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FIXED-TIME SAFE TRACKING CONTROL OF UNCERTAIN HIGH-ORDER NONLINEAR PURE-FEEDBACK SYSTEMS VIA UNIFIED TRANSFORMATION FUNCTIONS

Chaoqun Guo, Jiangping Hu, Jiasheng Hao, Sergej Čelikovský, and Xiaoming Hu

In this paper, a fixed-time safe control problem is investigated for an uncertain high-order nonlinear pure-feedback system with state constraints. A new nonlinear transformation function is firstly proposed to handle both the constrained and unconstrained cases in a unified way. Further, a radial basis function neural network is constructed to approximate the unknown dynamics in the system and a fixed-time dynamic surface control (FDSC) technique is developed to facilitate the fixed-time control design for the uncertain high-order pure-feedback system. Combined with the proposed unified transformation function and the FDSC technique, an adaptive fixed-time control strategy is proposed to guarantee the fixed-time tracking. The novel original results of the paper allow to design the independent unified flexible fixed-time control strategy taking into account the actual possible constraints, either present or missing. Numerical examples are presented to demonstrate the proposed fixed-time tracking control strategy.

Keywords: fixed-time safe control, nonlinear pure-feedback systems, state constrains, dynamic surface control, unified transformation function

Classification: 93D15, 70K20

1. INTRODUCTION

Convergence rate has been an important performance index of control systems. Finite-time control can ensure that the system state reaches the desired equilibrium in finite time [2, 12, 14] and it has become widely employed in many practical scenarios, such as variable length pendulum swing [4], vehicle tracking [6, 24] and finite-time consensus in dynamic networks [10]. However, the so-called **settling time**, needed to reach the equilibrium, is generally dependent on initial states and no finite bound of settling times is guaranteed for noncompact sets of initial conditions. To overcome this drawback, the concept of *homogeneity in bi-limit* was introduced in [1] to provide conditions for the so-called fixed-time stability, i. e. the existence of a finite bound of the settling time.

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Unfortunately, homogeneous approach does not allow to adjust or even estimate the settling time. To overcome this problem, [25] introduced a special modification of the so-called nested (terminal) second order sliding mode control algorithm that provided fixed-time stability of the origin and allowed to adjust the global settling time of the closed-loop system. An indirect method based on a comparison principle was developed in [13] to compute the upper bound of the settling time of uncertain integrator systems.

In this paper, the so-called **pure-feedback systems** will be considered. Pure-feedback systems were introduced and thoroughly studied in [17] along with their more specific version – the so-called **strict feedback system**. Alternatively, the terminology "triangular form system" was used in the literature, see *e.g* [5] and references within there. Note that, the references [5, 17] provided "classical" asymptotical stabilization only, though [5] used nonsmooth homogeneous approximation, yet with a positive degree, unlike the negative degree homogeneity used for finite-time stability.

Up to now, only a few studies were presented for the fixed-time control of pure-feedback systems having nonaffine connections between cascades, unlike less general strict-feedback systems where the cascade connection is affine and therefore easier to handle. In particular, [27, 28] provided the design of the fixed-time controllers for high-order integrator systems and strict-feedback systems only. At the same time, as correctly noted already in [17], many practical systems are commonly modelled as pure-feedback systems.

Besides the fast convergence rate, safety is also a crucial requirement for control systems. In recent years, safe controls have attracted much attention with the development of practical safety-critical systems, such as robotic systems, chemical plants, and autonomous vehicles [11, 23]. The output or state variables in safety-critical systems are usually constrained to ensure safeties [15]. In order to address safe control problems, barrier Lyapunov function (BLF) methods were commonly applied in the control design. For example, log-type BLF [18], tan-type BLF [8] and log-type integral BLF (IBLF) [20] methods were proposed to deal with static output/full-state constraints. Moreover, reference [7] employed BLF method to tackle a dynamic full-state constraint problem. However, BLF-based controls often depend on some feasibility conditions [26], which need extra complex offline calculations and even have no solution due to small thresholds of output/state constraints. In order to overcome such drawbacks, a nonlinear transformation function (NTF) technique was developed in [30], which can transform the original system with state constraints into an unconstrained system. Then, the boundedness of the transformed system can ensure that the constraints of the original system were satisfied. Moreover, NTF methods do not need additional feasibility conditions. Therefore, NTF methods have been widely concerned so far. In [31], a new NTF structure was proposed to solve a tracking control problem for state-constrained strict-feedback systems.

It is worthy of noting that in some scenarios, control systems have constrained and unconstrained states, simultaneously. Unfortunately, most of the existing safe control strategies are just proposed for control systems with state constraints. Reference [16] introduced a barrier function to tackle this situation, which relied on some complex feasibility conditions. Recently, reference [3] proposed a nonlinear transformation function to deal with the constrained and unconstrained cases in a unified way. Additionally, another unified nonlinear transformation function was proposed in [21] for stochastic systems. However, the safety-critical methods presented in the references mentioned above can only achieve asymptotic convergence or finite-time convergence. At the same time, the existing fixed-time controls established for state constrained systems fail in the cases with both constrained and unconstrained states. Although reference [22] designed a unified fixed-time controller for robotic systems with state constraints, it was only applicable to second-order systems.

As far as we know, fixed-time control of pure-feedback nonlinear systems with both constrained and unconstrained states has not been studied and suffers from the following two difficulties: construction of nonlinear transformation functions and design of fixed-time controllers. Motivated by the above discussions, this paper attempts to study the fixed-time control of uncertain high-order pure-feedback nonlinear systems with or without state constraints. To overcome the first difficulty, we construct a unified nonlinear transformation function to handle both constrained and unconstrained cases. For the second difficulty, we develop a new fixed-time dynamic surface control (FDSC) technique to facilitate fixed-time control design and reduce computational complexity.

In view of the previous exposition, the contributions of this paper are as follows:

1. A unified nonlinear transformation function is proposed to transform the original constrained system into an unconstrained one. The proposed nonlinear transformation can ensure that an unconstrained system is a special case of the constrained system. At the same time, the safe fixed-time control problem of the constrained system can be transformed to a fixed-time control problem of an unconstrained system.

2. A new FDSC technique is developed to facilitate fixed-time control design for the high-order pure-feedback nonlinear system. In contrast to dynamic surface control (DSC) technique proposed in [30], the new FDSC technique developed in this paper not only reduces computational cost, but also achieves fixed-time convergence. Compared to the DSC technique proposed in [19], the new FDSC technique can simplify the fixed-time convergence analysis.

3. Based on the proposed unified nonlinear transformation function and FDSC technique, an adaptive neural network based fixed-time control strategy is proposed for the high-order pure-feedback nonlinear system with unknown dynamics. Thus, the proposed control strategy, in addition to handling the constrained and unconstrained cases in a unified way, even ensures practical fixed-time tracking for uncertain systems.

The rest of the paper is organized as follows. Section 2 presents some preliminaries and problem formulation. In Section 3, the construction of a unified nonlinear transformation function and the design of an adaptive fixed-time control strategy are given. Simultaneously, practical fixed-time convergence is analyzed for the closed-loop system under the proposed control strategy. Simulation examples are given to validate the proposed fixed-time safe control strategy in Section 4. Conclusions are presented in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1. Preliminaries

Throughout the paper, the following dynamical system having the equilibrium at the origin will be considered

$$\dot{x} = f(t,x), f(t,0) \equiv 0, x \in \mathcal{D} \subset \mathbb{R}^n, \tag{1}$$

where the right hand side $f(t,x) : \mathbb{R}_+ \times \mathcal{D} \mapsto \mathbb{R}^n$ satisfies the assumptions for the existence of the solution in the Fillipov's sense [9]. More specifically, $f(t,x)$ is piecewise continuous with respect to $t \geq 0$ for any fixed $x \in \mathcal{D}$ and for any fixed $t > 0$ it is continuous with respect to $x \in \mathcal{D}$ except some smooth submanifolds of $\mathcal{D}$ where it is discontinous and has a finite collection of limit points when x approaches that discontinuity manifold. Here, $\mathcal{D}$ is a domain (open simply connected subset) in $\mathbb{R}^n$ containing its origin. Further, denote by $x(t,x_0)$ the solution of (1) such that $x(t_0,x_0) = x_0$, where $t_0 \geq 0$ is the initial time. Unless stated otherwise, in the sequel $t_0 = 0$ and $\mathcal{D} = \mathbb{R}^n$.

Definition 2.1. (Polyakov [25]) The origin of system (1) is said to be globally finite-time stable on $\mathbb{R}^n$, if it is globally asymptotically stable and $\forall x_0 \in \mathbb{R}^n$ there exists a positive constant $T(x_0)$ such that $x(t,x_0) = 0, \forall t \geq T(x_0)$. The function $T(x_0) : \mathbb{R}^n \mapsto \mathbb{R}_+$ is further referred to as the **settling function**.

Definition 2.2. (Polyakov [25]) The origin of system (1) is said to be fixed-time stable if it is globally finite-time stable and the settling time function $T(x_0)$ is globally bounded on R^n, *i. e.*, $\exists T_{\max} > 0$: $T(x_0) \leq T_{\max}$, $\forall x_0 \in \mathbb{R}^n$.

Definition 2.3. The origin of system (1) is said to be practically fixed-time stable on $\mathbb{R}^n$, if it is stable and $\forall \epsilon > 0$ there exists a positive constant $T_{\max}(\epsilon)$ such that $\forall x_0 \in \mathbb{R}^n$ there exists $T(x_0)$, $T_{\max}(\epsilon) > T(x_0) \geq 0$ and $\forall t > T(x_0)$ it holds that $\|x(t,x_0)\| \leq \epsilon$. The function $T(x_0)$ is further referred as the **practical settling time**.

Lemma 2.4. (Liu et al. [22]) The system (1) is practically fixed-time stable if $\forall \delta > 0$ there exist a positive definite function $V_\delta(t,x)$ and parameters $k_1 > 0$, $k_2 > 0$, $0 < \gamma < 1$, $\beta > 1$, and $0 < \theta < 1$ such that

$$\dot{V}_\delta(t,x) \leq -k_1 V_\delta(t,x)^\gamma - k_2 V_\delta(t,x)^\beta + \delta.$$

Furthermore, there exists a settling time T such that

$$V_\delta(t,x) \leq \min\Big\{\big(\frac{\delta}{k_1\theta}\big)^{\frac{1}{\gamma}}, \big(\frac{\delta}{k_2\theta}\big)^{\frac{1}{\beta}}\Big\},$$

when $t \geq T$, and the upper bound of the settling time T is given by:

$$T \leq \frac{1}{k_1(1-\theta)(1-\gamma)} + \frac{1}{k_2(1-\theta)(\beta-1)}.$$

The following Lemmas are straightforward and were used e. g. in [29].

Lemma 2.5. (Yang and Ye [29]) For arbitrary constants $x_1 > 0$, $x_1 \geq x_2$, and $p > 1$, it holds:

$$(x_1 - x_2)^p \geq x_2^p - x_1^p.$$

Lemma 2.6. (Yang and Ye [29]) For arbitrary constants $p > 0$, $x_1 \geq 0$, and $x_2 > 0$, it holds:

$$x_1^p(x_2 - x_1) \leq \frac{1}{1+p}(x_2^{1+p} - x_1^{1+p}).$$

Lemma 2.7. (Yang and Ye [29]) For arbitrary constants $x_i \in \mathbb{R}$ and $p > 0$, it holds:

$$(\sum_{i=1}^{n} |x_i|)^p \leq \max(n^{p-1}, 1) \sum_{i=1}^{n} |x_i|^p.$$

Radial basis function neural networks are widely employed to approximate the unknown continuous nonlinear functions in the fields of adaptive control and machine learning. A linearly parameterized model can be used to approximate an unknown continuous function $F(x) \in \mathbb{R}$ as follows:

$$F(x) = W^\top S(x) + \varepsilon(x),\ x \in \mathbb{R}^n, \tag{2}$$

where $W \in \mathbb{R}^N$ is the weight vector of a radial basis function neural network and $S(x) = [S_1(x), \ldots, S_N(x)]^T \in \mathbb{R}^N$ is the basis function vector. More specifically,

$$S_i(x) = \exp\big[-\frac{(x-\tau_i)^T(x-\tau_i)}{\psi_i^2}\big], \quad i = 1, \ldots, N, \tag{3}$$

where $\psi_i \in \mathbb{R}$, $\tau_i \in \mathbb{R}^n$ are the so-called width and the so-called center of the basis function, respectively. Finally, $\varepsilon \in \mathbb{R}$ is the estimation error.

Assumption 2.8. In the linearly parameterized model (2), $\|W\| \leq \bar{W}$, $|\varepsilon| \leq \varepsilon_1$, where $\bar{W}$ and ε_1 are unknown positive constants. In the sequel, denote $w = \max\{\bar{W}, \varepsilon_1\}$.

2.2. Problem formulation

Consider the following pure-feedback system [5]:

$$\begin{cases} \dot{x}_i = f_i(\bar{x}_i, x_{i+1}), i = 1, \ldots, n-1 \\ \dot{x}_n = f_n(\bar{x}_n, u), \\ y = x_1, \end{cases} \tag{4}$$

where $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is the state; $\bar{x}_i$ denotes $[x_1, \ldots, x_i]^T \in \mathbb{R}^i$; $y \in \mathbb{R}$ is the scalar output; $u \in \mathbb{R}$ is the the scalar control input and $f_i(\cdot)$ $(i = 1, \ldots, n)$ are unknown continuous nonlinear functions. The system is required to satisfy the following state constraints:

$$-h_{i1}(t) < x_i(t) < h_{i2}(t),\ i = 1, \ldots, n, \tag{5}$$

where the time-varying bounds $h_{i1}(t)$ and $h_{i2}(t)$ are strictly positive functions.

Assumption 2.9. The functions $h_{i1}(t), h_{i2}(t)$ and their derivatives are uniformly bounded on $\mathbb{R}_+$.

Assumption 2.10. The initial states satisfy $-h_{i1}(0) < x_i(0) < h_{i2}(0)$ for $i = 1, \ldots, n$.

Assumption 2.11. $f_i(\bar{x}_i, x_{i+1})$ $(i = 1, \ldots, n-1)$ and $f_n(\bar{x}_n, u)$ are continuously differentiable for all $x \in \mathbb{R}^n$.

Assumption 2.12. Let $g_n(\bar{x}_n, u) := \frac{\partial f_n(\bar{x}, u)}{\partial u}$. It holds that $\underline{g}_n \le g_n(\cdot) \le \overline{g}_n$, where $\underline{g}_n$, $\overline{g}_n$ are unknown positive constants.

Further, assume that the reference output y_d is given to be followed by the output of the nonlinear system (4).

Assumption 2.13. The reference output $y_d(t)$ satisfies the constraint (5), i. e., $-h_{11}(t) < y_d(t) < h_{12}(t)$. Moreover, the reference output $y_d(t)$, its first and its second derivatives are uniformly bounded on $\mathbb{R}_+$.

The aim of this paper is to design a practical fixed-time controller for the system (4) providing the given reference output tracking, that is,

$$|y(t) - y_d(t)| \le \zeta, \quad \forall t > T,$$

where T, ζ are some positive constants and at the same time the designed controller guarantees that the state constraints (5) are not violated at any time.

3. MAIN RESULTS

This section is divided into two parts. Subsection 3.1 introduces a unified nonlinear transformation function to transform the original constrained system (4) to a new unconstrained system. Subsection 3.2 gives the practical fixed-time controller to achieve the previously formulated tracking goal.

3.1. Unified nonlinear transformation function

Consider the system (4) and the constraint (5). A unified nonlinear transformation function is proposed as follows:

$$\xi_i(t) = \frac{h_{i1}(t) + h_{i2}(t)}{4} \ln \frac{h_{i1}(t) + x_i(t)}{h_{i2}(t) - x_i(t)}. \tag{6}$$

The nonlinear transformation (6) has the following property. For any $t \ge 0$ it holds:

$$\begin{cases} \text{(i)} \lim\limits_{x_i(t) \to -h_{i1}(t)} \xi_i(t) = -\infty; \\ \text{(ii)} \lim\limits_{x_i(t) \to h_{i2}(t)} \xi_i(t) = +\infty; \\ \text{(iii)} \lim\limits_{h_{i1}(t) = h_{i2}(t) \to +\infty} \xi_i(t) = x_i(t). \end{cases} \tag{7}$$

The properties (i) and (ii) can be easily obtained. When $h_{i1}(t) = h_{i2}(t) \to +\infty$, by straightforward computations, we can also verify that the property (iii) holds.

According to the properties (1) and (2), one can obtain that if $\xi_i(t)$ is bounded, then the condition $-h_{i1}(t) < x_i(t) < h_{i2}(t)$ holds for any $-h_{i1}(0) < x_i(0) < h_{i2}(0)$. Therefore, in order to ensure that the constraints are not violated, just ensure that $\xi_i(t)$ is bounded. For the property (3), $h_{i1}(t) = h_{i2}(t) = +\infty$ means that there is no state constraint and thus $\xi_i(t) = x_i(t)$. Consequently, the proposed transformation (6) can deal with the cases with and without state constraints in a unified manner.

Remark 3.1. The proposed transformation function (6) is different from the existing transformation functions given in [30, 31], which can only deal with the case with state constraints. In this paper, the proposed transformation function (6) can be applied to safe control problems in the following situations: 1) All states are constrained; 2) All states are not constrained; 3) Partial state variables are constrained while the remaning state variables are not constrained. From this perspective, the proposed NTF (6) can deal with the case with constrained and unconstrained states in a unified way.

Using (4), (6), and differentiating $\xi = [\xi_1, \dots, \xi_n]^T$ yields:

$$\begin{cases} \dot{\xi}_i = \varphi_i[F_i(\bar{x}_{i+1}, \xi_{i+1}) + \xi_{i+1}] + \psi_i, \ i = 1, \dots, n-1, \\ \dot{\xi}_n = \varphi_n[F_n(\bar{x}_n) + g_n u] + \psi_n, \end{cases} \tag{8}$$

where

$$\varphi_i = \frac{\partial \xi_i}{\partial x_i} = \frac{\big(h_{i1}(t) + h_{i2}(t)\big)^2}{4\big(h_{i1}(t) + x_i(t)\big)\big(h_{i2}(t) - x_i(t)\big)},$$

$$\psi_i = \frac{h_{i1}(t) + h_{i2}(t)}{4}\Big(\frac{\dot{h}_{i1}(t)}{h_{i1}(t) + x_i(t)} - \frac{\dot{h}_{i2}(t)}{h_{i2}(t) - x_i(t)}\Big) + \frac{\dot{h}_{i1}(t) + \dot{h}_{i2}(t)}{4} \ln \frac{h_{i1}(t) + x_i(t)}{h_{i2}(t) - x_i(t)},$$

$$F_i(\bar{x}_{i+1}, \xi_{i+1}) = f_i(\bar{x}_i, x_{i+1}) - \xi_{i+1}, \ i = 1, \dots, n-1,$$

$$F_n(\bar{x}_n) = f_n(\bar{x}_n),$$

and $f_n(\bar{x}_n) = f_n(\bar{x}_n, 0)$.

Obviously, the original system (4) with the state constraint (5) is transformed to an unconstrained system (8). The constraints on the state x can be guaranteed by ensuring the boundedness of the variable ξ. In sequel, based on the transformed system (8), we need to design a fixed-time controller to not only ensure that the state constraints are not violated, but also realize the practical fixed-time output tracking.

3.2. Control design and convergence analysis

According to the unified nonlinear transformation function (6) and Assumption 2.13, a nonlinear transformation function is given for the reference output y_d as follows:

$$\xi_d(t) = \frac{h_{i1}(t) + h_{i2}(t)}{4}[\ln(h_{i1}(t) + y_d) - \ln(h_{i2}(t) - y_d)]. \tag{9}$$

Define an error system as follows:

$$\begin{cases} \zeta_1 = \xi_1 - \xi_d, \\ \zeta_i = \xi_i - \alpha_{if}, \quad i = 2, \ldots, n, \end{cases} \tag{10}$$

where α_{if} is a dynamic variable designed by the following **fixed-time dynamic surface control (FDSC) technique**:

$$\lambda_i \dot{\alpha}_{if} = (\alpha_{i-1} - \alpha_{if})^{r_1} + (\alpha_{i-1} - \alpha_{if})^{r_2} + \alpha_{i-1} - \alpha_{if}, \quad i = 2, \ldots, n, \tag{11}$$

where $r_1 = \frac{m}{n}$, $r_2 = \frac{p}{q}$, and $m < n$, $p > q$ are positive odd numbers, λ_i is a positive constant, and α_{i-1}, $i = 2, \ldots, n$, are the virtual controllers to be designed latter.

Remark 3.2. Obviously, the equilibrium point of (11) is α_{i-1}, thus the state α_{if} will converge to α_{i-1}. Then we can regard α_{if} in (11) as an estimation of α_{i-1}, and its derivative $\dot{\alpha}_{if}$ as an estimation of $\dot{\alpha}_{i-1}$. Here, the FDSC technique is used to generate the derivative $\dot{\alpha}_{if}$and replace $\dot{\alpha}_{i-1}$ appearing in the backstepping design process. Thus there is no need to calculate the derivative of the virtual controller α_{i-1}. In contrast to the dynamic surface control (DSC) method proposed in [29], the fixed-time dynamic surface control (FDSC) (11) not only reduces computational cost in the control design, but also ensures the fixed-time convergence. Moreover, compared with the DSC technique proposed in [30], the FDSC (11) had an additional term $\alpha_{i-1} - \alpha_{if}$, which can bring a faster estimation speed and improve the fixed-time convergence rate of the closed-loop system.

According to the error system (10) and backstepping-like design method, we design the controller u as follows:

$$\begin{aligned} u = &- \rho_n \varphi_n \hat{a}_n \mu_n^2 \zeta_n - \frac{\psi_n^2 \zeta_n}{\varphi_n} - \frac{\dot{\alpha}_{nf}^2 \zeta_n}{\varphi_n} \\ &- \frac{\varphi_{n-1}^2 \zeta_{n-1}^2 \zeta_n}{\varphi_n} - \frac{k_{n1}}{\varphi_n} \zeta_n^{r_1} - \frac{k_{n2}}{\varphi_n} \zeta_n^{r_2}, \end{aligned} \tag{12}$$

$$\dot{\hat{a}}_n = \rho_n \varphi_n^2 \mu_n^2 \zeta_n^2 - \sigma_{n1} \hat{a}_n^{r_1} - \sigma_{n2} \hat{a}_n^{r_2}, \tag{13}$$

where ρ_n, σ_{n1}, and σ_{n2} are positive constants, μ_n is a positive variable defined in controller design process. The design process of the controller u is shown in Appendix A.

Under the controller (12), the main control results are shown in Theorem 3.3

Theorem 3.3. Consider the uncertain high-order pure-feedback system (4) with the state constraints given by (5). Under the Assumptions 2.8 – 2.11, the proposed controller (12) can achieve the following objectives:

(1) The closed-loop system is practically fixed-time stable.

(2) The output tracking error $e = y - y_d$ is bounded in fixed time.

(3) The state constraints are satisfied all the time.

P r o o f. See the Appendix B. □

4. SIMULATION EXAMPLE

In this section, two examples are presented to demonstrate the proposed fixed-time safe control strategy. Example 4.1 verifies the effectiveness of the fixed-time control strategy for the system with full state constraints. Example 4.2 shows that the proposed control strategy can still work for the system with partial state constraints.

Example 4.1. Consider the following pure-feedback nonlinear system:

$$\begin{cases} \dot{x}_1 = x_1 + x_2 + x_2^2, \\ \dot{x}_2 = x_1 x_2 + u + 0.1\sin(u), \\ y = x_1. \end{cases} \tag{14}$$

The state constraint functions are given by $h_{11}(t) = 0.5 + 0.3\sin(t)$, $h_{12}(t) = 0.6 - 0.2\sin(t)$, $h_{21}(t) = 0.5 + 0.2\cos(t)$, and $h_{22}(t) = 0.5 + 0.2\cos(t)$. The reference output is given by $y_d = 0.1\sin(0.5t)$.

The initial states are given as $x(0) = [0.2, -0.2]^T$. The parameters in the controller (43) are selected as follows: $k_{11} = k_{12} = k_{21} = k_{22} = 2$, $\sigma_{11} = 0.2$, $\sigma_{12} = 0.5$, $\sigma_{21} = 0.3$, $\sigma_{22} = 0.4$, $\rho_1 = \rho_2 = 0.0005$, $\lambda_2 = 0.1$, $r_1 = 97/99$, and $r_2 = 99/97$. The parameters of the neural network approximation of $F_1(X_1)$ are choosen as follows: $\psi_i = 3$, $\tau_i = [-\tau_{ii}, 0, \tau_{ii}]^T$ $(i = 1, \ldots, 5)$. The parameters of the neural network approximation of $F_2(X_2)$ are choosen as follows: $\psi_i = 3$, $\tau_i = [-\tau_{ii}, \tau_{ii}]^T$ $(i = 1, \ldots, 5)$. Here, τ_{ii} are selected as $1, 2, 3, 4$ and 5, respectively.

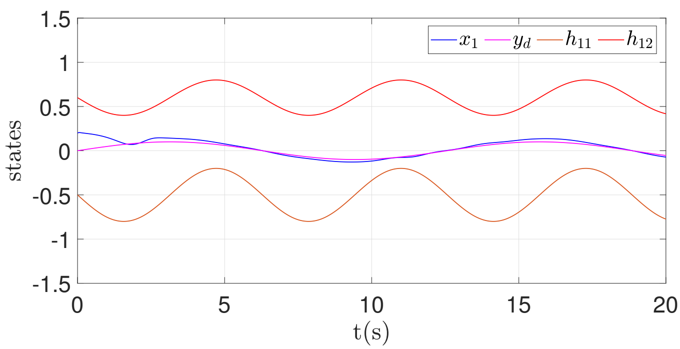

Fig. 1. The trajectories of the state x_1 and the reference output y_d.

Figure 1 illustrates the trajectories of the state x_1 and the desired output y_d. In Figure 1, we can see that state x_1 can track the reference output y_d at about $T = 5s$ under the proposed safe controller (43). Moreover, the state x_1 satisfies the given constraint all the time. Figure 2 illustrates the trajectory of the state x_2. From Figure 2, we can see that x_2 is always within the constraint. Figure 3 shows the trajectory of the tracking error and demonstrates that the output tracking error is bounded in fixed time, that is, $|e| \leq 0.05$ for $t \geq 5s$. Additionally, the evolution of the controller u is shown in Figure 4, which shows that the controller changes smoothly.

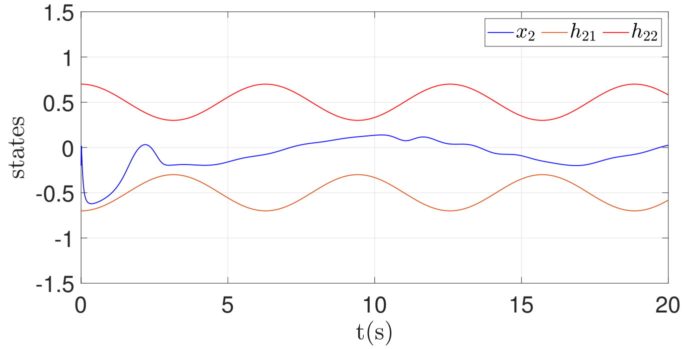

Fig. 2. The trajectory of the state x_2.

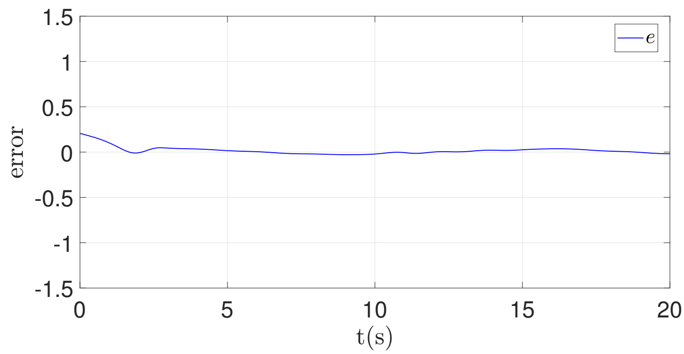

Fig. 3. The trajectory of the output tracking error e.

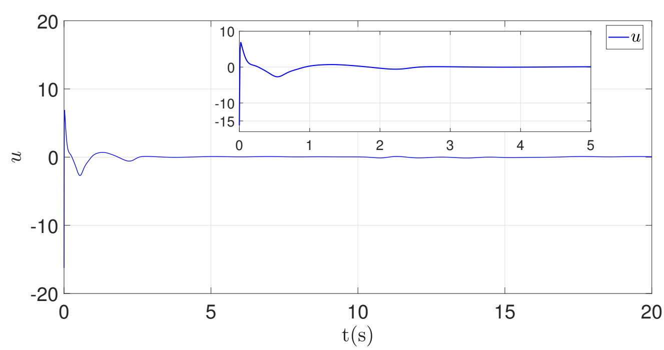

Fig. 4. The trajectory of controller u.

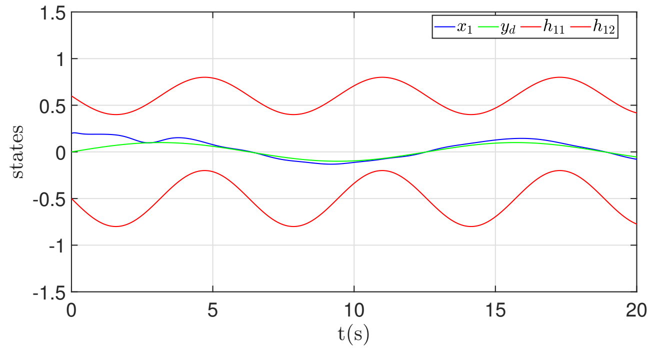

Fig. 5. The trajectories of the state x_1 and the reference output y_d.

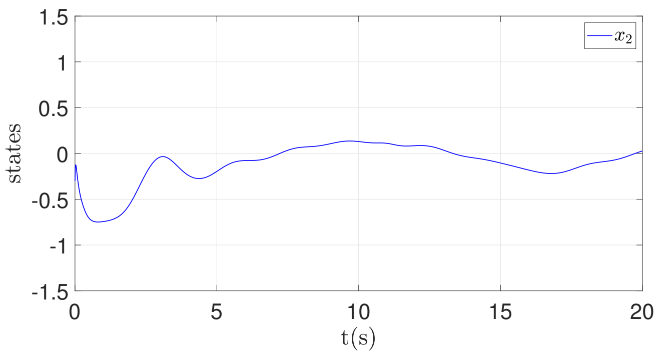

Fig. 6. The trajectory of the state x_2.

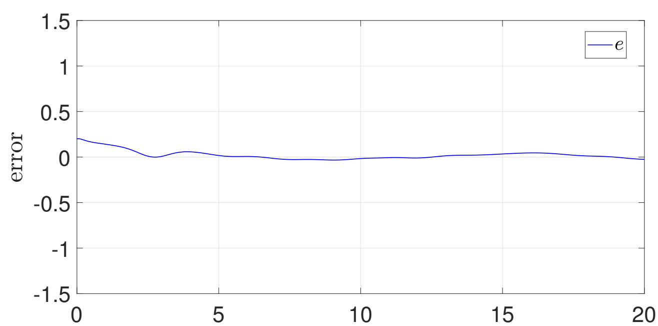

Fig. 7. The trajectory of the output racking error e.

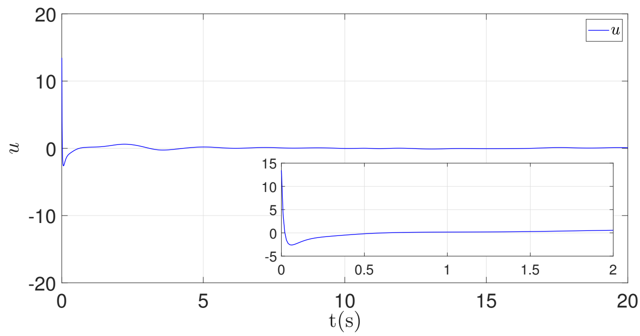

Fig. 8. The trajectory of the control input u.

Example 4.2. For the pure-feedback nonlinear system (14), we assume that the state x_1 is constrained by $h_{11}(t) = 0.5 + 0.3\sin(t)$ and $h_{12}(t) = 0.6 - 0.2\sin(t)$, while the state x_2 has no constraint, that is $h_{21}(t) = h_{22}(t) \equiv +\infty$. The initial state is given by $x(0) = [0.2, -0.3]^T$. Set $k_{11} = 2$, $k_{12} = 1$, $k_{21} = k_{22} = 2$. Other parameters are similar with those in Example 4.1.

The trajectories of the states x_1 and x_2 are shown in Figure 5 and Figure 6, respectively. From Figure 5 and Figure 6, it can be found that x_1 and x_2 are bounded while x_1 does not violate its constraint all the time. Moreover, the output tracking error $|e| \leq 0.05$ for $t \geq 4.3s$, which is illustrated in Figure 7. Figure 8 shows the smooth trajectory of the controller u.

The above simulation results show that the proposed fixed-time safe controller can deal with the output tracking problem for the uncertain pure-feedback nonlinear system with and without state constraints while keeping one control structure.

5. CONCLUSIONS

This paper has solved the fixed-time output tracking problem for uncertain high-order pure-feedback systems with and without state constraints in a unified way. A nonlinear transformation function method has been proposed to deal with the cases with and without state constraints. With the help of the unified nonlinear transformation, the fixed-time safe control problem has been transformed to just a fixed-time control problem. At the same time, a fixed-time dynamic surface control technique has been developed to facilitate the fixed-time controller design. Thus, an adaptive neural network based fixed-time control strategy has been proposed for the uncertain pure-feedback nonlinear system. Theoretical results have shown that the fixed-time convergence of the output tracking error can be achieved and all the state constraints can always be satisfied under the proposed control strategy. Moreover, the effectiveness of the proposed control strategy has been validated by numerical simulations.

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APPENDIX

A. DESIGN PROCESS OF THE CONTROLLER U

In this subsection, we present the design process of the controller u by **backstepping-like design method**. First, define

$$y_i = \alpha_{if} - \alpha_{i-1}, \quad i = 2, \dots, n, \tag{15}$$

where α_{if} is generated by the **FDSC technique** in (11), α_{i-1} is the virtual controller designed latter. Next, we design the virtual controller α_{i-1} at the $(i-1)$th step, and the controller u at the nth step.

Step 1: Differentiating ζ_1 yields:

$$\dot{\zeta}_1 = \varphi_1[F_1(x_1, x_2, \xi_2) + \xi_2] + \psi_1 - \dot{\xi}_d. \tag{16}$$

According to (10) and (15), it follows that $\xi_2 = \zeta_2 + y_2 + \alpha_1$. Then, we have

$$\dot{\zeta}_1 = \varphi_1[F_1(x_1, x_2, \xi_2) + \zeta_2 + y_2 + \alpha_1] + \psi_1 - \dot{\xi}_d. \tag{17}$$

Using the neural network based approximation (2), $F_1(x_1, x_2, \xi_2) = F_1(X_1)$ can be approximated as follows:

$$F_1(X_1) = W_1^T S(X_1) + \varepsilon_1(X_1), \tag{18}$$

where $X_1 = [x_1, x_2, \xi_2]^T$, $W_1 \in \mathbb{R}^N$ and $S(X_1) \in \mathbb{R}^N$ are the ideal weight vector and basis function vector respectively, and $\varepsilon_1(X_1)$ is the estimation error. According to Assumption 2.8, we have $||W_1|| \leq \bar{W}_1$, $|\varepsilon_1| \leq \varepsilon_{11}$, where $\bar{W}_1$ and ε_{11} are unknown positive constants.

Let $w_1 = \max\{\bar{W}_1, \varepsilon_{11}\}$, then one has

$$F_1(X_1) \leq w_1 \mu_1(X_1),$$

where $\mu_1(X_1) = ||S(X_1)|| + 1$.

Next, the virtual controller α_1 is designed as follows:

$$\begin{aligned}\alpha_1 = & - \rho_1 \hat{a}_1 \varphi_1 \mu_1^2 \zeta_1 - \varphi_1 \zeta_1 + \frac{\dot{\xi}_d}{\varphi_1} - \frac{\psi_1}{\varphi_1} \\ & - \frac{k_{11}}{\varphi_1} \zeta_1^{r_1} - \frac{k_{12}}{\varphi_1} \zeta_1^{r_2},\end{aligned} \tag{19}$$

where k_{11}, k_{12}, and ρ_1 are positive constants, r_1 and r_2 are defined in (11). Let $\hat{a}_1$ be the estimation of $a_1 = w_1^2$, which is determined by the following adaptive law:

$$\dot{\hat{a}}_1 = \rho_1 \varphi_1^2 \mu_1^2 \zeta_1^2 - \sigma_{11} \hat{a}_1^{r_1} - \sigma_{12} \hat{a}_1^{r_2}, \tag{20}$$

where σ_{11} and σ_{12} are positive constants.

According to Young's inequality, we have the following inequalities:

$$\varphi_1\zeta_1 F_1 \leq |\varphi_1\zeta_1 w_1\mu_1| \leq \rho_1 w_1^2\varphi_1^2\mu_1^2\zeta_1^2 + \frac{1}{4\rho_1}, \tag{21}$$

$$\varphi_1\zeta_1 y_2 \leq \varphi_1^2\zeta_1^2 + \frac{y_2^2}{4}, \tag{22}$$

where ρ_1 is defined in (19).

Then, we construct a Lyapunov function as follows:

$$V_1 = \frac{1}{2}\zeta_1^2 + \frac{1}{2}\tilde{a}_1^2 + \frac{1}{2}y_2^2, \tag{23}$$

where $\tilde{a}_1 = a_1 - \hat{a}_1$.

Calculating the derivative of V_1 and employing the virtual controller (19), and inequalities (21), (22), we have

$$\begin{aligned}\dot{V}_1 \leq& \rho_1\tilde{a}_1\varphi_1^2\mu_1^2\zeta_1^2 + \varphi_1\zeta_1\zeta_2 - k_{11}\zeta_1^{1+r_1} - k_{12}\zeta_1^{1+r_2} + \frac{y_2^2}{4}\\ &+ \frac{1}{4\rho_1} + \tilde{a}_1\dot{\tilde{a}}_1 + y_2\dot{y}_2\\ =&\rho_1\tilde{a}_1\varphi_1^2\mu_1^2\zeta_1^2 + \varphi_1\zeta_1\zeta_2 - k_{11}\zeta_1^{1+r_1} - k_{12}\zeta_1^{1+r_2} + \frac{y_2^2}{4}\\ &+ \frac{1}{4\rho_1} - \tilde{a}_1\dot{\hat{a}}_1 + y_2\dot{y}_2.\end{aligned} \tag{24}$$

Using the adaptive law (20), we have

$$-\tilde{a}_1\dot{\hat{a}}_1 = -\rho_1\tilde{a}_1\varphi_1^2\mu_1^2\zeta_1^2 + \sigma_{11}\tilde{a}_1\hat{a}_1^{r_1} + \sigma_{12}\tilde{a}_1\hat{a}_1^{r_2}.$$

Since $\tilde{a}_1 = a_1 - \hat{a}_1$, thus $\sigma_{11}\tilde{a}_1\hat{a}_1^{r_1}$ and $\sigma_{12}\tilde{a}_1\hat{a}_1^{r_2}$ can be written as $\sigma_{11}\hat{a}^{r_1}(a_1 - \hat{a}_1)$ and $\sigma_{12}\hat{a}_1^{r_2}(a_1 - \hat{a}_1)$, respectively. By analyzing the adaptive law (20), it can be verified that for any given initial value $\hat{a}_1(0) \geq 0$, one has $\hat{a}_1(t) \geq 0$ if $\dot{\hat{a}}_1(t) \geq 0$. It is noted that σ_{11} and σ_{12} are positive constants. If $\dot{\hat{a}}_1(t) < 0$, then $\hat{a}_1(t)$ will decrease until $\hat{a}_1(t) = 0$ at a certain time t_d. Due to the fact that $\rho_1\varphi_1^2\mu_1^2\zeta_1^2 \geq 0$, it can be found that $\dot{\hat{a}}_1(t) \geq 0$ when $\hat{a}_1(t) = 0$. Therefore, $\hat{a}_1(t) \geq 0$ after $t \geq t_d$. Thus, if we choose an initial value $\hat{a}_1(0) \geq 0$, then we have $\hat{a}_1(t) \geq 0$.

According to Lemma 2.6, we have

$$\begin{aligned}\sigma_{11}\hat{a}_1^{r_1}\tilde{a}_1 &= \sigma_{11}\hat{a}_1^{r_1}(a_1 - \hat{a}_1)\\ &\leq \sigma_{11}\frac{1}{1+r_1}(a_1^{1+r_1} - \hat{a}_1^{1+r_1})\\ &= \sigma_{11}\frac{1}{1+r_1}(a_1^{1+r_1} - [a_1 - \tilde{a}_1]^{1+r_1}).\end{aligned}$$

When $\hat{a}_1(t) > 0$, we have $a_1(t) > \tilde{a}_1(t)$. From Lemma 2.5, we have

$$\begin{aligned}\sigma_{11}\hat{a}_1^{r_1}\tilde{a}_1 &\leq \sigma_{11}\frac{1}{1+r_1}(a_1^{1+r_1} + a_1^{1+r_1} - \tilde{a}_1^{1+r_1})\\ &= \frac{2\sigma_{11}}{1+r_1}a_1^{1+r_1} - \frac{\sigma_{11}}{1+r_1}\tilde{a}_1^{2\times\frac{1+r_1}{2}}.\end{aligned} \tag{25}$$

Similarly, we also have

$$\sigma_{12}\hat{a}_1^{r_2}\tilde{a}_1 \leq \frac{2\sigma_{12}}{1+r_2}a_1^{1+r_2} - \frac{\sigma_{12}}{1+r_2}\tilde{a}_1^{2\times\frac{1+r_2}{2}}.$$

Thus we have

$$\begin{aligned} -\tilde{a}_1\dot{\hat{a}}_1 \leq &- \frac{\sigma_{11}}{1+r_1}\tilde{a}_1^{2\times\frac{1+r_1}{2}} - \frac{\sigma_{12}}{1+r_2}\tilde{a}_1^{2\times\frac{1+r_2}{2}} \\ &+ \frac{2\sigma_{11}}{1+r_1}a_1^{1+r_1} + \frac{2\sigma_{12}}{1+r_2}a_1^{1+r_2} \\ &- \rho_1\tilde{a}_1\varphi_1^2\mu_1^2\zeta_1^2. \end{aligned} \tag{26}$$

From equations (11) and (15), we have

$$\begin{aligned} &\frac{y_2^2}{4} + y_2\dot{y}_2 \\ &= \frac{y_2^2}{4} + y_2[-\frac{1}{\lambda_2}y_2^{r_1} - \frac{1}{\lambda_2}y_2^{r_2} - \frac{1}{\lambda_2}y_2 - \dot{\alpha}_1] \\ &= \frac{y_2^2}{4} - \frac{1}{\lambda_2}|y_2|^{1+r_1} - \frac{1}{\lambda_2}|y_2|^{1+r_2} - \frac{1}{\lambda_2}y_2^2 - y_2\nu_2 \\ &\leq \frac{y_2^2}{4} - \frac{1}{\lambda_2}|y_2|^{1+r_1} - \frac{1}{\lambda_2}|y_2|^{1+r_2} - \frac{1}{\lambda_2}y_2^2 + \frac{y_2^2}{4} + \nu_2^2 \\ &= -\frac{1}{\lambda_2}|y_2|^{1+r_1} - \frac{1}{\lambda_2}|y_2|^{1+r_2} - (\frac{1}{\lambda_2} - \frac{1}{2})y_2^2 + \nu_2^2 \\ &\leq -\frac{1}{\lambda_2}(y_2^2)^{\frac{1+r_1}{2}} - \frac{1}{\lambda_2}(y_2^2)^{\frac{1+r_2}{2}} + \nu_2^2 \end{aligned} \tag{27}$$

for $0 < \lambda_2 < 2$, where $\nu_2 = \dot{\alpha}_1 = \frac{\partial\alpha_1}{\partial\zeta_1}\dot{\zeta}_1 + \frac{\partial\alpha_1}{\partial\varphi_1}\dot{\varphi}_1 + \frac{\partial\alpha_1}{\partial\psi_1}\dot{\psi}_1 + \frac{\partial\alpha_1}{\partial\hat{a}_1}\dot{\hat{a}}_1 + \frac{\partial\alpha_1}{\partial\mu_1}\dot{\mu}_1 + \frac{\partial\alpha_1}{\partial\dot{\xi}_d}\ddot{\xi}_d$.

Then, applying the inequalities (26) and (27) to (24) yields:

$$\begin{aligned} \dot{V}_1 \leq &\varphi_1\zeta_1\zeta_2 - k_{11}(\zeta_1^2)^{\frac{1+r_1}{2}} - k_{12}(\zeta_1^2)^{\frac{1+r_2}{2}} \\ &- \frac{\sigma_{11}}{1+r_1}(\tilde{a}_1^2)^{\frac{1+r_1}{2}} - \frac{\sigma_{12}}{1+r_2}(\tilde{a}_1^2)^{\frac{1+r_2}{2}} \\ &- \frac{1}{\lambda_2}(y_2^2)^{\frac{1+r_1}{2}} - \frac{1}{\lambda_2}(y_2^2)^{\frac{1+r_2}{2}} + \Lambda_1, \end{aligned} \tag{28}$$

where $\Lambda_1 = \frac{2\sigma_{11}}{1+r_1}a_1^{1+r_1} + \frac{2\sigma_{12}}{1+r_2}a_1^{1+r_2} + \nu_2^2 + \frac{1}{4\rho_1}$.

Step i (i=2,...,n-1): Differentiating ζ_i results in:

$$\dot{\zeta}_i = \varphi_i[F_i(\bar{x}_{i+1}, \xi_{i+1}) + \xi_{i+1}] + \psi_i - \dot{\alpha}_{if}. \tag{29}$$

Since $\xi_i = \zeta_i + y_i + \alpha_{i-1}$, thus $\dot{\zeta}_i$ can be rewritten as follows:

$$\dot{\zeta}_i = \varphi_i[F_i(\bar{x}_{i+1}, \xi_{i+1}) + \zeta_{i+1} + y_{i+1} + \alpha_i] + \psi_i - \dot{\alpha}_{if}. \tag{30}$$

According to the neural network approximation (2), we approximate $F_i(\bar{x}_{i+1}, \xi_{i+1})$ as follows:

$$F_i(X_i) = W_i^T S(X_i) + \varepsilon_i(X_i), \tag{31}$$

where $X_i = [\bar{x}_{i+1}, \xi_{i+1}]^T$, $W_i \in \mathbb{R}^N$, $S(X_i) \in \mathbb{R}^N$, $\varepsilon_i(X_i) \in \mathbb{R}$. From Assumption 2.8, it follows that $||W_i|| \leq \bar{W}_i$, $|\varepsilon_i| \leq \varepsilon_{i1}$, where $\bar{W}_i$ and ε_{i1} are unknown positive constants. Define $w_i = \max\{\bar{W}_i, \varepsilon_{i1}\}$, then the following inequality holds:

$$F_i(X_i) \leq w_i\mu_i(X_i), \tag{32}$$

where $\mu_i(X_i) = ||S(X_i)|| + 1$.

Design the ith virtual controller α_i as follows:

$$\begin{aligned}\alpha_i =& -\rho_i\hat{a}_i\varphi_i\mu_i^2\zeta_i - \varphi_i\zeta_i - \frac{\psi_i}{\varphi_i} + \frac{\dot{\alpha}_{if}}{\varphi_i} - k_{i1}\zeta_i^{r_1} \\ & - k_{i2}\zeta_i^{r_2} - \frac{\varphi_{i-1}}{\varphi_i}\zeta_{i-1},\end{aligned} \tag{33}$$

where k_{i1}, k_{i2}, and ρ_i are positive constants, r_1 and r_2 are defined in (11). $\hat{a}_i$ is employed to estimate $a_i = w_i^2$ and determined by the following adaptive law:

$$\dot{\hat{a}}_i = \rho_i\varphi_i^2\mu_i^2\zeta_i^2 - \sigma_{i1}\hat{a}_i^{r_1} - \sigma_{i2}\hat{a}_i^{r_2}, \tag{34}$$

where σ_{i1} and σ_{i2} are positive constants.

Using the Young's inequation, the following inequalities can be obtained:

$$\begin{aligned}&\varphi_i\zeta_iF_i \le |\varphi_i\zeta_iw_i\mu_i| \le \rho_iw_i^2\varphi_i^2\mu_i^2\zeta_i^2 + \frac{1}{4\rho_i}, \\ &\varphi_i\zeta_iy_{i+1} \le \varphi_i^2\zeta_i^2 + \frac{y_{i+1}^2}{4},\end{aligned} \tag{35}$$

where ρ_i is defined in (33).

Next, we construct the Lyapunov function V_i as follows:

$$V_i = \frac{1}{2}\zeta_i^2 + \frac{1}{2}\tilde{a}_i^2 + \frac{1}{2}y_{i+1}^2, \tag{36}$$

where $\tilde{a}_i = a_i - \hat{a}_i$.

Using the virtual controller α_i (33), the adaptive law (34), and inequalities (35), and calculating the derivative of V_i, we have

$$\begin{aligned}\dot{V}_i \le& \rho_i\tilde{a}_i\varphi_i^2\mu_i^2\zeta_i^2 + \varphi_i\zeta_i\zeta_{i+1} - \varphi_{i-1}\zeta_{i-1}\zeta_i - k_{i1}\zeta_i^{1+r_1} \\ & - k_{i2}\zeta_i^{1+r_2} + \frac{y_{i+1}^2}{4} + \frac{1}{4\rho_i} + \tilde{a}_i\dot{\tilde{a}}_i + y_{i+1}\dot{y}_{i+1} \\ \le& \varphi_i\zeta_i\zeta_{i+1} - \varphi_{i-1}\zeta_{i-1}\zeta_i - k_{i1}\zeta_i^{1+r_1} - k_{i2}\zeta_i^{1+r_2} \\ & + \frac{y_{i+1}^2}{4} + \frac{1}{4\rho_i} + \sigma_{i1}\tilde{a}_i\hat{a}_i^{r_1} + \sigma_{i2}\tilde{a}_i\hat{a}_1^{r_2} + y_{i+1}\dot{y}_{i+1}.\end{aligned} \tag{37}$$

For the adaptive law (34), it follows that $\hat{a}_i(t) > 0$, $\forall t > 0$ for a positive initial value $\hat{a}_i(0)$. Thus $a_i > \tilde{a}_i$ when $\hat{a}_i(0) > 0$. According to Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}\sigma_{i1}\hat{a}_i^{r_1}\tilde{a}_i &= \sigma_{i1}\hat{a}_i^{r_1}(a_i - \hat{a}_i) \\ &\le \sigma_{i1}\frac{1}{1+r_1}(a_i^{1+r_1} - \hat{a}_i^{1+r_1}) \\ &= \sigma_{i1}\frac{1}{1+r_1}(a_i^{1+r_1} - [a_i - \tilde{a}_i]^{1+r_1}) \\ &\le \sigma_{i1}\frac{1}{1+r_1}(a_i^{1+r_1} + a_i^{1+r_1} - \tilde{a}_i^{1+r_1}) \\ &= \frac{2\sigma_{i1}}{1+r_1}a_i^{1+r_1} - \frac{\sigma_{i1}}{1+r_1}\tilde{a}_i^{2\times\frac{1+r_1}{2}}.\end{aligned} \tag{38}$$

Similarly, we also have

$$\sigma_{i2}\hat{a}_i^{r_2}\tilde{a}_i \le \frac{2\sigma_{i2}}{1+r_2}a_i^{1+r_2} - \frac{\sigma_{i2}}{1+r_2}\tilde{a}_i^{2\times\frac{1+r_2}{2}}. \tag{39}$$

Furthermore, from the definitions of y_i in (15) and the dynamic variable α_{if} in (11), we have

$$\dot{y}_i = -\frac{1}{\lambda_i} y_i^{r_1} - \frac{1}{\lambda_i} y_i^{r_2} - \frac{1}{\lambda_i} y_i - \nu_i,$$

where $\nu_i = \dot{\alpha}_{i-1}$. Then we have

$$\begin{aligned}
&\frac{y_{i+1}^2}{4} + y_{i+1}\dot{y}_{i+1} \\
&= -\frac{1}{\lambda_{i+1}}\left(|y_{i+1}|^{1+r_1} + |y_{i+1}|^{1+r_2} + y_{i+1}^2\right) \\
&\quad + \frac{y_{i+1}^2}{4} - y_{i+1}\nu_{i+1} \\
&\le -\frac{1}{\lambda_{i+1}}|y_{i+1}|^{1+r_1} - \frac{1}{\lambda_{i+1}}|y_{i+1}|^{1+r_2} \\
&\quad - (\frac{1}{\lambda_{i+1}} - \frac{1}{2})y_{i+1}^2 + \nu_{i+1}^2 \\
&\le -\frac{1}{\lambda_{i+1}}(y_{i+1}^2)^{\frac{1+r_1}{2}} - \frac{1}{\lambda_{i+1}}(y_{i+1}^2)^{\frac{1+r_2}{2}} + \nu_{i+1}^2
\end{aligned} \tag{40}$$

when $0 < \lambda_{i+1} < 2$.

Applying the inequalities(38)-(40) to (37), we have

$$\begin{aligned}
\dot{V}_i \le &\varphi_i\zeta_i\zeta_{i+1} - \varphi_{i-1}\zeta_{i-1}\zeta_i - k_{i1}(\zeta_i^2)^{\frac{1+r_1}{2}} - k_{i2}(\zeta_i^2)^{\frac{1+r_2}{2}} \\
&- \frac{\sigma_{i1}}{1+r_1}(\tilde{a}_i^2)^{\frac{1+r_1}{2}} - \frac{\sigma_{i2}}{1+r_2}(\tilde{a}_i^2)^{\frac{1+r_2}{2}} \\
&- \frac{1}{\lambda_{i+1}}(y_{i+1}^2)^{\frac{1+r_1}{2}} - \frac{1}{\lambda_{i+1}}(y_{i+1}^2)^{\frac{1+r_2}{2}} + \Lambda_i,
\end{aligned} \tag{41}$$

where $\Lambda_i = \frac{2\sigma_{i1}}{1+r_1}a_i^{1+r_1} + \frac{2\sigma_{i2}}{1+r_2}a_i^{1+r_2} + \nu_{i+1}^2 + \frac{1}{4\rho_i}$.

Step n: In this step, a fixed-time safe controller will be given. Differentiating ζ_n yields:

$$\dot{\zeta}_n = \varphi_n[F_n(\bar{x}_n) + g_n u] + \psi_n - \dot{\alpha}_{nf}. \tag{42}$$

It is noted that $F_n(\bar{x}_n)$ can be approximated by:

$$F_n(\bar{x}_n) = W_n^T S(\bar{x}_n) + \varepsilon_n(\bar{x}_n) \le w_n\mu_n(\bar{x}_n),$$

where $||W_n|| \le \bar{W}_n$, $|\varepsilon_n| \le \varepsilon_{n1}$, $w_n = \max\{\bar{W}_n, \varepsilon_{n1}\}$, and $\bar{W}_n$, ε_{n1} are unknown positive constants. According to Young's inequality, we have the following inequalities:

$$\varphi_n\zeta_n F_n \le \underline{g}_n\rho_n\varphi_n^2 w_n^2\mu_n^2\zeta_n^2 + \frac{1}{4\rho_n\underline{g}_n},$$

$$\psi_n\zeta_n \le \underline{g}_n\psi_n^2\zeta_n^2 + \frac{1}{4\underline{g}_n},$$

$$-\zeta_n\dot{\alpha}_{nf} \le \underline{g}_n\dot{\alpha}_{nf}^2\zeta_n^2 + \frac{1}{4\underline{g}_n},$$

where $\rho_n > 0$, $\underline{g}_n > 0$ is the lower bound of g_n. Define $a_n = w_n^2$, then we have

$$\begin{aligned}\zeta_n\dot{\zeta}_n &= \varphi_n\zeta_n F_n + \varphi_n\zeta_n g_n u + \zeta_n\psi_n - \zeta_n\dot{\alpha}_{nf} \\ &\le \underline{g}_n\rho_n\varphi_n^2 a_n\mu_n^2\zeta_n^2 + \frac{1}{4\rho_n\underline{g}_n} + \underline{g}_n\psi_n^2\zeta_n^2 + \underline{g}_n\dot{\alpha}_{nf}^2\zeta_n^2 \\ &\quad + \frac{1}{2\underline{g}_n} + \varphi_n\zeta_n g_n u.\end{aligned}$$

It is time to design the controller u as follows:

$$\begin{aligned}u = &-\rho_n\varphi_n\hat{a}_n\mu_n^2\zeta_n - \frac{\psi_n^2\zeta_n}{\varphi_n} - \frac{\dot{\alpha}_{nf}^2\zeta_n}{\varphi_n} \\ &- \frac{\varphi_{n-1}^2\zeta_{n-1}^2\zeta_n}{\varphi_n} - \frac{k_{n1}}{\varphi_n}\zeta_n^{r_1} - \frac{k_{n2}}{\varphi_n}\zeta_n^{r_2},\end{aligned} \tag{43}$$

where $\hat{a}_n$ is the estimation of a_n and determined by the following adaptive law:

$$\dot{\hat{a}}_n = \rho_n\varphi_n^2\mu_n^2\zeta_n^2 - \sigma_{n1}\hat{a}_n^{r_1} - \sigma_{n2}\hat{a}_n^{r_2}, \tag{44}$$

and σ_{n1}, σ_{n2} are positive constants.

When we choose $\hat{a}_n(0) \ge 0$, we have $\hat{a}_n \ge 0$. Then, according to Assumption 2.12, we have

$$\begin{aligned}&\varphi_n\zeta_n g_n u \\ &= -g_n\rho_n\varphi_n^2\hat{a}_n\mu_n^2\zeta_n^2 - g_n\psi_n^2\zeta_n^2 - g_n\dot{\alpha}_{nf}^2\zeta_n^2 \\ &\quad - g_n\varphi_{n-1}^2\zeta_{n-1}^2\zeta_n^2 - g_n k_{n1}\zeta_n^{1+r_1} - g_n k_{n2}\zeta_n^{1+r_2} \\ &\le -\underline{g}_n\rho_n\varphi_n^2\hat{a}_n\mu_n^2\zeta_n^2 - \underline{g}_n\psi_n^2\zeta_n^2 - \underline{g}_n\dot{\alpha}_{nf}^2\zeta_n^2 \\ &\quad - \underline{g}_n\varphi_{n-1}^2\zeta_{n-1}^2\zeta_n^2 - \underline{g}_n k_{n1}\zeta_n^{1+r_1} - \underline{g}_n k_{n2}\zeta_n^{1+r_2}.\end{aligned}$$

Substituting the above inequality to $\zeta_n\dot{\zeta}_n$ yields:

$$\begin{aligned}\zeta_n\dot{\zeta}_n \le\ &\underline{g}_n\rho_n\tilde{a}_n\varphi_n^2\mu_n^2\zeta_n^2 - \underline{g}_n k_{n1}\zeta_n^{1+r_1} - \underline{g}_n k_{n2}\zeta_n^{1+r_2} \\ &- \underline{g}_n\varphi_{n-1}^2\zeta_{n-1}^2\zeta_n^2 + \frac{1}{4\rho_n\underline{g}_n} + \frac{1}{2\underline{g}_n}.\end{aligned} \tag{45}$$

Now we construct a Lyapunov function V_n as follows:

$$V_n = \frac{1}{2}\zeta_n^2 + \frac{1}{2}\tilde{a}_n^2. \tag{46}$$

Using the inequality (45) and the adaptive law (44), we have

$$\begin{aligned}\dot{V}_n \le &-\underline{g}_n k_{n1}\zeta_n^{1+r_1} - \underline{g}_n k_{n2}\zeta_n^{1+r_2} - \underline{g}_n\varphi_{n-1}^2\zeta_{n-1}^2\zeta_n^2 \\ &+ \frac{1}{4\rho_n\underline{g}_n} + \frac{1}{2\underline{g}_n} - \underline{g}_n\sigma_{n1}\tilde{a}_n\hat{a}_n^{r_1} - \underline{g}_n\sigma_{n2}\tilde{a}_n\hat{a}_n^{r_2} \\ \le &-\underline{g}_n k_{n1}(\zeta_n^2)^{\frac{1+r_1}{2}} - \underline{g}_n k_{n2}(\zeta_n^2)^{\frac{1+r_2}{2}} - \underline{g}_n\varphi_{n-1}^2\zeta_{n-1}^2\zeta_n^2 \\ &- \underline{g}_n\frac{\sigma_{n1}}{1+r_1}(\tilde{a}_n^2)^{\frac{1+r_1}{2}} - \underline{g}_n\frac{\sigma_{n2}}{1+r_2}(\tilde{a}_n^2)^{\frac{1+r_2}{2}} + \Lambda_n,\end{aligned} \tag{47}$$

where $\Lambda_n = \underline{g}_n\frac{2\sigma_{n1}}{1+r_1}a_n^{1+r_1} + \underline{g}_n\frac{2\sigma_{n2}}{1+r_2}a_n^{1+r_2} + \frac{1}{4\rho_n\underline{g}_n} + \frac{1}{2\underline{g}_n}$.

Define

$$V = V_1 + \ldots, V_n.$$

Using the inequalities (28), (41), and (47), we have

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{n} \dot{V}_i \\
&\leq -\sum_{i=1}^{n-1} k_{i1}(\zeta_i^2)^{\frac{1+r_1}{2}} - \sum_{i=1}^{n-1} k_{i2}(\zeta_i^2)^{\frac{1+r_2}{2}} \\
&\quad - \sum_{i=1}^{n-1} \frac{\sigma_{i1}}{1+r_1}(\tilde{a}_i^2)^{\frac{1+r_1}{2}} - \sum_{i=1}^{n-1} \frac{\sigma_{i2}}{1+r_2}(\tilde{a}_i^2)^{\frac{1+r_2}{2}} \\
&\quad - \sum_{i=2}^{n} \frac{1}{\lambda_i}(y_i^2)^{\frac{1+r_1}{2}} - \sum_{i=2}^{n} \frac{1}{\lambda_i}(y_i^2)^{\frac{1+r_2}{2}} - \underline{g}_n k_{n1}(\zeta_n^2)^{\frac{1+r_1}{2}} \\
&\quad - \underline{g}_n k_{n2}(\zeta_n^2)^{\frac{1+r_2}{2}} - \frac{\underline{g}_n \sigma_{n1}}{1+r_1}(\tilde{a}_n^2)^{\frac{1+r_1}{2}} - \frac{\underline{g}_n \sigma_{n2}}{1+r_2}(\tilde{a}_n^2)^{\frac{1+r_2}{2}} \\
&\quad - \underline{g}_n \varphi_{n-1}^2 \zeta_{n-1}^2 \zeta_n^2 + \varphi_{n-1}\zeta_{n-1}\zeta_n + \sum_{i=1}^{n} \Lambda_i.
\end{aligned}$$

It is noted that

$$\varphi_{n-1}\zeta_{n-1}\zeta_n \leq \underline{g}_n \varphi_{n-1}^2 \zeta_{n-1}^2 \zeta_n^2 + \frac{1}{4\underline{g}_n},$$

then we have

$$\begin{aligned}
\dot{V} &\leq -\sum_{i=1}^{n-1} k_{i1}(\zeta_i^2)^{\frac{1+r_1}{2}} - \sum_{i=1}^{n-1} k_{i2}(\zeta_i^2)^{\frac{1+r_2}{2}} \\
&\quad - \sum_{i=1}^{n-1} \frac{\sigma_{i1}}{1+r_1}(\tilde{a}_i^2)^{\frac{1+r_1}{2}} - \sum_{i=1}^{n-1} \frac{\sigma_{i2}}{1+r_2}(\tilde{a}_i^2)^{\frac{1+r_2}{2}} \\
&\quad - \sum_{i=2}^{n} \frac{1}{\lambda_i}(y_i^2)^{\frac{1+r_1}{2}} - \sum_{i=2}^{n} \frac{1}{\lambda_i}(y_i^2)^{\frac{1+r_2}{2}} - \underline{g}_n k_{n1}(\zeta_n^2)^{\frac{1+r_1}{2}} \\
&\quad - \underline{g}_n k_{n2}(\zeta_n^2)^{\frac{1+r_2}{2}} - \frac{\underline{g}_n \sigma_{n1}}{1+r_1}(\tilde{a}_n^2)^{\frac{1+r_1}{2}} - \frac{\underline{g}_n \sigma_{n2}}{1+r_2}(\tilde{a}_n^2)^{\frac{1+r_2}{2}} \\
&\quad + \Lambda,
\end{aligned} \tag{48}$$

where $\Lambda = \frac{1}{4\underline{g}_n} + \sum_{i=1}^{n} \Lambda_i$.

B. PROOF OF THEOREM 1

P r o o f. Define

$$\Xi_1 = 2^{\frac{1+r_1}{2}} \min_{i=1,\ldots,n-1}\{k_{i1}, \underline{g}_n k_{n1}, \frac{\sigma_{i1}}{1+r_1}, \frac{\underline{g}_n \sigma_{n1}}{1+r_1}, \frac{1}{\lambda_i}\}, \tag{49}$$

$$\Xi_2 = 2^{\frac{1+r_2}{2}} \min_{i=1,\ldots,n-1}\{k_{i2}, \underline{g}_n k_{n2}, \frac{\sigma_{i2}}{1+r_2}, \frac{\underline{g}_n \sigma_{n2}}{1+r_2}, \frac{1}{\lambda_i}\}, \tag{50}$$

where parameters k_{i1}, k_{i2}, σ_{i1}, σ_{i2}, and λ_i are all positive constants. $r_1 = \frac{m}{n}$, $r_2 = \frac{p}{q}$, where $m < n$, $p > q$ are positive odd numbers.

From the inequality (48), we have

$$\begin{aligned}
\dot{V} \leq & -\Xi_1 \sum_{i=1}^{n-1} \Big[(\frac{1}{2}\zeta_i^2)^{\frac{1+r_1}{2}} + (\frac{1}{2}\tilde{a}_i^2)^{\frac{1+r_1}{2}} + (\frac{1}{2}y_{i+1}^2)^{\frac{1+r_1}{2}} \Big] \\
& -\Xi_1 \Big[(\frac{1}{2}\zeta_n^2)^{\frac{1+r_1}{2}} + (\frac{1}{2}\tilde{a}_n^2)^{\frac{1+r_1}{2}} \Big] \\
& -\Xi_2 \sum_{i=1}^{n-1} \Big[(\frac{1}{2}\zeta_i^2)^{\frac{1+r_2}{2}} + (\frac{1}{2}\tilde{a}_i^2)^{\frac{1+r_2}{2}} + (\frac{1}{2}y_{i+1}^2)^{\frac{1+r_2}{2}} \Big] \\
& -\Xi_2 \Big[(\frac{1}{2}\zeta_n^2)^{\frac{1+r_2}{2}} + (\frac{1}{2}\tilde{a}_n^2)^{\frac{1+r_2}{2}} \Big] + \Lambda.
\end{aligned}$$

Furthermore, according to Lemma 2.7, we have:

$$\begin{aligned}
\dot{V} \leq & -\Xi_1 \sum_{i=1}^{n} \Big(V_i^{\frac{1+r_1}{2}} \Big) - \Xi_2 \sum_{i=1}^{n-1} \Big(3^{\frac{1-r_2}{2}} V_i^{\frac{1+r_2}{2}} \Big) \\
& -\Xi_2 \Big(2^{\frac{1-r_2}{2}} V_n^{\frac{1+r_2}{2}} \Big) + \Lambda \\
\leq & -\Xi_1 V^{\frac{1+r_1}{2}} - 3^{\frac{1-r_2}{2}} \Xi_2 \Big(n^{\frac{1-r_2}{2}} V^{\frac{1+r_2}{2}} \Big) + \Lambda \\
\leq & -\Xi_1 V^{\frac{1+r_1}{2}} - \bar{\Xi}_2 V^{\frac{1+r_2}{2}} + \Lambda,
\end{aligned} \tag{51}$$

where $\bar{\Xi}_2 = (3n)^{\frac{1-r_2}{2}} \Xi_2$.

Finally, from Lemma 2.4, there exists a settling time T such that

$$V \leq \mathcal{R} = \min \Big\{ (\frac{\Lambda}{\Xi_1 \theta})^{\frac{2}{1+r_1}}, (\frac{\Lambda}{\bar{\Xi}_2 \theta})^{\frac{2}{1+r_2}} \Big\},$$

when $t \geq T$, where

$$T \leq \frac{2}{\Xi_1(1-\theta)(1-r_1)} + \frac{2}{\bar{\Xi}_2(1-\theta)(r_2-1)},$$

and $0 < \theta < 1$ is a constant.

Therefore, the closed-loop system (8) is practically fixed-time stable. Since $V \in \mathcal{L}_\infty$, then all the states of the closed-loop system are bounded, that is, $\zeta_i \in \mathcal{L}_\infty$, $\tilde{a}_i \in \mathcal{L}_\infty$, and $y_i \in \mathcal{L}_\infty$.

Furthermore, it is noted that $V \leq \mathcal{R}$ when $t \geq T$, thus we have $|\zeta_1| \leq \sqrt{2\mathcal{R}}$ when $t \geq T$. Recalling the transformation functions (6) and (9) and applying the mean value theorem, there exists a constant $\hat{\xi}$ such that

$$\begin{aligned}
|x_1 - y_d| = & \Big| \frac{h_{11}(t) + h_{12}(t)}{2} \tanh \Big(\frac{2\xi_1}{h_{11}(t) + h_{12}(t)} \Big) \\
& - \frac{h_{11}(t) + h_{12}(t)}{2} \tanh \Big(\frac{2\xi_d}{h_{11}(t) + h_{12}(t)} \Big) \Big| \\
= & \frac{h_{11}(t) + h_{12}(t)}{2} \Big| \frac{1}{cosh^2(\hat{\xi})} \times \frac{2(\xi_1 - \xi_d)}{h_{11}(t) + h_{12}(t)} \Big| \\
\leq & \frac{h_{11}(t) + h_{12}(t)}{2} \times \frac{2}{h_{11}(t) + h_{12}(t)} |\xi_1 - \xi_d| \\
= & |\xi_1 - \xi_d| \\
= & \zeta_1,
\end{aligned}$$

where $\hat{\xi} \in (\frac{2\xi_1}{h_{11}(t)+h_{12}(t)}, \frac{2\xi_d}{h_{11}(t)+h_{12}(t)})$. Thus the output tracking error satisfies $|e| = |x_1 - y_d| \leq |\zeta_1| \leq \sqrt{2\mathcal{R}}$ when $t \geq T$, which means that the tracking error is bounded in fixed time.

Finally, we verify that all the state constraints are satisfied all the time. According to Assumption 2.13, we know that ξ_d is bounded. It is noted that $\zeta_i \in \mathcal{L}_\infty$, thus ζ_1 is also bounded. Therefore, $\xi_1 = \zeta_1 + \xi_d$ is bounded, which implies that $-h_{11}(t) < x_1(t) < h_{12}(t)$ is satisfied when $-h_{11}(0) < x_1(0) < h_{12}(0)$. Additionally, according to the fact that $y_2 \in \mathcal{L}_\infty$, $\alpha_{2f} = y_2 + \alpha_1$ is bounded, and $\dot{\alpha}_{2f}$ is bounded. Then, the boundedness of ζ_2 and α_{2f} ensures that $\xi_2 = \zeta_2 + \alpha_{2f}$ is bounded, which implies that the constraint $-h_{21}(t) < x_2(t) < h_{22}(t)$ is satisfied when $-h_{21}(0) < x_2(0) < h_{22}(0)$. Using similar analysis, it can be shown that all the state constraints are satisfied all the time.

The proof is thus complete. □

Chaoqun Guo, School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, Sichuan, P.R. China; Yangtze Delta Region Institute (Huzhou), University of Electronic Science and Technology of China, Huzhou 313001. P.R. China.

e-mail: guochaoqunlg@126.com

Jiangping Hu, Corresponding author, School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, Sichuan, P.R. China; Yangtze Delta Region Institute (Huzhou), University of Electronic Science and Technology of China, Huzhou 313001. P.R. China.

e-mail: hujp@uestc.edu.cn

Jiasheng Hao, School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, Sichuan. P.R. China.

e-mail: hao@uestc.edu.cn

Sergej Čelikovský, The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod Vodárenskou věží 4, 182 08 Praha 8. Czech Republic.

e-mail: celikovs@utia.cas.cz

Xiaoming Hu, Optimization and Systems Theory, Royal Institute of Technology, Stockholm SE-10044. Sweden.

e-mail: hu@kth.se