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GENERALIZED SYNCHRONIZATION IN THE NETWORKS WITH DIRECTED ACYCLIC STRUCTURE

Sergej Čelikovský, Volodymyr Lynnyk, Anna Lynnyk and Branislav Rehák

Generalized synchronization in the direct acyclic networks, i.e. the networks represented by the directed tree, is presented here. Network nodes consist of copies of the so-called generalized Lorenz system with possibly different parameters yet mutually structurally equivalent. The difference in parameters actually requires the generalized synchronization rather than the identical one. As the class of generalized Lorenz systems includes the well-known particular classes such as (classical) Lorenz system, Chen system, or Lü system, all these classes can be synchronized using the presented approach as well. The main theorem is rigorously mathematically formulated and proved in detail. Extensive numerical simulations are included to illustrate and further substantiate these theoretical results. Moreover, during these numerical experiments, the so-called duplicated system approach is used to double-check the generalized synchronization.

Keywords: generalized Lorenz system, generalized synchronization, chaos, networks

Classification: 93C10, 05C82, 34D06

1. INTRODUCTION

1.1. State of the art

The generalized synchronization of the direct acyclic network (DAG) having copies of the so-called generalized Lorenz system (GLS) at its nodes is investigated here. GLS is a three-dimensional system of ordinary differential equations presenting for a wide range of its parameters chaotic behavior. Connections between nodes correspond to scalar information transmissions enabling synchronization of all states of all nodes. Since nodes may have different parameters, only the so-called generalized synchronization is achieved, which is weaker than identical synchronization. Note, that DAG topology is represented by a special graph, the **directed tree**, where all edges are directed and each node has no more than one incoming and out-coming edge; moreover, any tree is by definition a connected graph. Obviously, there should be a unique node without an incoming edge, called in the sequel as the **master node** of the network or the **root** of the tree.

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During the last decades, there has been widespread and intense interest in studying complex networks (CN) consisting of dynamical systems. The network may serve as the diagrammatic representation of the structure or physical systems [9]. CN structure is the graph of the nodes and links with a specific topology, represented by different types of typologies, depending on the structure of the physical system or structure. The main topologies, for example, are star, chain, tree, ring, lattice, etc. Natural and artificial systems, like social networks, metabolic networks, electric power grids, neural networks, biochemical networks, communication networks, etc., can be described by CN. Among problems to be addressed when studying CN, the synchronization phenomena have played a prominent role.

The synchronization can be viewed as a presence of a collective state of interconnected systems [5] and has various specific and mathematically exact formulated types. The studies of the so-called identical synchronization of the couples systems were first presented in [10, 20]. Interconnected chaotic systems and their identical synchronization phenomena were first investigated in [19]. Identical synchronization (IS), when the states of the interconnected systems mutually asymptotically converge to each other, represents one of the most studied types of synchronization. Among the other kinds of synchronization in coupled chaotic systems or complex networks, there is generalized synchronization (GS) [12, 29], projective synchronization [17], lag synchronization [28], phase synchronization [27], among others. It is known that disturbances and/or time delays, quantization, and bandwidth limitation can heavily influence the quality and speed of the synchronization between nodes of the CN. These obstacles that may naturally occur in the communication channels were studied in [3, 22, 23, 25, 26], and references within there.

This paper focuses on the study of the specific case of the synchronization of structurally equivalent systems (identical systems or identical systems with a parameter mismatch rectifiable by the state transformation, representing either smooth or topological equivalence). The respective area is still rather wide, and it is studied by a large portion of the existing literature in the field, cf. [3, 29, 31] and references within there. Yet, even for structurally equivalent, but non-identical systems, identical synchronization is not possible. The suitable alternative is the so-called **generalized synchronization (GS)** [4] where the state of one system is synchronized with an image of the state of other system under suitable smooth, or non-smooth mappings, possibly even non-invertible ones. The functional relationship between coupled systems is the primary synchronization condition in GS of structurally equivalent systems. The stability of the synchronization manifold of GS can be such as in the case of IS, i. e., by the negativity of conditional Lyapunov exponents [21]. GS of two interconnected systems in the master-slave configuration was first reported in [2]. Later, in [29], the method of mutual false neighbor for detecting the GS was introduced. Then in [1], the auxiliary system approach (ASA) was proposed and widely used to verify the presence of the GS between the unidirectional coupled chaotic systems. The so-called duplicated systems approach (DSA), the simplified method based on the ASA method, was introduced by authors in [16]. This method required fewer additional nodes than the ASA method.

As we noted before, many theoretical results about synchronization focus on structurally equivalent coupled systems, but, at the same time, the study of the synchronization phenomena between strictly different (including systems with different dimensions) dynamical systems is also essential, especially in the case of biological and social sciences [3, 13, 18, 30].

In this paper, the main theoretical approach is based on the explicit construction of a diffeomorphism between systems that allows demonstrating directly GS between them. Practically, the GS is double-checked using the DSA in the numerical simulations.

1.2. Purpose and outline of the paper

This paper studies the generalized synchronization in DAG complex networks having structurally equivalent coupled chaotic systems as their nodes. Each node of the CN is a chaotic system described by the generalized Lorenz system (GLS), introduced in [6, 7] and repeated later on in this paper for the reader's convenience. Many famous chaotic systems, for example, the Chen system [8], the Lorenz system [14], the Lü system [15], etc., are actually subclasses of GLS. The detailed classification of these chaotic systems with respect to GLS, is summed up in Table I. More specifically, the theoretical result addressing the generalized synchronization of the DAG complex network with nodes being GLS is rigorously formulated and proved. This result uses some synchronizing gains l_1 and l_2 identical for each node incoming connection. Using the same gains for all nodes opens some interesting application avenues in secure communications between nodes. Further, the duplicated system approach is applied to detecting the GS in the DAG complex network in numerical experiments.

The rest of the paper is organized as follows. Section 2 collects some definitions and preliminary results. Main theoretical results are given in section 3 and numerical simulations and experiments are presented in section 4. The conclusions and future research directions are discussed in the final section.

Notations. Some abbreviations are collected here: DAG – direct acyclic graph, CN – complex network, GS – generalized synchronization, GLS – generalized Lorenz system, DSA – duplicated system approach, ASA – auxiliary system approach, IS – identical synchronization.

2. DEFINITIONS AND PRELIMINARY RESULTS

Some known definitions and results are repeated here for further exposition, including proof where appropriate for the self-contained reading.

Definition 2.1. The GLS is the three-dimensional dynamical system with real parameters a_{11}, a_{12}, a_{21}, a_{22}, λ_3 given by the following ordinary differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \tag{1}$$

$$a_{11}a_{22} - a_{12}a_{21} < 0, \quad a_{11} + a_{22} < 0, \quad \lambda_3 < 0\,. \tag{2}$$

Remark 2.2. GLS (1) – (2) possesses the main structural characteristics of the Lorenz system; in particular, it includes the "classical" Lorenz system for $a_{11} = -\sigma$, $a_{12} = \sigma$, $a_{21} = r$, $a_{22} = -1$, $\lambda_3 = -b$. The right-hand side of GLS (1) consists of two parts: the linear one and the quadratic one. The quadratic part is the same as in the classical Lorenz system, but the linear one has the same more general block triangular structure mimicking the classical Lorenz system. Condition (2) guarantees that the respective linear part has two negative eigenvalues and one positive real eigenvalue.

Theorem 2.3. (Lynnyk et al. [16]) GLS (1) with $a_{12} \neq 0$ is state equivalent to

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -w_1 w_3 - \frac{w_1^3}{2a_{12}} \\ K w_1^2 \end{bmatrix}, \quad K = \frac{\lambda_3 - 2a_{11}}{2a_{12}}, \tag{3}$$

where the corresponding state transformation and its inverse are

$$w_1 = x_1,\ w_2 = x_2,\ w_3 = x_3 - \frac{x_1^2}{2a_{12}}, \quad x_1 = w_1,\ x_2 = w_2,\ x_3 = w_3 + \frac{w_1^2}{2a_{12}}. \tag{4}$$

Proof. Straightforward computations using (3) and (4) give

$$\dot{w}_1 = \dot{x}_1 = a_{11}x_1 + a_{12}x_2 = a_{11}w_1 + a_{12}w_2,$$

$$\dot{w}_2 = \dot{x}_2 = a_{12}x_1 + a_{22}x_2 - x_1x_3 = a_{12}w_1 + a_{22}w_2 - w_1w_3 - \frac{w_1^3}{2a_{12}},$$

and one has by differentiation of the definition of w_3 in (4) that

$$\dot{w}_3 = \dot{x}_3 - \frac{2x_1}{2a_{12}}\dot{x}_1 = \lambda_3 x_3 + x_1x_2 - \frac{2x_1}{2a_{12}}(a_{11}x_1 + a_{12}x_2) = \lambda_3 x_3 - \frac{a_{11}x_1^2}{a_{12}}.$$

Substituting in the last expression x_1, x_2, x_3 from (4) gives

$$\dot{w}_3 = \lambda_3 x_3 - \frac{a_{11}x_1^2}{a_{12}} = \lambda_3 \left(w_3 + \frac{w_1^2}{2a_{12}} \right) - \frac{a_{11}w_1^2}{a_{12}} = \lambda_3 w_3 + K w_1^2, \tag{5}$$

where K is as in (3) and proof is completed. □

Theorem 2.3 provides the construction of the synchronization of two GLS's in the master-slave configuration which is facilitated by the special form 3. First, synchronization is described for the respective transformed system by the following

Lemma 2.4. Consider (3) and the notation used there. Define the following system

$$\begin{bmatrix} \dot{\widehat{w}}_1 \\ \dot{\widehat{w}}_2 \\ \dot{\widehat{w}}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \widehat{w}_1 \\ \widehat{w}_2 \\ \widehat{w}_3 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \\ 0 \end{bmatrix} (w_1 - \widehat{w}_1) + \begin{bmatrix} 0 \\ -w_1 \widehat{w}_3 - \frac{w_1^3}{2a_{12}} \\ K w_1^2 \end{bmatrix}. \tag{6}$$

Let there exist $W > 0$ such that for solution of (3) it holds $|w_1(t)| \le W,\ \forall t \ge 0$, let

$$a_{11} + a_{22} - l_1 < 0, \quad a_{11}a_{22} - a_{12}a_{21} - l_1 a_{22} + l_2 a_{12} > 0, \tag{7}$$

and let $\lambda_3 < 0$. Then there exist $M > 0,\ L > 0,\ \forall t \in \mathbb{R}^+$ such that

$$\hat{w} := \begin{bmatrix} \widehat{w}_1 \\ \widehat{w}_2 \\ \widehat{w}_3 \end{bmatrix}, \ w := \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \ \|\hat{w}(t) - w(t)\| \le M \exp(-Lt)\|\hat{w}(0) - w(0)\|. \tag{8}$$

Proof. Denote $\epsilon := \hat{w} - w = [\epsilon_1\ \epsilon_2\ \epsilon_3]^\top$, subtracting (3) from (6) gives straightforwardly

$$\begin{bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \end{bmatrix} = \begin{bmatrix} a_{11} - l_1 & a_{12} & 0 \\ a_{21} - l_2 & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \epsilon - \begin{bmatrix} 0 \\ \epsilon_3 \\ 0 \end{bmatrix} w_1(t).$$

Since the above equations obviously give that $\epsilon_3(t) = \epsilon_3(0)\exp(\lambda_3 t)$, one has

$$\begin{bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \end{bmatrix} = E \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \epsilon_3(0)\exp(\lambda_3 t) w_1(t) \end{bmatrix}, \quad E := \begin{bmatrix} a_{11} - l_1 & a_{12} \\ a_{21} - l_2 & a_{22} \end{bmatrix},$$

$$\begin{bmatrix} \epsilon_1(t) \\ \epsilon_2(t) \end{bmatrix} = \exp(Et) \begin{bmatrix} \epsilon_1(0) \\ \epsilon_2(0) \end{bmatrix} + \int_0^t \exp(E(t-s)) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \epsilon_3(0)\exp(\lambda_3 s) w_1(s) \mathrm{d}s. \tag{9}$$

Straightforward computations using (7) show that the (2×2) matrix E has eigenvalues with the negative real parts and therefore by Theorem 4.11 of [11] there exist $\widetilde{M} > 0, \widetilde{L} > 0$, such that $\|\exp(Et)\| \le \widetilde{M}\exp(-\widetilde{L}t),\ \forall t \ge 0$. Moreover, $\widetilde{L} > 0$ can be taken such that $\widetilde{L} \ne -\lambda_3$. Indeed, if existing by Theorem 4.11 of [11] number $\widetilde{L} = -\lambda_3 > 0$, it can be replaced by any number $\widetilde{L} \in (0, -\lambda_3)$ obviously keeping the respective inequality valid as well. Applying the inequality $\|\exp(Et)\| \le \widetilde{M}\exp(-\widetilde{L}t),\ \forall t \ge 0$, the assumption $|w_1(t)| \le W,\ \forall t \ge 0$, and the obvious $\|[\epsilon_1(0), \epsilon_2(0)]^\top]\| = 1$ to (9) gives

$$\forall t \ge 0: \quad \|[\epsilon_1(t), \epsilon_2(t)]^\top\| \le \widetilde{M}\exp(-\widetilde{L}t)\|[\epsilon_1(0), \epsilon_2(0)]^\top]\| + W\widetilde{M}|\epsilon_3(0)|$$

$$\times \int_0^t \exp(-\widetilde{L}(t-s))\|[0,1]^\top\|\exp(\lambda_3 s)\mathrm{d}s \le \widetilde{M}\exp(-\widetilde{L}t)\|[\epsilon_1(0), \epsilon_2(0)]^\top]\|$$

$$+W\widetilde{M}\exp(-\widetilde{L}t)|\epsilon_3(0)| \int_0^t \exp((\lambda_3 + \widetilde{L})s))\mathrm{d}s$$

$$= \widetilde{M}\exp(-\widetilde{L}t)\left[\|[\epsilon_1(0), \epsilon_2(0)]^\top]\| + W|e_3(0)|\frac{\exp((\lambda_3 + \widetilde{L})t) - 1}{\lambda_3 + \widetilde{L}}\right]$$

$$= \widetilde{M}\exp(-\widetilde{L}t)\Big[\|[\epsilon_1(0),\epsilon_2(0)]^\top]\| - |e_3(0)|\frac{W}{\lambda_3+\widetilde{L}}\Big] + \frac{\widetilde{M}W}{\lambda_3+\widetilde{L}}\exp(\lambda_3 t)|e_3(0)|$$
$$\leq \widetilde{M}\|\epsilon(0)\|\Big[\exp(-\widetilde{L}t) + \exp(-\widetilde{L}t)\frac{W}{|\lambda_3+\widetilde{L}|} + \frac{W}{|\lambda_3+\widetilde{L}|}\exp(\lambda_3 t)\Big]$$
$$\leq \widehat{M}\exp(-\widehat{L}t)\|\epsilon(0)\|, \quad \widehat{L} := \min(-\lambda_3, \widetilde{L}), \ \widehat{M} := \widetilde{M}\big(1+2\frac{W}{|\lambda_3+\widetilde{L}|}\big).$$

The penultimate inequality above is due to the obvious inequalities $\|[\epsilon_1(0),\epsilon_2(0)]^\top]\| \leq \|\epsilon(0)\|$ and $|e_3(0)| \leq \|\epsilon(0)\|$ while the subsequent inequality is by definitions of $\widehat{M}, \widehat{L}$ given after it. In such a way, it has been proved that $\forall t \geq 0$ it holds

$$\|[\epsilon_1(t),\epsilon_2(t)]^\top\| \leq \widehat{M}\exp(-\widehat{L}t)\|\epsilon(0)\|, \quad \widehat{L} := \min(-\lambda_3, \widetilde{L}), \ \widehat{M} := \widetilde{M}\Big(1+2\frac{W}{|\lambda_3+\widetilde{L}|}\Big).$$

To complete the proof of Lemma, recall that $\epsilon := \hat{w} - w = [\epsilon_1, \epsilon_2, \epsilon_3]^\top$ and $\epsilon_3(t) = \epsilon_3(0)\exp(\lambda_3 t)$. Triangle inequality gives $\forall t \geq 0$

$$\|\epsilon(t)\| \leq \|[\epsilon_1(t),\epsilon_2(t)]^\top\| + |e_3(t)| \leq \widehat{M}\exp(-\widehat{L}t)\|\epsilon(0)\| + |e_3(0)|\exp(\lambda_3 t)$$
$$\leq \exp(-\widehat{L}t)\big(\widehat{M}\|\epsilon(0)\| + |\epsilon_3(0)|\big) \leq \exp(-\widehat{L}t)\big(\widehat{M}+1\big)\|\epsilon(0)\|,$$

where the penultimate inequality is due to $\widehat{L} := \min(-\lambda_3, \widetilde{L})$ while the subsequent one is due to $|e_3(0)| \leq \|\epsilon(0)\|$. Summarizing, it has been proven that there exists $M, L > 0$ such that $\forall t \geq 0$ it holds

$$\|\epsilon(t)\| \leq M\exp(-Lt)\|\epsilon(0)\|, \quad L := \min(-\lambda_3, \widetilde{L}), \ M := 1 + \widetilde{M}\Big(1+2\frac{W}{|\lambda_3+\widetilde{L}|}\Big).$$

Proof has been completed. □

Remark 2.5. Constants $M, L > 0$ can be estimated using the constructive proof just presented. They depend on constants $\widetilde{M} > 0, \widetilde{L} > 0$, existing by Theorem 4.11 of [11], where one can find also ideas on how to estimate them for a particular matrix E. Note, that their quality can be affected by the observer gains l_1, l_2 that may be used as free design parameters. Further, they depend on $W > 0$ bounding the component $w_1(t)$ of the master system (3) and its linear part eigenvalue $\lambda_3 < 0$, which can not be in any way affected by the mentioned observer gains l_1, l_2 selection.

Corollary 2.6. Let any $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ be given such that $a_{12} \neq 0$. Then, the pair of conditions given in (7) holds if and only if $l_1 \in \mathbb{R}, l_2 \in \mathbb{R}$ are such that

$$l_1 > a_{11} + a_{22},$$
$$l_2 \begin{cases} > \frac{-a_{11}a_{22}+a_{12}a_{21}+l_1a_{22}}{a_{12}} & \text{if} \quad a_{12} > 0, \\ < \frac{-a_{11}a_{22}+a_{12}a_{21}+l_1a_{22}}{a_{12}} & \text{if} \quad a_{12} < 0. \end{cases} \tag{10}$$

Remark 2.7. Corollary 2.6 shows that the assumption (7) of Lemma 2.4 can be always satisfied by the proper selection of gains $l_1, l_2 \in \mathbb{R}$ when $a_{12} \neq 0$. The latter is quite reasonable as it is necessary for some complex nontrivial behavior of the master system (1) and its observer form (3). Indeed, the first equation in system (1) is independent for $a_{12} = 0$, so x_1 either goes to zero or an infinity or stays constant, and in this case, the behavior of GLS (1) is uninteresting (see Table I).

The following lemma was originally presented in [16]. It is recalled here including its proof as its ideas are used later on to prove other results of the current paper as well.

Lemma 2.8. Consider the system with states $\widehat{x}_1, \widehat{x}_2, \widehat{x}_3$ and other states $\widehat{w}_1, \widehat{w}_2, \widehat{w}_3$ as:

$$\begin{bmatrix} \dot{\widehat{x}}_1 \\ \dot{\widehat{x}}_2 \\ \dot{\widehat{x}}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{bmatrix} + \widehat{x}_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{bmatrix}$$
$$+ (x_1 - \widehat{x}_1) \begin{bmatrix} l_1 \\ l_2 - \widehat{x}_3 - \frac{x_1(\widehat{x}_1 + x_1)}{2a_{12}} \\ K(\widehat{x}_1 + x_1) + \frac{l_1 \widehat{x}_1}{a_{12}} \end{bmatrix}, \ K = \frac{\lambda_3 - 2a_{11}}{2a_{12}}, \tag{11}$$

$$\widehat{w}_1 = \widehat{x}_1, \ \widehat{w}_2 = \widehat{x}_2, \ \widehat{w}_3 = \widehat{x}_3 - \frac{\widehat{x}_1^2}{2a_{12}}, \quad \widehat{x}_1 = \widehat{w}_1, \ \widehat{x}_2 = \widehat{w}_2, \ \widehat{x}_3 = \widehat{w}_3 + \frac{\widehat{w}_1^2}{2a_{12}}. \tag{12}$$

Then (4)–(12) mapping $\mathbb{R}^6 \mapsto \mathbb{R}^6$ transforms the system (3)–(6) having the state $w_1, w_2, w_3, \widehat{w}_1, \widehat{w}_2, \widehat{w}_3$ into the system (1)–(11) having the state $x_1, x_2, x_3, \widehat{x}_1, \widehat{x}_2, \widehat{x}_3$.

Proof. One has by (12) and by (6) that

$$\dot{\widehat{x}}_1 = \dot{\widehat{w}}_1 = a_{11}\widehat{w}_1 + a_{12}\widehat{w}_2 + l_1(w_1 - \widehat{w}_1) = a_{11}\widehat{x}_1 + a_{12}\widehat{x}_2 + l_1(x_1 - \widehat{x}_1),$$

$$\dot{\widehat{x}}_2 = \dot{\widehat{w}}_2 = a_{21}\widehat{w}_1 + a_{22}\widehat{w}_2 + l_2(w_1 - \widehat{w}_1) - w_1\widehat{w}_3 - \frac{w_1^3}{2a_{12}} = a_{21}\widehat{x}_1 + a_{22}\widehat{x}_2 + l_2(x_1 - \widehat{x}_1)$$
$$-x_1(\widehat{x}_3 - \frac{\widehat{x}_1^2}{2a_{12}}) - \frac{x_1^3}{2a_{12}} = a_{21}\widehat{x}_1 + a_{22}\widehat{x}_2 - \widehat{x}_1\widehat{x}_3 + (x_1 - \widehat{x}_1)\left(l_2 - \widehat{x}_3 - \frac{x_1(\widehat{x}_1 + x_1)}{2a_{12}}\right),$$

$$\dot{\widehat{x}}_3 = \dot{\widehat{w}}_3 + \frac{\widehat{w}_1}{a_{12}}\dot{\widehat{w}}_1 = \lambda_3\widehat{w}_3 + Kw_1^2 + \frac{\widehat{w}_1}{a_{12}}\left(a_{11}\widehat{w}_1 + a_{12}\widehat{w}_2 + l_1(w_1 - \widehat{w}_1)\right)$$
$$= \lambda_3\left(\widehat{x}_3 - \widehat{x}_1^2/(2a_{12})\right) + Kx_1^2 + \frac{\widehat{x}_1}{a_{12}}\left(a_{11}\widehat{x}_1 + a_{12}\widehat{x}_2 + l_1(x_1 - \widehat{x}_1)\right)$$
$$= \lambda_3\widehat{x}_3 + \frac{\widehat{x}_1^2}{2a_{12}}(2a_{11} - \lambda_3 - 2l_1) + \widehat{x}_1\widehat{x}_2 + Kx_1^2 + \frac{l_1 x_1 \widehat{x}_1}{a_{12}} = \lambda_3\widehat{x}_3 + \widehat{x}_1\widehat{x}_2$$
$$+K(x_1^2 - \widehat{x}_1^2) - \frac{l_1\widehat{x}_1^2}{a_{12}} + \frac{l_1 x_1\widehat{x}_1}{a_{12}} = \lambda_3\widehat{x}_3 + \widehat{x}_1\widehat{x}_2 + (x_1 - \widehat{x}_1)\left(K(x_1 + \widehat{x}_1) + \frac{l_1\widehat{x}_1}{a_{12}}\right),$$

which, after comparison with (11), completes the proof. □

Remark 2.9. The right-hand side of the system (11) is the sum of the copy of GLS (1) with state $\widehat{x}$ and the connection to (1), the latter obviously vanishes when $x = \hat{x}$. As a matter of fact, it is actually the so-called **synchronizing connection**; this terminology is justified by Theorem 2.10 presented below.

Theorem 2.10. (Lynnyk et al. [16]) Consider system (1) and system (11) with l_1, l_2 satisfying (7). Let for a solution $x(t)$ of system (1) there exists $W > 0$ such that $\|x(t)\| \le W, \ \forall t \ge 0$. Further, denote by $\widehat{x}(t)$ be the solution of (11). Then $\widehat{x}(t)$ exists for all $t \ge 0$ and there exist positive constants $\overline{M}$ and $\overline{L}$ such that

$$\|\widehat{x}(t) - x(t)\| \le \overline{M} \exp(-\overline{L}t) \|\widehat{x}(0) - x(0)\|. \tag{13}$$

P r o o f . Note that the existence of $\widehat{x}(t)$ for all $t \ge 0$ is the straightforward consequence of (13) and the assumption that there is $W > 0$ such that $\|x(t)\| \le W, \ \forall t \ge 0$. To prove (13), realize by Lemma 2.8 that (1) – (11) is state equivalent to (3) – (6) and by Lemma 2.4 the inequality (8) holds for $w, \widehat{w}$ in (3) – (6). Since $x, \widehat{x}$ are the images of $w, \widehat{w}$ via the globally smooth one-to-one mapping (4) – (12), the inequality (13) should hold for some suitable positive constants $\overline{M}$ and $\overline{L}$ as well. □

Generalized Lorenz system (1) – (2)
$a_{12} = 0$: trivial case
$a_{21} = 0 \wedge a_{11} \ge 0$: trivial case
$a_{21} = 0 \wedge a_{11} < 0$: Lü system [15]
$a_{12}a_{21} > 0$: Lorenz system [14]
$a_{12}a_{21} < 0$: Chen system [8]

Tab. 1. Classification of the special cases of GLS (1) – (2) [**?**].

This section is concluded by recalling some basics from complex networks and graph theory. The complex network structure is usually identified with either a directed or undirected graph, having N nodes and some connections between them, called edges. The overall graph structure can be described by the so-called **adjacency** matrix

$$C = \begin{bmatrix} c_{11} & \cdots & c_{1N} \\ \vdots & \ddots & \vdots \\ c_{N1} & \cdots & c_{NN} \end{bmatrix}, \quad c_{ij} \in \{0, 1\}, \ i, j = 1, \dots, N.$$

If there exists edge going from the ith to the jth node, then $c_{ij} = 1$, otherwise $c_{ij} = 0$. The node j_M is called **root (master)** if $c_{ij_M} = 0, \ \forall j = 1, \dots, N$. The **path** connecting nodes i_0 and i_M is a sequence of $c_{i_0 i_1} = c_{i_1 j_2} = \cdots = c_{i_{M-2} i_{M-1}} = c_{i_{M-1} j_M} = 1$, M is called the **path length**. The path is called **cycle (or loop)**, if $i_0 = i_M$. The graph is called **connected** if there is a path connecting any pair of nodes, the connected graph may have only one root. **Direct acyclic graph (or tree)**, see Figure 1, is a connected graph having a master and a unique path between any two nodes, or, equivalently, with no loops. Most importantly (crucially used later on): **each node of the tree has exactly one incoming edge except the root with no incoming edge**. More detailed exposition can be found e. g. in [9]).

3. MAIN RESULT

Generalized synchronization (GS) in the unidirectional complex networks is studied in this section to present the main novel theoretical result of the current paper, namely, Theorem 3.2 characterizing the generalized synchronization of a complex network with directed acyclic graph topology. First, the so-called generalized synchronization and auxiliary system approach are recalled, and some important results are repeated.

3.1. Generalized synchronization in the master-slave configuration

Consider two interconnected continuous time systems given by the vector fields f and g, where g includes the coupling, namely

$$\dot{x} = f(x) \tag{14a}$$

$$\dot{\hat{x}} = g(\hat{x}, x). \tag{14b}$$

Here, $x \in \mathbb{R}^n$ and $\hat{x} \in \mathbb{R}^m$ are called the states of the drive and response systems, respectively [**?**]. The system (14) is said to be **globally generalized** synchronized if there exists continuous mapping $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that for all solutions of (14) it holds

$$\lim_{t\to\infty} \big(\hat{x}(t) - \Phi(x(t)\big) = 0. \tag{15}$$

The presence of the asymptotically stable limit set of the response system is the essential condition of the generalized synchronization [12]. The GS between interconnected chaotic systems could be detected by various methods, among them the so-called **auxiliary system approach (ASA)** and **duplicated system approach (DSA)**.

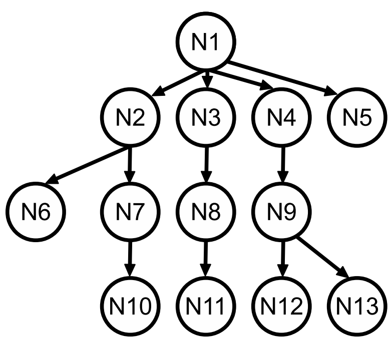

Fig. 1. Example of the network with topology described by the directed acyclic graph.

3.2. Generalized synchronization and auxiliary system approach

The block scheme of the auxiliary system approach (ASA) [1] used for detection of the generalized synchronization between two unidirectional coupled chaotic systems is shown in Figure 2. This method uses the identical response system copy, influenced by the same driving signal, i. e. N2 and N2' in Figure 2 are driven by the same master driving signal. In [31], the necessary conditions of the application of the auxiliary system approach for detecting the GS regime in CN with directional coupling are reported. Consequently, the auxiliary system approach can be applied for the GS detection in the CN with the directed acyclic graph structure (Figure 1). However, the ASA competence was shown as well in the case of the directed CNs with ring topology that is not described by DAG topology [16].

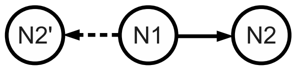

Fig. 2. Auxiliary system approach scheme [1]. N1 is a drive node, N2 is a response node, and N2' is an auxiliary node that copies the response node N2.

3.3. Generalized synchronization of generalized Lorenz systems in master-slave configuration

For the sake of self-complete exposition, the following theorem is repeated, [16].

Theorem 3.1. Consider system (1) and the system

$$\begin{aligned} \begin{bmatrix} \dot{\widehat{x}}_1 \\ \dot{\widehat{x}}_2 \\ \dot{\widehat{x}}_3 \end{bmatrix} &= \begin{bmatrix} \widehat{a}_{11} & \widehat{a}_{12} & 0 \\ \widehat{a}_{21} & \widehat{a}_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \widehat{x}_1 \\ \widehat{x}_2 \\ \widehat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\widehat{x}_1\widehat{x}_3 \\ \widehat{x}_1\widehat{x}_2 \end{bmatrix} \\ &+ (x_1 - \widehat{x}_1) \begin{bmatrix} l_1 \\ l_2 - \widehat{x}_3 - \frac{x_1(\widehat{x}_1 + x_1)}{2\widehat{a}_{12}} \\ K(\widehat{x}_1 + x_1) + \frac{l_1\widehat{x}_1}{\widehat{a}_{12}} \end{bmatrix}, \quad K = \frac{\lambda_3 - 2\widehat{a}_{11}}{2\widehat{a}_{12}}, \end{aligned} \tag{16}$$

where $l_1, l_2 \in \mathbb{R}$ satisfy

$$l_1 > \widehat{a}_{11} + \widehat{a}_{22}, \qquad l_2 \begin{cases} > \frac{-\widehat{a}_{11}\widehat{a}_{22} + \widehat{a}_{12}\widehat{a}_{21} + l_1\widehat{a}_{22}}{\widehat{a}_{12}} & \text{if} \quad \widehat{a}_{12} > 0, \\ < \frac{-\widehat{a}_{11}\widehat{a}_{22} + \widehat{a}_{12}\widehat{a}_{21} + l_1\widehat{a}_{22}}{\widehat{a}_{12}} & \text{if} \quad \widehat{a}_{12} < 0. \end{cases} \tag{17}$$

Further, let there exists $\nu \in \mathbb{R}, \nu \neq 0$ such that

$$\widehat{a}_{11} = a_{11}, \quad \widehat{a}_{22} = a_{22}, \quad a_{12} = \nu\widehat{a}_{12}, \tag{18}$$

and denote

$$\mathcal{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \nu \end{bmatrix}. \tag{19}$$

Let for a given solution $x(t)$ of system (1) there exist a constant $W > 0$, such that $\|x(t)\| \leq W \; \forall t \in \mathbb{R}^+$. Then there exists real positive constants $\overline{M}$ and $\overline{L}$ such that

$$\|\widehat{x}(t) - \mathcal{D}x(t)\| \leq \overline{M} \exp(-\overline{L}t)\|\widehat{x}(0) - \mathcal{D}x(0)\|. \tag{20}$$

Proof. Introduce the transformed state of GLS (1) and denote it $\widetilde{x}$, namely $\widetilde{x} = \mathcal{D}x$, where $\mathcal{D}$ is given by (19). Straightforward computations using GLS (1) give that

$$\dot{\widetilde{x}} = \mathcal{D}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \mathcal{D}\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \mathcal{D}x_1\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

$$\dot{\widetilde{x}} = \mathcal{D}\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\mathcal{D}^{-1}\begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \end{bmatrix} + \widetilde{x}_1\mathcal{D}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\mathcal{D}^{-1}\begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \end{bmatrix}.$$

This gives

$$\dot{\widetilde{x}} = \begin{bmatrix} \widehat{a}_{11} & \widehat{a}_{12} & 0 \\ \widehat{a}_{21} & \widehat{a}_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \end{bmatrix} + \widetilde{x}_1\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \\ \widetilde{x}_3 \end{bmatrix}. \tag{21}$$

Applying Theorem 2.10 with GLS (1) replaced by (21) completes the proof. □

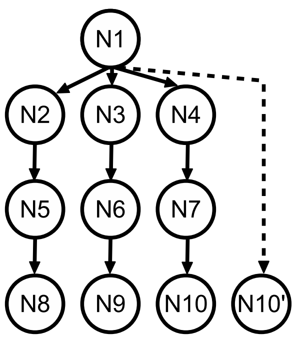

Fig. 3. Example of DAG network subject to DSA using the duplicated node.

3.4. Generalized synchronization of a complex network with topology described by directed acyclic graph

The main theoretical contribution of the present paper is the following

Theorem 3.2. Consider the direct acyclic graph with N nodes and $(N \times N)$ adjacency matrix $C = [c_{ij}],\ i, j = 1, \ldots, N$ and the respective complex network having at each node a copy of GLS with possibly different parameters node by node. Let the synchronizing connections are as in (16) with unique pair of gains l_1, l_2 along all existing edges and let the first node ($i = 1$) be the root. That is, consider the following $(3N \times 3N)$-dimensional system

$$\begin{bmatrix} \dot{x}_1^i \\ \dot{x}_2^i \\ \dot{x}_3^i \end{bmatrix} = \begin{bmatrix} a_{11}^i x_1^i + a_{12}^i x_2^i \\ a_{21}^i x_1^i + a_{22}^i x_2^i - x_1^i x_3^i \\ \lambda_3 x_3^i + x_1^i x_2^i \end{bmatrix} + \sum_{j=1}^{N} c_{ji}(x_1^j - x_1^i) \begin{bmatrix} l_1 \\ l_2 - x_3^i - \frac{x_1^j(x_1^i + x_1^j)}{2a_{12}^i} \\ K^i(x_1^i + x_1^j) + \frac{l_1 x_1^i}{a_{12}^i} \end{bmatrix},$$

$$x^i := \begin{bmatrix} x_1^i \\ x_2^i \\ x_3^i \end{bmatrix} \in \mathbb{R}^3, \quad K^i := \frac{\lambda_3^i - 2a_{11}^i}{2a_{12}^i}, \ i = 1, \dots, N, \quad c_{i1} = 0, \ i = 1, \dots, N. \tag{22}$$

Let $a_{12}^i \neq 0$, let the synchronizing gains l_1, l_2 in (22) satisfy $\forall i = 1, \dots, N$

$$l_1 > a_{11}^i + a_{22}^i, \quad l_2 a_{12}^i > -a_{11}^i a_{22}^i + a_{12}^i a_{21}^i + l_1 a_{22}^i \tag{23}$$

and let for every $i = 1, \dots, N$ there exist $\nu^i \in \mathbb{R} \setminus \{0\}$, such that

$$a_{11}^1 = a_{11}^i, \ a_{22}^1 = a_{22}^i, \quad a_{12}^1 = \nu^i a_{12}^i, \ a_{21}^1 = a_{21}^i / \nu^i. \tag{24}$$

Let $x^1(t)$ be a solution of the master node GLS (i. e. (22) for $i = 1$ only) and let there exist $W > 0$ such that $\|x^1(t)\| \leq W, \ \forall t \geq 0$. Then there exist real positive constants M^i and L^i, $i = 1, \dots, N$, such that $\forall i = 1, \dots, N$ it holds

$$\|x^i(t) - \mathcal{D}_i x^1(t)\| \leq M^i \exp(-L^i t)\|x^i(0) - \mathcal{D}_i x^1(0)\|, \quad \mathcal{D}_i := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \nu^i & 0 \\ 0 & 0 & \nu^i \end{bmatrix}. \tag{25}$$

The following lemma is needed to prove the previous theorem.

Lemma 3.3. Assume that $\phi(\eta) : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $\theta(t) : \mathbb{R} \mapsto \mathbb{R}^n$ are continuously differentiable and consider the system $\dot{\eta} = \phi(\eta), \quad \eta \in \mathbb{R}^n$. Assume its solutions $\eta(t)$ exist $\forall t \geq 0$ and let there exist positive real constants M, L, μ, λ such that

$$\|\eta(t)\| \leq M \exp(-Lt)\|\eta(0)\|, \ \|\theta(t)\| \leq \mu \exp(-\lambda t), \ \forall t \geq 0.$$

Then solutions $\widehat{\eta}(t), t \geq 0$, of the perturbed system $\dot{\widehat{\eta}} = \phi(\widehat{\eta}) + \theta(t), \ \widehat{\eta} \in \mathbb{R}^n$, exist $\forall t \geq 0$ and there exist positive real constants $\widehat{M}, \widehat{L}$ such that

$$\|\widehat{\eta}(t)\| \leq \widehat{M} \exp(-\widehat{L}t)\|\widehat{\eta}(0)\|, \ \forall t \geq 0.$$

Proof. By Converse Theorem on Exponential Stability, see e. g. [11], there exists continuously differentiable $V(\eta) : \mathbb{R}^n \mapsto \mathbb{R}$ and positive constants c_1, c_2, c_3, c_4, such that

$$c_1\|\eta\|^2 \leq V(\eta) \leq c_2\|\eta\|^2, \quad \frac{\partial V}{\partial \eta}\phi(\eta) \leq -c_3\|\eta\|^2, \quad \|\frac{\partial V}{\partial \eta}\| \leq c_4\|\eta\|, \ \eta \in \mathbb{R}^n.$$

Consider any solution $\widehat{\eta}(t), t \geq 0$, of the perturbed system $\dot{\widehat{\eta}} = \phi(\widehat{\eta}) + \theta(t)$. This solution exists uniquely locally $\forall t \in [0, \delta], \delta > 0$, since the perturbed system has continuously differentiable right-hand side. Moreover, by the last two inequalities of V above and the assumption on $\|\theta(t)\|$

$$\frac{\partial V}{\partial \eta}(\widehat{\eta}(t))[\phi(\widehat{\eta}(t)) + \theta(t)] \leq -c_3\|\widehat{\eta}(t)\|^2 + c_4\mu\|\widehat{\eta}(t)\| \exp(-\lambda t).$$

First of all, this inequality excludes finite-time escape, and therefore any solution $\widehat{\eta}(t)$ exists $\forall t \geq 0$. Further, let $[t_1^l, t_2^l]$ be some time segment such that $\forall t \in [t_1^l, t_2^l]$ it holds

$$\frac{\partial V}{\partial \eta}(\widehat{\eta}(t))[\phi(\widehat{\eta}(t)) + \theta(t)] \leq -c_3\|\widehat{\eta}(t)\|^2 + c_4\mu\|\widehat{\eta}(t)\| \exp(-\lambda t) \leq -(c_3/2)\|\widehat{\eta}(t)\|^2.$$

In such a way, using $c_1\|\eta\|^2 \leq V(\eta) \leq c_2\|\eta\|^2$, it holds $\forall t \in [t_1^l, t_2^l]$ that

$$\frac{\mathrm{d}V(\widehat{\eta}(t))}{\mathrm{d}t} = \frac{\partial V}{\partial \eta}(\widehat{\eta}(t))[\phi(\widehat{\eta}(t)) + \theta(t)] \leq -(c_3/2)\|\widehat{\eta}(t)\|^2 \leq -(c_3/2)V(\widehat{\eta}(t))/c_2$$

and using Comparison Lemma, see e.g. [11], it holds

$$V(\widehat{\eta}(t)) \leq V(\widehat{\eta}(t_1^l)) \exp(-\frac{c_3}{2c_2}t) \quad \Rightarrow \quad \|\widehat{\eta}(t)\|^2 \leq \frac{c_2}{c_1}\exp(-\frac{c_3}{2c_2}t)\|\widehat{\eta}(t_1^l)\|^2,$$

using $c_1\|\eta\|^2 \leq V(\eta) \leq c_2\|\eta\|^2$ again. So, it holds on $[t_1^l, t_2^l]$ that

$$\|\widehat{\eta}(t)\| \leq \sqrt{\frac{c_2}{c_1}} \exp(-\frac{c_3}{4c_2}t)\|\widehat{\eta}(t_1^l)\|^2.$$

Finally, let $[t_1^h, t_2^h]$ be some time segment such that $\forall t \in [t_1^h, t_2^h]$ it holds

$$-c_3\|\widehat{\eta}(t)\|^2 + c_4\mu\|\widehat{\eta}(t)\| \exp(-\lambda t) > -(c_3/2)\|\widehat{\eta}(t)\|^2.$$

Straightforward manipulations show that the above inequality is equivalent to

$$\|\widehat{\eta}(t)\| < (2c_4\mu/c_3)\exp(-\lambda t), \ \forall t \in [t_1^h, t_2^h].$$

Obviously, for every $t \geq 0$, it holds that it either belongs to some segment of type $[t_1^h, t_2^h]$, or to some segment of type $[t_1^l, t_2^l]$, which easily completes the proof. Indeed, an important crucial feature of both estimates obtained previously is that their constants depend on c_1, c_2, c_3, c_4 and K only; they do not depend on a particular interval of each type. □

P r o o f o f T h e o r e m 3.2. First, for clarity, note explicitly that by (22) and the first node being the master (root), i.e. $c_{j1} = 0, \ j = 1, \dots, N$, the solution $x^1(t)$ satisfies:

$$\begin{bmatrix} \dot{x}_1^1 \\ \dot{x}_2^1 \\ \dot{x}_3^1 \end{bmatrix} = \begin{bmatrix} a_{11}^1 x_1^1 + a_{12}^1 x_2^1 \\ a_{21}^1 x_1^1 + a_{22}^1 x_2^1 - x_1^1 x_3^1 \\ \lambda_3 x_3^1 + x_1^1 x_2^1 \end{bmatrix}, \quad x^1 := \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \end{bmatrix}. \tag{26}$$

Since (24) gives $\nu^1 = 1$, (25) trivially holds, indeed, $x^i(t) - \mathcal{D}_i x^1(t) = x^1(t) - x^1(t) = 0$. Furthermore, by the assumption that the respective graph is the direct acyclic one, or a tree, as already repeated in the review part of the paper before, every other node distinct from the root has exactly one incoming edge, i. e. exactly one synchronizing connection. That means that for every $i = 2, \ldots, N$ there exists unique $j = 1, \ldots, N-1$ such that

$$\begin{bmatrix} \dot{x}_1^i \\ \dot{x}_2^i \\ \dot{x}_3^i \end{bmatrix} = \begin{bmatrix} a_{11}^i x_1^i + a_{12}^i x_2^i \\ a_{21}^i x_1^i + a_{22}^i x_2^i - x_1^i x_3^i \\ \lambda_3 x_3^i + x_1^i x_2^i \end{bmatrix} + (x_1^j - x_1^i) \begin{bmatrix} l_1 \\ l_2 - x_3^i - \frac{x_1^j (x_1^i + x_1^j)}{2a_{12}^i} \\ K^i (x_1^i + x_1^j) + \frac{l_1 x_1^i}{a_{12}^i} \end{bmatrix}. \tag{27}$$

Introduce the so-called depth of any ith node, $i > 1$, namely, the integer d_i equal to a number of the edges connecting the root (the 1-st node) and the ith node. Obviously, $d_i \leq N-1$, $\forall i = 2, \ldots, N$ and d_i is uniquely and well-defined $\forall i = 2, \ldots, N$ due to the respective network being connected and acyclic. In such a way, the theorem claim can be proved by induction using the depth of the nodes.

I. (25) **is valid for any** i**th node with** $d_i = 1$**.** Indeed, by (27) it holds

$$\begin{bmatrix} \dot{x}_1^i \\ \dot{x}_2^i \\ \dot{x}_3^i \end{bmatrix} = \begin{bmatrix} a_{11}^i x_1^i + a_{12}^i x_2^i \\ a_{21}^i x_1^i + a_{22}^i x_2^i - x_1^i x_3^i \\ \lambda_3 x_3^i + x_1^i x_2^i \end{bmatrix} + (x_1^1 - x_1^i) \begin{bmatrix} l_1 \\ l_2 - x_3^i - \frac{x_1^j (x_1^i + x_1^1)}{2a_{12}^i} \\ K^i (x_1^i + x_1^1) + \frac{l_1 x_1^i}{a_{12}^i} \end{bmatrix}$$

and using Theorem 3.1 with $\widehat{x} := x^i$, $x := x^1$ and $\mathcal{D} := \mathcal{D}_i$ obviously proves (25).

II. Let $k > 1$ **and assume** (25) **is valid for any** j**th node with** $d_j = k-1$**.** As a matter of fact, the assumption just made means that there exist $\widetilde{M}, \widetilde{L} \in \mathbb{R}^+$ such that

$$\|x^j(t) - \mathcal{D}_j x^1(t)\| \leq \widetilde{M} \exp(-\widetilde{L} t) \|x^j(0) - \mathcal{D}_j x^1(0)\|, \ \forall t \in \mathbb{R}^+, \tag{28}$$

for every j such that $d_j = k-1$. Next, choose any $i \in \{2, \ldots, N-1\}$ such that for the ith node $d_i = k$. As already noted, the network is connected and there is a unique path from the root to that chosen ith node and therefore there exists some $j \in \{2, \ldots, N-1\}$ such that for the jth node $d_j = k-1$ and (27), (28) hold. Further, straightforward computations show that (26) is transformed using (24) as follows

$$\begin{bmatrix} \dot{\overline{x}}_1^1 \\ \dot{\overline{x}}_2^1 \\ \dot{\overline{x}}_3^1 \end{bmatrix} = \begin{bmatrix} a_{11}^i \overline{x}_1^1 + a_{12}^i \overline{x}_2^1 \\ a_{21}^i \overline{x}_1^1 + a_{22}^i \overline{x}_2^1 - \overline{x}_1^1 \overline{x}_3^1 \\ \lambda_3 \overline{x}_3^1 + \overline{x}_1^1 \overline{x}_2^1 \end{bmatrix}, \ \overline{x} := \mathcal{D}_i x^1 = \begin{bmatrix} x_1^1 \\ \nu_i x_2^1 \\ \nu_i x_3^1 \end{bmatrix}, \ \overline{x}^1 := \begin{bmatrix} \overline{x}_1^1 \\ \overline{x}_2^1 \\ \overline{x}_3^1 \end{bmatrix}. \tag{29}$$

Applying transformations (4) – (12) with $x := \overline{x}^1$, $\widehat{x} := x^i$ to (27) and (29) one gets straightforwardly[1] that the systems (27) and (29) are smoothly state transformed to

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -w_1 w_3 - \frac{w_1^3}{2a_{12}} \\ K w_1^2 \end{bmatrix}, \quad K = \frac{\lambda_3 - 2a_{11}}{2a_{12}}, \tag{30}$$

[1]Lemma 2.8 can not be straightly applied since (27) does not precisely match its formulation. Yet, the proof of Lemma 2.8 can be basically repeated giving (31) which is just (6) with w_1 replaced by x^j.

$$\begin{bmatrix} \dot{\widehat{w}}_1 \\ \dot{\widehat{w}}_2 \\ \dot{\widehat{w}}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \widehat{w}_1 \\ \widehat{w}_2 \\ \widehat{w}_3 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \\ 0 \end{bmatrix} (x_1^j - \widehat{w}_1) + \begin{bmatrix} 0 \\ -x_1^j \widehat{w}_3 - \frac{(x_1^j)^3}{2a_{12}} \\ K(x_1^j)^2 \end{bmatrix} \quad (31)$$

Equations (31) can be further represented as follows

$$\begin{bmatrix} \dot{\widehat{w}}_1 \\ \dot{\widehat{w}}_2 \\ \dot{\widehat{w}}_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \widehat{w}_1 \\ \widehat{w}_2 \\ \widehat{w}_3 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \\ 0 \end{bmatrix} (w_1 - \widehat{w}_1) + \begin{bmatrix} 0 \\ -w_1 \widehat{w}_3 - \frac{w_1^3}{2a_{12}} \\ K w_1^2 \end{bmatrix}$$

$$+ (w_1 - x_1^j)\omega(w_1, \widehat{w}_3, x_1^j) \ , \quad \|\omega(w_1, \widehat{w}_3, x_1^j)\| \le P, \quad (32)$$

where $\omega : \mathbb{R}^3 \mapsto \mathbb{R}$ is smooth and could be explicitly expressed, which is skipped for brevity, yet, it is clear that ω is smooth. Furthermore, the inequality in (32) holds for some suitable selected real constant $P > 0$ along all solutions of the above equations due to the following justification. As already noted, ω is a smooth function and the solution $w_1(t) = \overline{x}_1^1 = x_1^1$ is bounded by the assumption of the theorem being proved, $x_1^j(t)$ is bounded by the induction assumption that jth node globally exponentially converges to that bounded $w_1(t)$. The only less trivial part is to show that $\widehat{w}_3(t)$ is bounded which is by the equation $\dot{\widehat{w}}_3 = \lambda_3 \widehat{w}_3 + K(x_1^j)^2$ with $\lambda_3 < 0$ and already justified property x_1^j being bounded.

Next, note that $|w_1(t) - x_1^j(t)| \le \widetilde{M} \exp(-\widetilde{L}t)|w_1(t) - x_1^j(0)|$ due to the inductive assumption (28) and $w_1 := \mathcal{D}_j x_1$ [2]

$$\|(w_1 - x_1^j)\omega(w_1, \widehat{w}_3, x_1^j)\| \le P\widetilde{M} \exp(-\widetilde{L}t)|w_1(t) - x_1^j(0)|. \quad (33)$$

Denote $\widehat{\epsilon} := \hat{w} - w = [\widehat{\epsilon}_1 \ \widehat{\epsilon}_2 \ \widehat{\epsilon}_3]^\top$, subtracting (30) from (31) gives straightforwardly

$$\begin{bmatrix} \dot{\widehat{\epsilon}}_1 \\ \dot{\widehat{\epsilon}}_2 \\ \dot{\widehat{\epsilon}}_3 \end{bmatrix} = \begin{bmatrix} a_{11} - l_1 & a_{12} & 0 \\ a_{21} - l_2 & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \widehat{\epsilon} - \begin{bmatrix} 0 \\ \widehat{\epsilon}_3 \\ 0 \end{bmatrix} w_1(t) + (w_1 - x_1^j)\omega(w_1, \widehat{w}_3, x_1^j).$$

Further, during the proof of Lemma 2.4, it was shown that for the system

$$\begin{bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \end{bmatrix} = \begin{bmatrix} a_{11} - l_1 & a_{12} & 0 \\ a_{21} - l_2 & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \epsilon - \begin{bmatrix} 0 \\ \epsilon_3 \\ 0 \end{bmatrix} w_1(t)$$

it holds $\|\epsilon(t)\| \le M \exp(-Lt)\|\epsilon(0)\|$ for some suitable $M, L > 0$. Now, due to (33) and Lemma 3.3 one concludes that there exist positive real constants $\widehat{M}, \widehat{L}$ such that

$$\|\widehat{w}(t) - w(t)\| = \|\widehat{\epsilon}(t)\| \le \widehat{M} \exp(-\widehat{L}t)\|\widehat{\epsilon}(0)\|, \ \forall t \ge 0.$$

Finally, since $\widehat{w}(t), w(t)$ are smooth images of $x := \overline{x}^1$, $\widehat{x} := x^i$ applying transformations (4) – (12) to (27) and (29), one concludes that (28) holds with i, $d_i = k$ as well, possible with some different constants $\widetilde{M}, \widetilde{L}$.

[2] Note that the first component is always kept unchanged everywhere in all transformations used throughout the paper.

Summarizing, the validity of (25) for every j with $d_j = k-1$, $k > 1$, implies the validity of (25) for every i with $d_i = k$. The proof has been completed by induction.

□

Remark 3.4. Applying transformations (4) and (12) with $x := \overline{x}^1$, $\widehat{x} := x^i$ to (27) and (29) to $w(t)$ and $\widehat{w}(t)$ coordinates was needed since in original coordinates it is not guaranteed that $x_3^i(t)$ bounded which would prevent the use of Lemma 3.3. Indeed, dynamics of $x_3^i(t)$ dynamics is more complex, as a matter of fact, the simple dynamics of w_3 is a crucial useful point of nonlinearly transformed GLS.

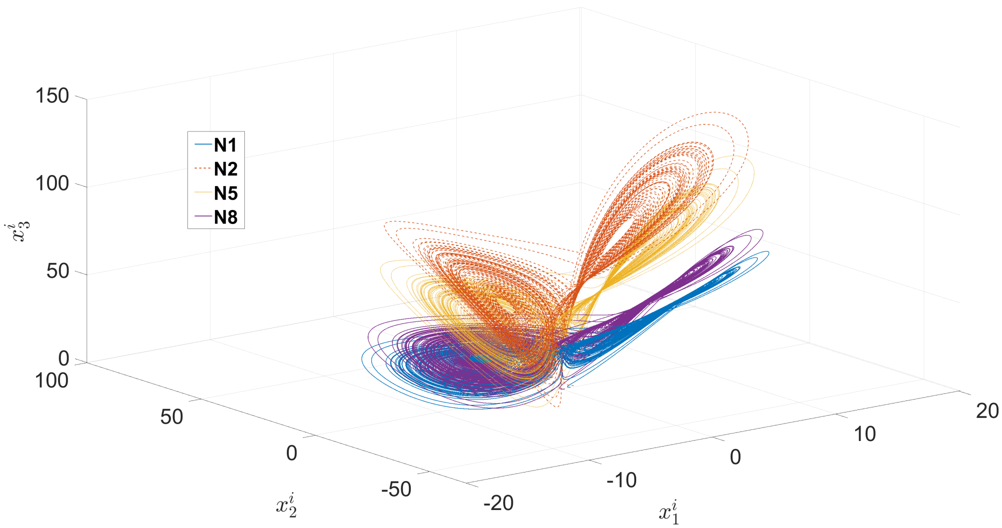

Fig. 4. The generalized synchronization of chaotic nodes – classical Lorenz system case.

Remark 3.5. The choice of the gains l_1, l_2, which should satisfy the condition (23), specifies the speed and accuracy of the network synchronization. Note that these gains are the same for all connections between nodes, while every node may have different parameters due to (24), indeed, the choice of nonzero ν_i, $i = 1, \ldots, N$ is arbitrary for each node. These numbers are not visible from synchronizing connections, and each particular node requires knowledge of its own ν_i, while other nodes, including the root master node, need not know this value. In other words, all ν_i, $i = 1, \ldots, N$ actually serve as kind of separate private keys for each node. The necessity to find l_1, l_2, satisfying (23) being $2N$ inequalities for just 2 free parameters may seem to be demanding. Yet, it was rather easy to satisfy them in simulations later on just manually, without engaging e. g. linear programming codes.

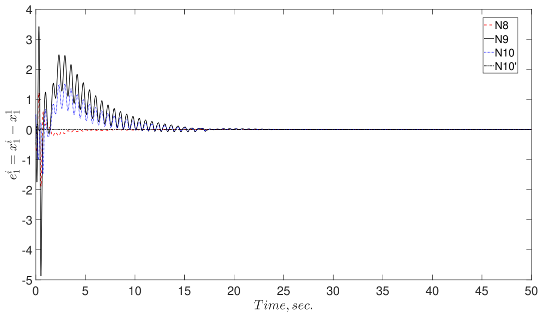

Fig. 5. Errors between N1 and N8, N9, N10, N10' – classical Lorenz system case.

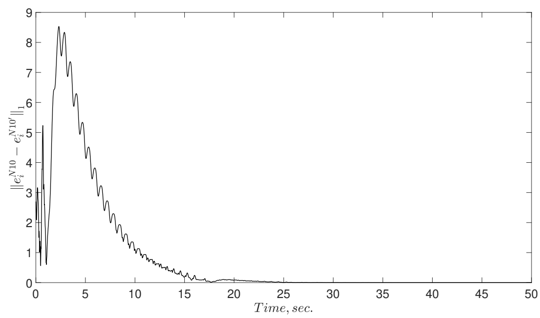

Fig. 6. Error between N10 and N10' – classical Lorenz system case.

4. NUMERICAL EXAMPLES

Theorem 3.2 will be illustrated by the following three examples of generalized synchronization. The structure of the respective DAG is the same in all three examples, and it is shown in Figure 3 while the nodes are generalized Lorenz system with different parameters representing classical Lorenz system, Chen system and Lü system. The root (master) node drives all the other nodes in the network directly or indirectly by the same synchronizing signal exactly as described in Theorem 3.2. Equivalently, topology in Figure 3 $c_{ij} = 1$ is described by (10×10) adjacency matrix with $c_{ij} = 1$ for $(i,j) = (1,2),(1,3),(1,4),(2,5),(3,6),(4,7),(5,8),(6,9),(7,10)$ and $c_{ij} = 0$ elsewhere. The node denoted as N10' is the duplicated mode to be used for additional practical synchronization tests via DSA.

Obviously, this network topology satisfies the assumptions of Theorem 3.2. Further, for each particular case later on (classical Lorenz, Chen and Lü system), the specific (cf. Table I) $a^i_{11}, a^i_{12}, a^i_{21}, a^i_{22}, \nu^i,\ i = 1,\dots,N$ satisfying (24) are selected. The particular systems at each node are different, yet, in each of the cases, belong to the same specific

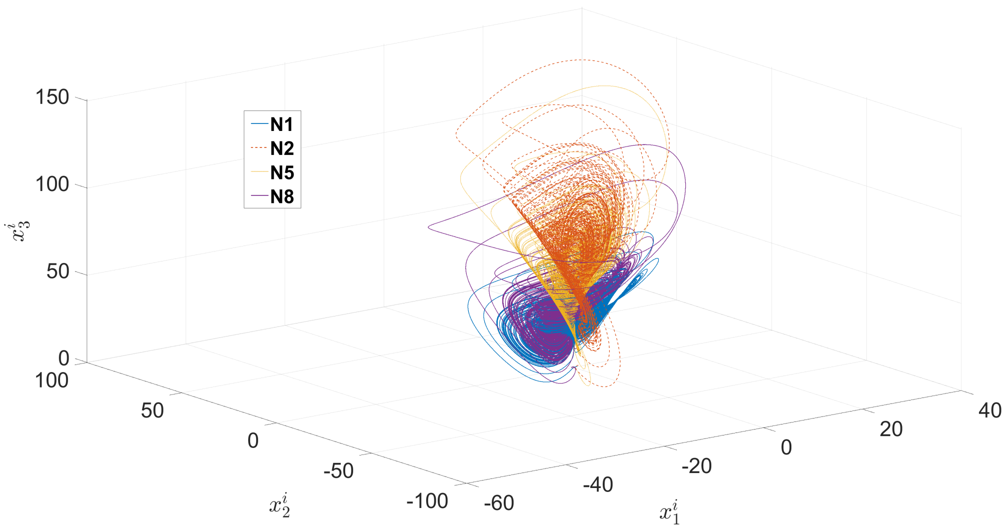

Fig. 7. The generalized synchronization of chaotic nodes – Chen system case.

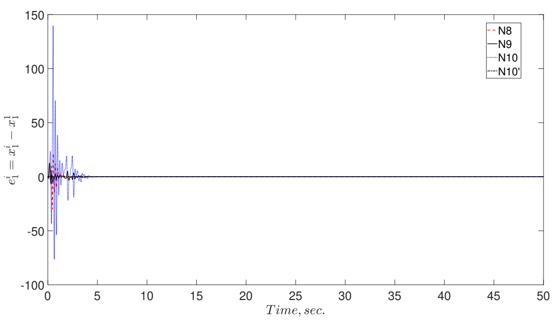

Fig. 8. Errors between N1 and N8, N9, N10, N10' – Chen system case.

subclass of GLS, i. e., either to classical Lorenz, or to Chen, or to Lü system. Indeed, all these classes are known not to be mutually topologically equivalent, and therefore (24) can not be satisfied for pair of systems from two different mentioned subclasses of GLS.

Next, for each case later on, the gains l_1, l_2 are rather easily found to satisfy (23). For each case and each node, including the duplicated one, some initial conditions $[x_1^i(0), x_2^i(0), x_3^i(0)]^\top$, $i = 1, \dots, N$, are selected. They are all mutually different, but all of them have the norm less than one.

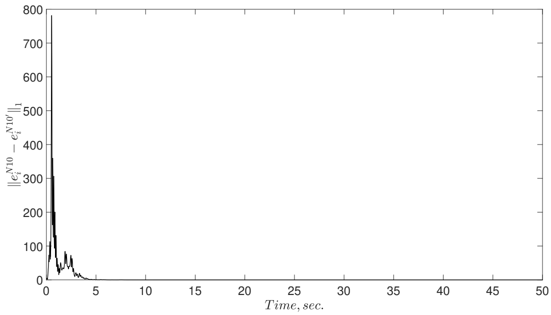

Fig. 9. Error between N10 and N10' – Chen system case.

With all that settings, the following three examples can be presented in detail. In all examples, the duplicated node N10' is the identical copy of the respective node N10, but it is initialized with different initial conditions, which is obviously a key ingredient to test synchronization using the DSA.

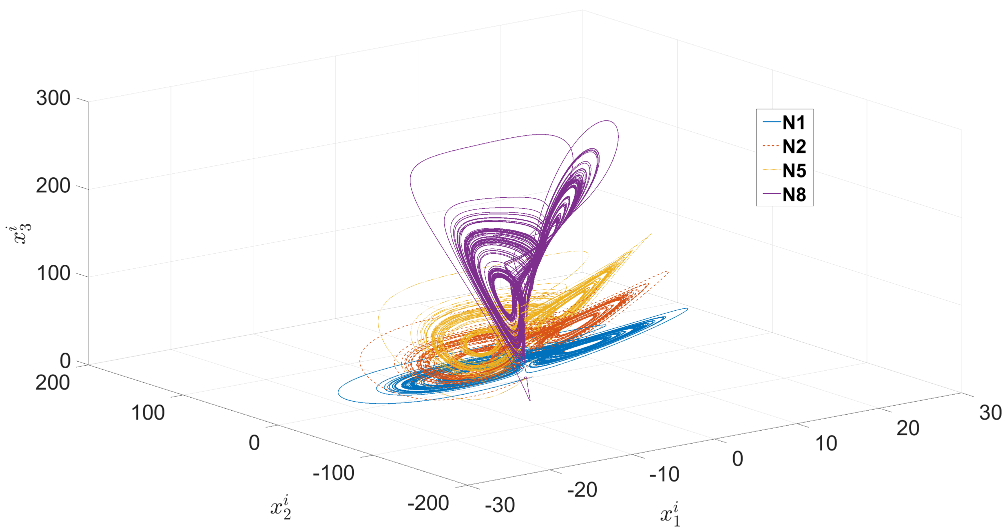

Fig. 10. The generalized synchronization of chaotic nodes – Lü system case.

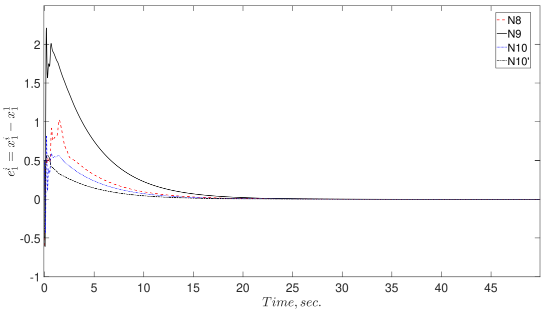

Fig. 11. Errors between N1 and N8, N9, N10, N10' – Lü system case.

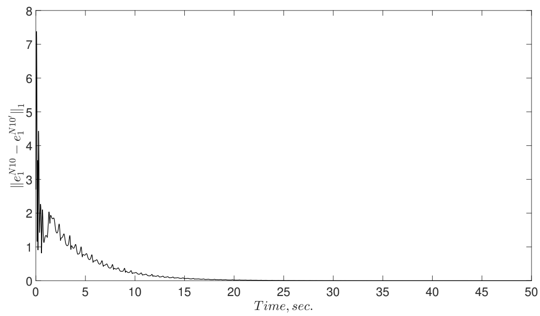

Fig. 12. Error between N10 and N10' – Lü system case.

4.1. Example 1. Lorenz system case.

In the first example, the nodes are represented by the generalized Lorenz systems with the next parameters: $a_{11}^{1,\ldots,10} = -10$, $a_{22}^{1,\ldots,10} = -1$, $a_{12}^{1} = 10$, $a_{21}^{1} = 28$, $a_{12}^{2,3,4} = 4$, $a_{21}^{2,3,4} = 70$, $a_{12}^{5,6,7} = 5$, $a_{21}^{5,6,7} = 56$, $a_{12}^{8,9,10} = 8$, $a_{21}^{8,9,10} = 35$, $\lambda_3 = -2.667$. The values of ν^i can be calculated to satisfy (24) as $\nu^{2,3,4} = 2.5$, $\nu^{5,6,7} = 2$, $\nu^{8,9,10} = 1.25$. As a matter of fact, the master node parameters are equal to the respective classical Lorenz system parameters [14]. Based on the conditions (23) the gains l_1 and l_2 were selected, namely, $l_1 = 1 > -11$ and $l_2 = 68 > 67.25$. The chaotic attractors generated by the master node N1 of the network and by the influenced nodes N2, N5, and N8 are shown in Figure 4, while Figure 5 represent the synchronization errors between master node N1 and nodes N8, N9, N10, N10'. All these errors go to zero as time goes to infinity, yet, one can see that the error between N1 and N10' goes to zero much faster than the others. This feature is due to the fact that there is a direct connection between N1 and N10', while N8, N9, and N10 are influenced indirectly.

Finally, to apply DSA, the error between node N10 and its duplicated copy N10' has been computed, see Figure 6. As this error goes to zero as time goes to infinity and the node N10' is the duplicated one influenced directly by the master node, the DSA thereby confirms synchronization of the whole network.

4.2. Example 2. Chen system case.

In the second example, the nodes are represented by the Chen systems with the next parameters: $a_{11}^{1,\ldots,10} = -35$, $a_{22}^{1,\ldots,10} = 28$, $a_{12}^{1} = 35$, $a_{21}^{1} = -7$, $a_{12}^{2,3,4} = 14$, $a_{21}^{2,3,4} = -17.5$, $a_{12}^{5,6,7} = 17.5$, $a_{21}^{5,6,7} = -14$, $a_{12}^{8,9,10} = 28$, $a_{21}^{8,9,10} = -8.75$, $\lambda_3 = -3$. The values of ν^i can be calculated to satisfy (24) as $\nu^{2,3,4} = 2.5$, $\nu^{5,6,7} = 2$, $\nu^{8,9,10} = 1.25$. The master node parameters are equal to the respective Chen system parameters [8]. Based on the conditions (23) the gains l_1 and l_2 were selected, namely, $l_1 = 1 > -7$ and $l_2 = 55 > 54.5$. The chaotic attractors generated by the master node N1 of the network and by the influenced nodes N2, N5, and N8 are shown in Figure 7. Figure 8 represent the synchronization errors between master node N1 and nodes N8, N9, N10, N10'. All these errors go to zero as time goes to infinity, yet, one can see that the error between N1 and N10' goes to zero much faster than the others. This feature is due to the fact that there is a direct connection between N1 and N10', while N8, N9, and N10 are influenced indirectly.

Finally, to apply DSA, the error between node N10 and its duplicated copy N10' has been computed, see Figure 9. As this error goes to zero as time goes to infinity and the node N10' is the duplicated one influenced directly by the master node, the DSA thereby confirms synchronization of the whole network.

4.3. Example 3. Lü systems case.

In the third example, the nodes are represented by the Lü systems with the next parameters:: $a_{11}^{1,\ldots,10} = -36$, $a_{22}^{1,\ldots,10} = 20$, $a_{21}^{1,\ldots,10} = 0$, $a_{12}^{1} = 36$, $a_{12}^{2,3,4} = 6$, $a_{12}^{5,6,7} = 12$, $a_{12}^{8,9,10} = 18$, $\lambda_3 = -3$. The values of ν^i can be calculated to satisfy (24) as $\nu^{2,3,4} = 6$, $\nu^{5,6,7} = 3$, $\nu^{8,9,10} = 2$. The master node parameters are equal to the respective Lü system parameters [15]. Based on the conditions (23) the gains l_1 and l_2 were selected, namely, $l_1 = 1 > -16$ and $l_2 = 124 > 123.3$. The chaotic attractors generated by the master node N1 of the network and by the influenced nodes N2, N5, and N8 are shown in Figure 10. Figure 11 represent the synchronization errors between master node N1 and nodes N8, N9, N10, N10'. All these errors go to zero as time goes to infinity, yet, one can see that the error between N1 and N10' goes to zero much faster than the others. This feature is due to the fact that there is a direct connection between N1 and N10', while N8, N9, and N10 are influenced indirectly.

Finally, to apply DSA, the error between node N10 and its duplicated copy N10' has been computed, see Figure 12. As this error goes to zero as time goes to infinity and the node N10' is the duplicated one influenced directly by the master node, the DSA thereby confirms synchronization of the whole network.

5. CONCLUSIONS AND OUTLOOKS

The generalized synchronization of the complex network having the directed acyclic graph structure and generalized Lorenz systems at its nodes have been presented. Theoretical results showing the generalized synchronization were rigorously formulated and proved. Numerical experiments confirm the viability of theoretical results. Moreover, during the numerical experiments duplicated system approach was used to double-check the generalized synchronization of the respective network. Ongoing and future related research is focused on complex networks with time delays and/or more general network topology, e. g. with more incoming edges to one node, or even networks with directed loops.

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Sergej Čelikovský, The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod Vodárenskou věží 4, 182 00 Praha 8. Czech Republic.
e-mail: celikovs@utia.cas.cz

Volodymyr Lynnyk, The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod Vodárenskou věží 4, 182 00 Praha 8. Czech Republic.
e-mail: volodymyr.lynnyk@utia.cas.cz

Anna Lynnyk, The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod Vodárenskou věží 4, 182 00 Praha 8. Czech Republic.
e-mail: lynnyk.a@utia.cas.cz

Branislav Rehák, The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod Vodárenskou věží 4, 182 00 Praha 8. Czech Republic.
e-mail: rehakb@utia.cas.cz