Zbigniew Lipecki

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Abstract. Let K be a compact space and let $C(K)$ be the Banach lattice of real-valued continuous functions on K. We establish eleven conditions equivalent to the strong compactness of the order interval $[0, x]$ in $C(K)$, including the following ones:

(i) $\{x > 0\}$ consists of isolated points of K;
(ii) $[0, x]$ is pointwise compact;
(iii) $[0, x]$ is weakly compact;
(iv) the strong topology and that of pointwise convergence coincide on $[0, x]$;
(v) the strong and weak topologies coincide on $[0, x]$.

Moreover, the weak topology and that of pointwise convergence coincide on $[0, x]$ if and only if $\{x > 0\}$ is scattered. Finally, the weak topology on $[0, x]$ is metrizable if and only if the topology of pointwise convergence on $[0, x]$ is such if and only if $\{x > 0\}$ is countable.

Keywords: real linear lattice; order interval; locally solid; Banach lattice $C(K)$; strongly compact; weakly compact; pointwise compact; coincidence of topologies; metrizable; scattered; Čech–Stone compactification

Classification: 46A40, 46B42, 46E05, 54C35, 54D30

1. Introduction

Not much seems to be known about topological properties of order intervals in Banach lattices and more general linear lattices equipped with a linear topology. It is the purpose of this paper as well as of two other papers [6] and [7] by the author to improve the situation. In contrast to those papers, we are concerned below with rather concrete linear lattices. They are the linear lattice $C(K)$ of real-valued continuous functions on a completely regular space K and its linear sublattice $C_b(K)$ consisting of bounded functions. We consider $C(K)$ equipped with the topology p of pointwise convergence on K and $C_b(K)$ equipped, in addition, with the strong topology s given by the uniform norm and the weak topology w.

Let $x \in C(K)$ be positive. We establish four conditions equivalent to the pointwise compactness of $[0, x]$, including the following one: $\{x > 0\}$ consists of

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isolated points of K (see Proposition 1 in Section 3). We also show that the topology of pointwise convergence is metrizable when restricted to $[0, x]$ if and only if $\{x > 0\}$ is countable (see Proposition 3 in Section 5).

Let x be, moreover, bounded, i.e., $x \in C_b(K)$. We then show that $[0, x]$ is strongly compact if and only if it is weakly compact if and only if $\{x \geqslant \varepsilon\}$ is finite for every $\varepsilon > 0$ (if and only if $[0, x]$ is pointwise compact provided K is compact). Four more equivalent conditions are also presented, two of which are in terms of coincidence of pairs of topologies on $[0, x]$ (see Theorem 1 in Section 3 for details).

Let now K be compact. The weak topology and that of pointwise convergence then coincide on $[0, x]$ if and only if $\{x > 0\}$ is scattered (see Theorem 2 in Section 4). Moreover, the weak topology restricted to $[0, x]$ is metrizable if and only if $\{x > 0\}$ is countable (see Theorem 5 in Section 5).

The proofs use standard results from general topology, measure theory, and functional analysis. Among those results are the Tychonoff product theorem, the existence of the Čech–Stone compactification of a completely regular space, and the Riesz representation theorem in the compact case. Our notation and terminology are also mostly standard; see Section 2 for some explanations.

Sections 3 and 5 are essentially independent of each other, as far as the proofs are concerned.

2. Preliminaries

We start with some terminology and notation concerning general linear lattices (Riesz spaces in another terminology) with or without a compatible topology. They will be applied below almost exclusively in the context of linear lattices of real-valued continuous functions on a topological space, equipped with some standard topologies.

Let X be a real linear lattice, with the order denoted by "$\leqslant$". As usual, X_+ stands for the positive cone of X. The *order interval* $[0, x]$, where $x \in X_+$, is the set

$$\{y \in X : 0 \leqslant y \leqslant x\}.$$

Let $\{x_\alpha\}$ be a net in X. We write $x_\alpha \downarrow 0$ if the net is directed downwards and its order limit $\inf_\alpha x_\alpha$ equals 0.

Let τ be a (Hausdorff) linear topology on X and let $x \in X_+$. We say that τ is *order continuous* (*o.c.*, for short) or *has the Lebesgue property on* $[0, x]$ if, for every net $\{x_\alpha\}$ in $[0, x]$ with $x_\alpha \downarrow 0$, we have $x_\alpha \to 0$ with respect to τ (cf. [1, Definition 12.7] and [2, Definition 3.1(2)]). In the case where τ is given by a lattice norm and this condition holds for every $x \in X_+$, the norm is often called order continuous.

Let τ_1 and τ_2 be linear topologies on X and let $x \in X_+$. We write

$$\tau_1 = \tau_2 \qquad \text{on } [0, x],$$

if the restrictions of τ_1 and τ_2 to $[0, x]$ coincide.

The terminology concerning topological spaces we use is standard and mostly follows Engelking's monograph [4]. This is, in particular, the case of the notion of a completely regular space, but not that of a compact space, where we assume the Hausdorff axiom, following N. Bourbaki.

Throughout the rest of the paper, K stands for a completely regular space. This includes, of course, the case where K is compact.

The set of accumulation points of K is denoted by K^d, and its complement by K^{dc}. Thus, $t \in K^{dc}$ if and only if t is an isolated point of K.

We denote by $C(K)$ the linear lattice of real-valued continuous functions on K, equipped with the usual algebraic operations and pointwise order. The linear sublattice of $C(K)$ consisting of bounded functions is denoted by $C_b(K)$. Equipped with the uniform norm $\|\cdot\|$, $C_b(K)$ becomes a Banach lattice. Its strong and weak topologies are denoted by s and w, respectively. The linear lattice $C(K)$ carries the topology of pointwise convergence, which we denote by p. We note that, in the literature, the notation $C_p(K)$ is often used in this connection (see, e.g., [3] or [8]).

The following lemma will be used repeatedly below. The simple proof is left to the reader.

Lemma 1. *Let $x \in C(K)_+$, let $t_1, \dots, t_n$ be different points of K, and let $s_1, \dots, s_n \in \mathbb{R}_+$ be such that $s_i \leqslant x(t_i)$ for all i. Then there exists*

$$y \in [0, x] \qquad \text{with } y(t_i) = s_i \text{ for all } i.$$

Let K be compact and let $\mathcal{M}(K)$ stand for the Banach lattice of real-valued Radon measures on K. By the Riesz representation theorem, $\mathcal{M}(K)$ can be identified with the dual of $C(K)$ (see, e.g., [9, § 18]). We shall apply this theorem, sometimes tacitly, many times below. In particular, this is the case when the weak topology is involved (see the proofs of Theorem 1, (vii) ⇒ (ii), Theorem 2 and Lemma 2). We set

$$\mathcal{S}(K) = \{\mu \in \mathcal{M}(K)_+ \colon\ \mu(K) = 1\}.$$

3. Compactness. Coincidence of the strong topology with the weak topology or with the topology of pointwise convergence

Our first concern is compactness with respect to the topology of pointwise convergence in the general case.

Proposition 1. *Let $x \in C(K)_+$. The following five conditions are then equivalent:*

(i) $([0,x],p)$ *is compact;*
(ii) $\{x>0\} \subset K^{dc}$*;*
(iii) p *is o.c. on* $[0,x]$*;*
(iv) $[0,x]=\{y \in \mathbb{R}^K : 0 \leqslant y \leqslant x\}$*;*
(v) $([0,x],p)$ *is complete.*

In connection with condition (v), recall that a subset E of a topological linear space is called *complete* if every Cauchy net of elements of E converges to an element of E (see [2, page 52]).

Proof: By Lemma 1, (i) implies (iv). The converse implication is a consequence of the Tychonoff product theorem. Clearly, (iv) implies (iii).

To derive (iv) from (ii), consider $y \in \mathbb{R}^K$ with $0 \leqslant y \leqslant x$ and $t \in K$. If $x(t)=0$, then t is, clearly, a continuity point of y. Otherwise, t is an isolated point of K, by (ii). Thus, y is continuous.

Suppose (ii) fails. We shall show that (iii) then fails, too. Let $t_0 \in K^d$ be such that $x(t_0)>0$. The set

$$\{y \in [0,x] \colon y(t_0)=x(t_0)\}$$

is then directed downwards by "$\leqslant$" and the corresponding net converges pointwise to 0 for $t \in K$ with $t \neq t_0$, and so its order limit in $C(K)$ equals the constant 0 function. Thus, (iii) fails.

Clearly, (iv) implies (v). Finally, (v) implies (iv), by Lemma 1. □

Remark 1. Clearly, the equivalent conditions (i)–(v) of Proposition 1 imply that $[0,x]$ is order complete and atomic. (The latter means that $x=0$ or x is the supremum of the set of all atoms c of $C(K)$ with $c \leqslant x$; see [6, Section 2].) The converse does not hold even if K is compact. Indeed, if $K=\beta\mathbb{N}$ and $x=1_K$, then $[0,x]$ is order complete (see [9, Example 24.2.6 (a) and Theorem 24.7.1]). Moreover, x is the supremum of the set

$$\{1_{\{n\}} \colon n \in \mathbb{N}\}$$

in $C(K)$, but (ii), obviously, fails.

Conditions (i)–(iii) of the following theorem are, clearly, stronger versions of the corresponding conditions of Proposition 1. Moreover, they are equivalent to them when K is compact. In particular, condition (iii) of Proposition 1 then implies condition (iii) of Theorem 1, by Dini's theorem.

Theorem 1. *Let $x \in C_b(K)_+$. The following seven conditions are then equivalent:*

(i) $([0,x],s)$ *is compact;*
(ii) $\{x \geqslant \varepsilon\}$ *is finite for every* $\varepsilon > 0$*;*
(iii) s *is o.c. on* $[0,x]$*;*
(iv) $s = p$ *on* $[0,x]$*;*
(v) $([0,x],w)$ *is compact;*
(vi) w *is o.c. on* $[0,x]$*;*
(vii) $s = w$ *on* $[0,x]$.

They imply the next two conditions:

(viii) $([0,x],p)$ *is compact;*
(ix) p *is o.c. on* $[0,x]$.

If K is compact, then conditions (i)–(ix) *are all equivalent.*

Proof: We start by establishing the equivalence of conditions (i)–(iv). Clearly, (i) implies (iv). We shall show the implications (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) and the equivalence (ii) $\Leftrightarrow$ (iii).

Suppose (ii) fails, i.e., $\{x \geqslant \varepsilon_0\}$ is infinite for some $\varepsilon_0 > 0$. Set

$$V = \{y \in [0,x] \colon \|y\| < \varepsilon_0\}.$$

We claim that there is no neighbourhood W of 0 in $([0,x],p)$ with $W \subset V$, and so (iv) fails, too. We may assume that

$$W = \{y \in [0,x] \colon \ y(t) < \eta \ \text{ for } \ t \in F\},$$

where F is a finite subset of K and $\eta > 0$. Choose $t_0 \in F^c$ with $x(t_0) \geqslant \varepsilon_0$. By Lemma 1, there exists $y \in [0,x]$ such that

$$y|F = 0 \qquad \text{and} \qquad y(t_0) = \varepsilon_0.$$

Hence $y \in W \setminus V$, proving our claim.

Suppose (ii) holds, and fix $\varepsilon > 0$. Let M be a finite ε-mesh in the product

$$\prod_{t \in \{x \geqslant \varepsilon\}} [0, x(t)]$$

equipped with the uniform metric. For every $z \in M$ choose $\tilde{z} \in [0,x]$ such that

$$\tilde{z}|\{x \geqslant \varepsilon\} = z$$

(see Lemma 1). Then $\{\tilde{z} \colon z \in M\}$ is a finite ε-mesh in $[0,x]$, and so (i) follows.

Suppose (ii) holds. We then have $\{x > 0\} \subset K^{dc}$. Let $\{x_\alpha\}$ be a net in $[0, x]$ with $x_\alpha \downarrow 0$. By Proposition 1, (ii) $\Rightarrow$ (iii), $x_\alpha \to 0$ pointwise on K. Fix $\varepsilon > 0$, and set $F = \{x \geqslant \varepsilon\}$. There exists α_0 such that

$$x_\alpha(t) \leqslant \varepsilon \qquad \text{for } t \in F \text{ and } \alpha \geqslant \alpha_0,$$

and so $\|x_\alpha\| \leqslant \varepsilon$ for $\alpha \geqslant \alpha_0$. Therefore, $\|x_\alpha\| \to 0$, and (iii) follows.

Suppose (iii) holds. By Proposition 1, (iii) $\Rightarrow$ (ii), we then have $\{x > 0\} \subset K^{dc}$. Assume, to get a contradiction, that $\{x \geqslant \varepsilon_0\}$ is infinite for some $\varepsilon_0 > 0$. For every finite $F \subset \{x \geqslant \varepsilon_0\}$ set

$$x_F = \varepsilon_0 1_{\{x \geqslant \varepsilon_0\} \setminus F}.$$

We have $x_F \in [0, x]$, since $\{x \geqslant \varepsilon_0\}$ and F are closed-and-open subsets of K. Consider the net $\{x_F\}$, where the index set is directed upwards by inclusion. We then have $x_F(t) \downarrow 0$ for every $t \in K$, but $\|x_F\| = \varepsilon_0$ for all F as above. This contradicts (iii).

The implications (i) $\Rightarrow$ (v) $\Rightarrow$ (viii) and (iii) $\Rightarrow$ (vi) are clear. So is the implication (i) $\Rightarrow$ (vii). The equivalence of (viii) and (ix) holds, by Proposition 1.

If K is compact, then (viii) implies (ii), by Proposition 1, (i) $\Rightarrow$ (ii). Summing up, in the compact case, conditions (i)–(vi), (viii) and (ix) are all equivalent.

Suppose K is compact and (ii) fails. We shall show that (vii) then fails, too. Let $\varepsilon_0 > 0$ be such that $\{x \geqslant \varepsilon_0\}$ is infinite, and set

$$V = \{y \in [0, x] \colon \|y\| < \varepsilon_0\}.$$

We shall show that V does not contain any weak neighbourhood W of 0 in $[0, x]$. We may assume that

$$W = \left\{y \in [0, x] \colon \int y \, \mathrm{d}\mu_i < \eta, \ i = 1, \ldots, n\right\},$$

where $\eta > 0$ and $\mu_1, \ldots, \mu_n \in \mathcal{M}(K)_+$. Let $\varkappa_i$ and λ_i be the atomic and nonatomic components of μ_i, respectively. There exists a finite subset F of K such that

$$\varkappa_i(F^c) < \frac{\eta}{2\varepsilon_0}, \qquad i = 1, \ldots, n.$$

Choose $t_0 \in F^c$ with $x(t_0) \geqslant \varepsilon_0$ and an open subset O of F^c such that

$$t_0 \in O \quad \text{and} \quad \lambda_i(O) < \frac{\eta}{2\varepsilon_0}, \qquad i = 1, \ldots, n.$$

It follows that $\mu_i(O) < \eta/\varepsilon_0$ for all i. Finally, let $y_0 \in [0, x]$ be such that

$$y_0 \leqslant \varepsilon_0, \qquad y_0(t_0) = \varepsilon_0 \qquad \text{and} \qquad y_0 | O^c = 0.$$

Clearly, $y_0 \in W \setminus V$.

Finally, we note that $C_b(K)$ and $C(\beta K)$ can be identified as Banach lattices via extension by continuity. Therefore, the equivalence of conditions (i) and (v)–(vii) carries over from the compact case to the general one. □

Condition (vii) of Theorem 1 also appears, in the context of Banach lattices with o.c. norm, in [6, Theorem 6]. For more conditions equivalent to the compactness of an order interval in an arbitrary linear lattice equipped with a Hausdorff locally solid (convex-solid, respectively) topology, see [7] ([2, Theorem 6.56], respectively).

Remark 2. (a) In the case where K is compact, the implication (viii) $\Rightarrow$ (v) of Theorem 1 holds for arbitrary bounded subsets of $C(K)$, by a result of Grothendieck (see [5, Theorem 4.2]).

(b) The implications (v) $\Rightarrow$ (iii) and (vi) $\Rightarrow$ (iii) of Theorem 1 hold in an arbitrary linear lattice equipped with a Hausdorff locally convex-solid topology (see [2, Theorem 6.46, (i) $\Rightarrow$ (iv)] and [1, Theorem 11.8], respectively).

(c) In the setting of Theorem 1, condition (iii) implies (vii), by a general classical result (see [2, Theorem 4.22]), since we have $p \subset w \subset s$.

(d) One of the steps in the proof of Theorem 1 is the derivation of (ii) from (vii) in the case where K is compact. Instead one can derive (v) from (vii) in that case, by applying [2, Theorem 6.46, (v) $\Rightarrow$ (i)], the Riesz representation theorem, and the Lebesgue convergence theorem.

The compactness assumption in the final part of Theorem 1 is essential. This follows from assertions (1)–(3) of the next proposition.

Proposition 2. *Let K be an infinite discrete space. In the Banach space $C_b(K)$ the following holds:*

(1) $([0, 1_K], s)$ *is not compact;*
(2) $([0, 1_K], p)$ *is compact;*
(3) p *is o.c. on* $[0, 1_K]$*;*
(4) $([0, 1_K], w)$ *is not metrizable;*
(5) $([0, 1_K], p)$ *is metrizable provided K is countable.*

Proof: Assertions (1), (3) and (5) are plain. Assertion (2) holds, by the Tychonoff product theorem.

Since $C_b(K)$ is not separable, neither is its dual space, by a well-known result. Therefore, (4) follows from [1, Theorem 10.8]. □

Remark 3. The equivalence of conditions (i) and (vii) of Theorem 1 also holds in an arbitrary Banach lattice X with o.c. norm, because $([0, x], w)$ is then compact for every $x \in X_+$ (see [1, Theorem 12.9, (1) $\Rightarrow$ (5)]). The author does not know whether the assumption of order continuity is essential.

4. Coincidence of the weak topology with that of pointwise convergence

Our next result should be compared with Theorem 1 whose conditions (iv) and (ii) are, clearly, stronger than conditions (i) and (iii) below, respectively.

Theorem 2. *Let K be compact and let $x \in C(K)_+$. The following three conditions are then equivalent:*

(i) $w = p$ *on* $[0, x]$*;*
(ii) $\{x > 0\}$ *is scattered;*
(iii) $\{x \geqslant \varepsilon\}$ *is scattered for every* $\varepsilon > 0$*.*

Proof: Clearly, (ii) implies (iii). The converse is seen, since if D is a nonempty dense-in-itself subset of $\{x > 0\}$, then $D \cap \{x > \varepsilon\}$ is dense-in-itself and nonempty for ε small enough. We shall show that (i) and (iii) are also equivalent.

Suppose (iii) fails. Then there is an $\varepsilon_0 > 0$ such that $L = \{x \geqslant \varepsilon_0\}$ is not scattered. Let $\mu \in \mathcal{S}(K)$ be nonatomic and such that $\mu(L^c) = 0$ (see [9, Theorem 19.7.6]). Set

$$V = \left\{ y \in [0, x] \colon \int y \, \mathrm{d}\mu < \frac{1}{2}\varepsilon_0 \right\}.$$

Then V is a neighbourhood of 0 in $([0, x], w)$. We claim that $W \setminus V \neq \emptyset$ for every neighbourhood W of 0 in $([0, x], p)$. We may assume that

$$W = \{y \in [0, x] \colon \, y(t) < \eta \text{ for all } t \in F\},$$

where F is a finite subset of K and $\eta > 0$. To establish the claim, take a closed subset L_0 of L such that

$$\mu(L_0) \geqslant \frac{1}{2} \qquad \text{and} \qquad L_0 \cap F = \emptyset.$$

Let, further, $y_0 \in [0, x]$ be such that

$$y_0 | L_0 = \varepsilon_0 \qquad \text{and} \qquad y_0 | F = 0.$$

Clearly, $y_0 \in W$. On the other hand,

$$\int y_0 \, \mathrm{d}\mu \geqslant \int_{L_0} y_0 \, \mathrm{d}\mu \geqslant \frac{1}{2}\varepsilon_0,$$

and so $y_0 \notin V$. Thus, the claim is established. Therefore, (i) fails, too.

Suppose (iii) holds and $x \neq 0$. To derive (i), it is enough to show that, given $y_0 \in [0, x]$, $\mu \in \mathcal{S}(K)$ and $\eta > 0$, the neighbourhood V of y_0 in $([0, x], w)$ defined by

$$V = \left\{ y \in [0,x] \colon \left| \int (y - y_0)\, \mathrm{d}\mu \right| < \eta \right\}$$

contains a neighbourhood of y_0 in $([0,x],p)$. We assume, as we may, that $\eta < \|x\|$.

Set $L = \{x \geqslant \eta/3\}$. We then have

$$\left| \int_{L^c} (y - y_0)\, \mathrm{d}\mu \right| \leqslant \int_{L^c} |y - y_0|\, \mathrm{d}\mu \leqslant \frac{1}{3}\eta \qquad \text{for all } y \in [0,x].$$

Since L is scattered, we can find a (nonempty) finite subset F of L such that

$$\mu(L \setminus F) < \frac{\eta}{3\|x\|},$$

see [9, Theorem 19.7.6]. It follows that

$$\begin{aligned} \left| \int_{L\setminus F} (y - y_0)\, \mathrm{d}\mu \right| &\leqslant \int_{L\setminus F} |y - y_0|\, \mathrm{d}\mu \\ &\leqslant \|x\|\ \mu(L \setminus F) < \frac{1}{3}\eta \qquad \text{for all } y \in [0,x]. \end{aligned}$$

Set

$$W = \left\{ y \in [0,x] \colon |y(t) - y_0(t)| < \frac{\eta}{3n} \text{ for all } t \in F \right\},$$

where n is the cardinality of F. For $y \in W$ we then have

$$\left| \int_F (y - y_0)\, \mathrm{d}\mu \right| \leqslant \sum_{t\in F} |y(t) - y_0(t)| < \frac{1}{3}\eta.$$

It follows that $W \subset V$. Thus, (i) is established. □

Remark 4. The implication (i) ⇒ (ii) of Theorem 2 extends to an arbitrary completely regular space K and $x \in C_b(K)_+$. This is seen by considering the continuous extension x^β of x to βK and noting that the condition that $w = p$ on $[0,x]$ implies $w = p$ on $[0,x^\beta]$ in $C(\beta K)$. The converse implication fails, however, as assertions (4) and (5) of Proposition 2 show.

5. Metrizability

We start with a version of well-known results (see [3, Theorem I.1.1] or [8, Corollary 6.2.3]).

Proposition 3. *For $x \in C(K)_+$ the following two conditions are equivalent:*

(i) *$([0,x],p)$ is metrizable;*

(ii) *$\{x > 0\}$ is countable.*

Under these conditions, $([0,x],p)$ is separable.

Proof: The implication (ii) $\Rightarrow$ (i) and the additional assertion are seen, since the mapping

$$[0,x] \ni y \longmapsto y|\{x>0\} \in \mathbb{R}^{\{x>0\}}$$

is a homeomorphic embedding of $([0,x],p)$ into $(\mathbb{R}^{\{x>0\}},p)$.

Suppose (ii) fails. We shall show that (i) then fails, too. To this end, fix a sequence (V_n) of neighbourhoods of 0 in $([0,x],p)$. We shall find a neighbourhood V of 0 in $([0,x],p)$ with $V_n \setminus V \neq \emptyset$ for each n. We may assume that

$$V_n = \{y \in [0,x] \colon\ y(t) < \eta_n \ \text{ for all } \ t \in F_n\},$$

where $\eta_n > 0$ and F_n is a finite subset of K for each n. Choose

$$t_0 \in K \setminus \bigcup_{n=1}^{\infty} F_n \qquad \text{with } \ x(t_0) > 0,$$

and set

$$V = \Big\{y \in [0,x] \colon y(t_0) < \frac{1}{2}x(t_0)\Big\}.$$

By Lemma 1, we can find $y_n \in V_n$ with $y_n(t_0) = x(t_0)/2$. Thus, V is as desired. □

The equivalent conditions (i) and (ii) of Proposition 3 are not implied by its additional assertion even if K is compact. A counterexample is, e.g., $K = [0,1]$ and $x = 1_K$.

In the proof of Theorem 3 below we shall apply the following lemma.

Lemma 2. *Let K be compact, let $x \in C(K)_+$, and let $\{x>0\}$ be uncountable. If (V_n) is a sequence of neighbourhoods of 0 in $([0,x],w)$, then there exists a neighbourhood V of 0 in $([0,x],p)$ with $V_n \setminus V \neq \emptyset$ for each n.*

Proof: By assumption, $\{x \geqslant \varepsilon_0\}$ is uncountable for some $\varepsilon_0 > 0$. We may assume that

$$V_n = \Big\{y \in [0,x] \colon \int y \,\mathrm{d}\mu_n^i < \eta_n;\ i = 1,\dots,m_n\Big\},$$

where

$$\mu_n^i \in \mathcal{S}(K) \qquad \text{and} \qquad \eta_n > 0, \qquad i = 1,\dots,m_n \ \text{ and } \ n = 1,2,\dots$$

Let $\varkappa_n^i$ and λ_n^i be the atomic and nonatomic components of μ_n^i, respectively.

Let F_n be a finite subset of K such that

$$\varkappa_n^i(F_n^c) < \frac{\eta_n}{2\|x\|}, \qquad i = 1,\dots,m_n \ \text{ and } \ n = 1,2,\dots$$

Fix $t_0 \in \{x \geqslant \varepsilon_0\} \setminus \bigcup_{n=1}^{\infty} F_n$, and set

$$V = \{y \in [0,x] \colon y(t_0) < \varepsilon_0\}.$$

Fix n, and choose a neighbourhood O of t_0 with

$$O \cap F_n = \emptyset \qquad \text{and} \qquad \lambda_n^i(O) < \frac{\eta_n}{2\|x\|}, \qquad i = 1, \ldots, m_n.$$

Finally, let $y_0 \in [0, x]$ be such that

$$y_0(t_0) = \varepsilon_0 \qquad \text{and} \qquad y_0 | O^c = 0.$$

Clearly, $y_0 \notin V$. On the other hand, we have

$$\int y_0 \, \mathrm{d}\varkappa_n^i = \int_{F_n^c} y_0 \, \mathrm{d}\varkappa_n^i < \frac{1}{2}\eta_n,$$
$$\int y_0 \, \mathrm{d}\lambda_n^i = \int_O y_0 \, \mathrm{d}\lambda_n^i < \frac{1}{2}\eta_n$$

for $i = 1, \ldots, m_n$. It follows that $y_0 \in V_n$. □

Theorem 3. *Let K be compact and let $x \in C(K)_+$. The following two conditions are then equivalent:*

(i) $([0, x], w)$ *is metrizable;*
(ii) $\{x > 0\}$ *is countable.*

Under these conditions, $w = p$ on $[0, x]$.

Proof: The implication (i) ⇒ (ii) is a consequence of Lemma 2. Suppose (ii) holds. Then $\{x \geqslant \varepsilon\}$ is countable, and so scattered (see [9, Proposition 8.5.7]) for every $\varepsilon > 0$. By an application of Theorem 2, (iii) ⇒ (i), and Proposition 3, we get (i) and the additional assertion. □

The equivalent conditions (i) and (ii) of Theorem 3 are not implied by its additional assertion. This is seen, by taking for K an arbitrary uncountable compact scattered space and $x = 1_K$, and applying Theorem 2.

Remark 5. The implication (i) ⇒ (ii) of Theorem 3 extends to an arbitrary completely regular space X and $x \in C_b(K)_+$. This is seen, by using βK as in Remark 4. However, the converse implication fails, as assertion (4) of Proposition 2 shows.

In the special case where $x = 1_K$, the equivalence of conditions (i) and (ii) of Theorem 3 also follows from [1, Theorem 10.8] and the Riesz representation theorem.

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Z. Lipecki:

Institute of Mathematics, Polish Academy of Sciences, Wrocław Branch, Kopernika 18, 51-617 Wrocław, Poland

E-mail: lipecki@impan.pan.wroc.pl

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