k_j} - u\rangle \\
&= \limsup_{j\to\infty} \langle A(u_{k_j}), u_{k_j} - u\rangle - \lim_{j\to\infty} \langle B(u_{k_j}, u), u_{k_j} - u\rangle \leqslant 0.
\end{aligned}
$$

Then there is a subsequence $\{\alpha_{k'_j}\}$ which converges to 0 for a.e. $x \in \Omega$. Without loss of generality, we can assume that

$$
u_{k'_j}(x) \to u(x), \ \ \alpha_{k'_j}(x) \to 0 \quad \text{as } k'_j \to \infty \tag{4.11}
$$

for a.e. $x \in \Omega$. Take a measurable set $\Omega' \subset \Omega$ satisfying the following conditions: $|\Omega \setminus \Omega'| = 0$ and (1.2), (1.9) and (4.11) hold and $a(x, z, \xi)$ is continuous in $(z, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and $|\nabla u_{k'_j}(x)| < \infty$, $|\nabla u(x)| < \infty$, $|f_1(x)| < \infty$, $|f_2(x)| < \infty$, $|f_4(x)| < \infty$, $|f_5(x)| < \infty$ for any $x \in \Omega'$. Let $x \in \Omega'$ be fixed. Using (1.3), (1.10) and the Young inequality, we get

$$
a(x,z,\xi)\xi \geqslant \frac{c_0}{2}|\xi|^{p(x)} - \Big(b_0^{q(x)}\Big(\frac{2}{c_0}\Big)^{q(x)-1} + b_1\Big)|z|^{q(x)} - f_2(x)|z| - f_4(x)|z|^{r_2(x)} - f_5(x).
$$

If $\{\nabla u_{k'_j}(x)\}$ is unbounded, then there exists a subsequence $\{\nabla u_{k''_j}(x)\} \subset \{\nabla u_{k'_j}(x)\}$ such that $\nabla u_{k''_j}(x) \to \xi_0$ and $|\xi_0| = \infty$. Hence, by (1.2) and the above inequality it is easily seen that

$$
\begin{aligned}
\alpha_{k''_j}(x) &\geqslant a(x, u_{k''_j}(x), \nabla u_{k''_j}(x))\nabla u_{k''_j}(x) \\
&\qquad - C(|\nabla u_{k''_j}(x)|^{p(x)-1} + |\nabla u_{k''_j}(x)| + 1) \to \infty \quad \text{as } k''_j \to \infty,
\end{aligned}
$$

which contradicts the fact that $\{\alpha_{k''_j}(x)\}$ converges to 0. Hence $\{\nabla u_{k'_j}(x)\}$ is bounded. Let $\xi_0 \in \mathbb{R}^n$ be an accumulation point of $\{\nabla u_{k'_j}(x)\}$. Assume that $\nabla u_{k''_j}(x) \to \xi_0$ as $k''_j \to \infty$, where $\{\nabla u_{k''_j}(x)\}$ is a subsequence of $\{\nabla u_{k'_j}(x)\}$. Then by (4.11), it follows that

$$
\alpha_{k''_j}(x) \to (a(x, u(x), \xi_0) - a(x, u(x), \nabla u(x)))(\xi_0 - \nabla u(x)) = 0 \quad \text{as } k''_j \to \infty
$$

and, thus, from (1.9) we have $\xi_0 = \nabla u(x)$. This implies that $\{\nabla u_{k'_j}(x)\}$ has the unique accumulation point $\nabla u(x)$. Therefore, it follows that $\nabla u_{k'_j}(x) \to \nabla u(x)$ as $k'_j \to \infty$. Thus

$$
b(x, u_{k'_j}(x), \nabla u_{k'_j}(x)) \to b(x, u(x), \nabla u(x))
$$

for a.e. $x \in \Omega$. Hence, by Lemma 4.1 we have $g(x) = b(x, u(x), \nabla u(x))$ and this implies that (4.10) holds. Using (4.2), (4.7) and (4.8), it follows from (4.4) that

$$
\liminf_{k\to\infty} \langle A(u_k), u - v\rangle \geqslant \langle B(u, u_\varepsilon), u - v\rangle.
$$

However, since $u_\varepsilon \to u$ as $\varepsilon \to 0$ in X, it follows from Proposition 2.1 that

$$\langle B(u, u_\varepsilon), u - v\rangle \to \langle B(u, u), u - v\rangle \quad \text{as } \varepsilon \to 0,$$

so we have

$$\liminf_{k\to\infty} \langle A(u_k), u - v\rangle \geqslant \langle A(u), u - v\rangle. \tag{4.12}$$

Since it follows from (1.9) that $\langle B(u_k, u_k) - B(u_k, u), u_k - u\rangle \geqslant 0$ and we have $\lim\limits_{k\to\infty} \langle B(u_k, u), u_k - u\rangle = 0$ by replacing u_ε in (4.7) by u, we get

$$\liminf_{k\to\infty} \langle A(u_k), u_k - u\rangle \geqslant 0. \tag{4.13}$$

Combining (4.12) and (4.13), we have

$$\begin{aligned}
\liminf_{k\to\infty} \langle A(u_k), u_k - v\rangle &= \liminf_{k\to\infty} (\langle A(u_k), u_k - u\rangle + \langle A(u_k), u - v\rangle) \\
&\geqslant \liminf_{k\to\infty} \langle A(u_k), u_k - u\rangle + \liminf_{k\to\infty} \langle A(u_k), u - v\rangle \\
&\geqslant \langle A(u), u - v\rangle.
\end{aligned}$$

Therefore (4.3) is proved. □

Proof of Theorem 1.2. From (H6) we have

$$\begin{aligned}
\langle Au, u\rangle &= \int_\Omega (a(x, u, \nabla u)\nabla u + b(x, u, \nabla u)u)\,\mathrm{d}x \\
&\geqslant \int_\Omega (c_0 |\nabla u|^{p(x)} - |f_4(x)||u|^{r_2(x)} - f_5(x))\,\mathrm{d}x \\
&\geqslant (c_0\|u\|^{p^- - r_2^+} - C\|f_4\|_{q(\cdot)/(q(\cdot)-r_2(\cdot))} - \|f_5\|_1)\|u\|,
\end{aligned}$$

provided $\|u\|$ is large enough (see the proof of Lemma 3.3 for detail). Since $p^- > r_2^+$, thus A is coercive. Therefore, by Proposition 2.7 and Lemma 4.2, the problem (1.1) has a weak solution. □

As seen in the proof of Lemma 4.2, the condition (H5) was used only to ensure that the subsequence $\{\nabla u_{k'_j}(x)\}$ converges to $\nabla u(x)$ for a.e. $x \in \Omega$ in order to prove the weak convergence (4.10). The following lemma shows that if the lower order term b is independent of ξ or linearly dependent on ξ, then it allows us to replace the condition (H5) in Lemma 4.2 by the monotonicity condition (H2).

Lemma 4.3. *Let* (H1) *and* (H2) *be satisfied except* (1.3) *and let one of the following two cases hold:*

(i) *b is independent of* ξ, *i.e.*, $b(x,z,\xi)\equiv b(x,z)$ *and* $b\colon\ \Omega\times\mathbb{R}\to\mathbb{R}$ *satisfies*

$$|b(x,z)|\leqslant b_1|z|^{q(x)-1}+f_2(x) \tag{4.14}$$

for a.e. $x\in\Omega$ *and all* $z\in\mathbb{R}$, *where the constant* b_1 *and functions* q, f_2 *are the same as in* (H1).

(ii) *b is linearly dependent on* ξ, *i.e.*, $b(x,z,\xi)\equiv\bar b(x,z)\cdot\xi$ *and* $\bar b\colon\ \Omega\times\mathbb{R}\to\mathbb{R}^n$ *is a Carathéodory function satisfying*

$$|\bar b(x,z)|\leqslant b_2|z|^{((m(x)-1)p(x)-m(x))/p(x)}+f(x) \tag{4.15}$$

for a.e. $x\in\Omega$ *and all* $z\in\mathbb{R}$, *where* b_2 *is a nonnegative constant and variable exponent* p *is the same as in* (H1) *and* m *is a function satisfying* $p'(x)<m(x)<p^*(x)$ *for any* $x\in\overline{\Omega}$ *and* $m\in C(\overline{\Omega})$ *and* $f\in L^{p(\cdot)m'(\cdot)/(p(\cdot)-m'(\cdot))}(\Omega)$. *Then the operator* A *defined by* (3.1) *is pseudomonotone.*

Proof. *Case (i):* Since the condition (4.14) on b is a special case of (1.3), it remains only to show that

$$\lim_{k\to\infty}\int_\Omega b(x,u_k)(u-v)\,\mathrm{d}x=\int_\Omega b(x,u)(u-v)\,\mathrm{d}x \tag{4.16}$$

instead of (4.9), where $\{u_k\}$, u and v are the same as in the proof of Lemma 4.2. Since $u_k\to u$ strongly in $L^{q(\cdot)}(\Omega)$ by Proposition 2.2, it follows from (4.14) and Proposition 2.1 that $b(x,u_k)\to b(x,u)$ strongly in $L^{q'(\cdot)}(\Omega)$. Hence, we use that $u-v\in L^{q(\cdot)}(\Omega)$ to get (4.16).

Case (ii): Using the Schwarz inequality and Young inequality, it follows from (4.15) that

$$|b(x,z,\xi)|=|\bar b(x,z)\cdot\xi|\leqslant(b_2+1)|\xi|^{p(x)/m'(x)}+b_2|z|^{m(x)-1}+|f(x)|^{p(x)/(p(x)-m'(x))}.$$

Since $f\in L^{p(\cdot)m'(\cdot)/(p(\cdot)-m'(\cdot))}(\Omega)$, putting $f_2(x):=|f(x)|^{p(x)/(p(x)-m'(x))}\in L^{m'(\cdot)}(\Omega)$, we obtain the inequality

$$|b(x,z,\xi)|\leqslant(b_2+1)|\xi|^{p(x)/m'(x)}+b_2|z|^{m(x)-1}+f_2(x),$$

which is similar to (1.3). Hence, like in Case (i), in order to prove Case (ii), it suffices to show that

$$\lim_{k\to\infty}\int_\Omega\bar b(x,u_k)\cdot\nabla u_k w\,\mathrm{d}x=\int_\Omega\bar b(x,u)\cdot\nabla u w\,\mathrm{d}x\quad\text{for any } w\in L^{m(\cdot)}(\Omega). \tag{4.17}$$

Since $u_k\rightharpoonup u$ in X, the sequence $\{\nabla u_k\}$ converges weakly to ∇u in $(L^{p(\cdot)}(\Omega))^n$. Therefore, if

$$\bar b(x,u_k)w\to\bar b(x,u)w\text{ strongly in }(L^{p'(\cdot)}(\Omega))^n, \tag{4.18}$$

then it follows that (4.17) holds. By the Hölder inequality we have

$$
\begin{aligned}
(4.19)\qquad &\int_\Omega |\bar b(x,u_k)w-\bar b(x,u)w|^{p'(x)}\,\mathrm{d}x\\
&\quad\leqslant 2\||w|^{p'(x)}\|_{m(\cdot)/p'(\cdot)}\||\bar b(x,u_k)-\bar b(x,u)|^{p'(x)}\|_{m(\cdot)/(m(\cdot)-p'(\cdot))}\\
&\quad\leqslant 2\max\{\|w\|_{m(\cdot)}^{(p')^-},\|w\|_{m(\cdot)}^{(p')^+}\}\\
&\qquad\times\max\{\||\bar b(x,u_k)-\bar b(x,u)|\|_{m(\cdot)p'(\cdot)/(m(\cdot)-p'(\cdot))}^{(p')^-},\\
&\qquad\qquad\||\bar b(x,u_k)-\bar b(x,u)|\|_{m(\cdot)p'(\cdot)/(m(\cdot)-p'(\cdot))}^{(p')^+}\}.
\end{aligned}
$$

It is clear that

$$\frac{m(x)p'(x)}{m(x)-p'(x)}=\frac{p(x)m'(x)}{p(x)-m'(x)}\quad\text{and}\quad\frac{(m(x)-1)p(x)-m(x)}{p(x)}=m(x)\frac{p(x)-m'(x)}{p(x)m'(x)}.$$

Thus, from (4.15) and Proposition 2.1, it follows that

$$\bar b(x,u_k)\to\bar b(x,u)\text{ strongly in }(L^{m(\cdot)p'(\cdot)/(m(\cdot)-p'(\cdot))}(\Omega))^n,$$

where we used the fact that $u_k\to u$ in $L^{m(\cdot)}(\Omega)$, which follows from Proposition 2.2. Lemma is proved. □

Theorem 4.4. *Let the hypothesis of Lemma* 4.3 *hold, and let, in addition, the assumption* (H6) *be satisfied. Then the problem* (1.1) *has at least one weak solution.*

Proof. In last part of the proof of Theorem 1.2, we showed that the operator $A\colon X\to X^*$ defined by (3.1) is coercive. Therefore, the proof of Theorem 4.4 is completed by using Proposition 2.7 and Lemma 4.3. □

Remark 4.5. In papers [19] and [28], the authors studied the regularity properties of weak solutions to elliptic equations with the general nonstandard growth conditions. According to [28], Theorem 1.1, under slightly stronger conditions on the data in (H1) and (H6), that is, $f_1\in L^{p'(\cdot)s(\cdot)}(\Omega)$, $f_2\in L^{q'(\cdot)s(\cdot)}(\Omega)$, $f_3\in L^{p(\cdot)s(\cdot)/(p(\cdot)-r(\cdot))}$, $f_4\in L^{q(\cdot)s(\cdot)/(q(\cdot)-r(\cdot))}(\Omega)$, $f_5\in L^{s(\cdot)}(\Omega)$ for some $s\in C(\overline{\Omega})$ satisfying $s(x)>p^*(x)/(p^*(x)-p(x))$ for all $x\in\overline{\Omega}$, the weak solution of (1.1) is bounded in Ω, where we use the convention $\infty/\infty=1$. If, in addition, p is log-Hölder continuous in Ω, i.e.,

$$-|p(x)-p(y)|\log|x-y|\leqslant C\quad\forall\,x,y\in\Omega,\ |x-y|\leqslant\frac12$$

and Ω is sufficiently smooth, for example, Ω satisfies a uniform exterior cone condition on $\partial\Omega$, and u is a weak solution of (1.1) then $u\in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$; see [28], Theorem 1.2.

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