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Mathematica Bohemica, Vol. 147 (2022), No. 4, 471–484

Persistent URL: http://dml.cz/dmlcz/151093

GLOBAL EXISTENCE AND STABILITY OF SOLUTION FOR A NONLINEAR KIRCHHOFF TYPE REACTION-DIFFUSION EQUATION WITH VARIABLE EXPONENTS

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Received July 8, 2020. Published online November 16, 2021.
Communicated by Ondřej Kreml

Abstract. We consider a class of Kirchhoff type reaction-diffusion equations with variable exponents and source terms

$$u_t - M\Bigl(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\Bigr)\Delta u + |u|^{m(x)-2}u_t = |u|^{r(x)-2}u.$$

We prove with suitable assumptions on the variable exponents $r(\cdot)$, $m(\cdot)$ the global existence of the solution and a stability result using potential and Nihari's functionals with small positive initial energy, the stability being based on Komornik's inequality.

Keywords: Kirchhoff equation; reaction-diffusion equation; variable exponent; global solution

MSC 2020: 35B40, 35L70, 35L10

1. Introduction

We consider the initial-boundary value problem

$$(1.1)\quad \begin{cases} u_t - M\Bigl(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\Bigr)\Delta u + |u|^{m(x)-2}u_t = |u|^{r(x)-2}u, & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n\geqslant 1$, with smooth boundary $\partial\Omega$ and $M(s) = a + bs^\gamma$ with positive parameters a, b, γ. Further, $r(\cdot)$ and $m(\cdot)$ are given measurable

DOI: 10.21136/MB.2021.0122-20

functions on Ω, satisfying

$$(1.2)\qquad \begin{aligned} 2 \leqslant q_1 \leqslant q(x) \leqslant q_2 < \frac{2n}{n-2} \quad & \text{if } n \geqslant 3,\\ q(x) \geqslant 2 \qquad\qquad & \text{if } n = 1,2. \end{aligned}$$

We also assume that $m(\cdot)$ and $r(\cdot)$ satisfy the log-Hölder continuity condition:

$$|q(x)-q(y)| \leqslant -\frac{A}{\log|x-y|} \quad \text{for a.e. } x,y\in\Omega,\ |x-y|<\delta \text{ with } A>0,\ 0<\delta<1.$$

Equation (1.1) appears in various physical contexts. In particular, this equation arises from the mathematical description of the reaction-diffusion or diffusion, heat transfer, and population dynamic processes (see [10]).

In the last few years, partial equations with different kinds of nonlocal terms have drawn more and more attention because of their wide applications in both physics and biology. For example, the hyperbolic equation with a nonlocal coefficient is

$$(1.3)\qquad \varepsilon u^{\varepsilon}_{tt} + u^{\varepsilon}_t - M\Big(\int_\Omega |\nabla u^\varepsilon|^p \,\mathrm{d}x\Big)\Delta_p u^\varepsilon = f(x,t,u^\varepsilon),$$

where $M(s)=a+bs$, $a>0$, $b>0$ and $p>1$. In a bounded domain $\Omega\subset\mathbb{R}^n$ it is a potential model for damped small transversal vibrations of an elastic string with uniform density ε (see [8]). For $p=2$, such nonlocal equations were first proposed by Kirchhoff in 1883 (see [11]) and therefore they were usually referred to as Kirchhoff equations. In the case $\varepsilon=0$, (1.3) becomes a Kirchhoff type parabolic equation

$$(1.4)\qquad u_t - M\Big(\int_\Omega |\nabla u|^p\,\mathrm{d}x\Big)\Delta_p u = f(x,t,u).$$

Equation (1.4) can also be used to describe the motion of a nonstationary fluid or gas in a nonhomogeneous and anisotropic medium, and the nonlocal term M appearing in (1.4) can describe a possible change in the global state of the fluid or gas caused by its motion in the considered medium (see [5]). In [16], Li and Han studied the p-Kirchhoff equation

$$(1.5)\qquad u_t - \Big(a + b\int_\Omega |\nabla u|^p\,\mathrm{d}x\Big)\Delta_p u = |u|^{q-1}u,$$

where a, b are two positive constants, $p>\max\{2n/(n+1),1\}$, $2p-1<q<p^*-1$, and p^* is the Sobolev conjugate of p. They proved the global existence and finite time blow-up of solutions. Also in [15], Haixia studied the same equation where the

source term is a function depending on u and satisfying some conditions. He proved the blow-up of solutions and the results generalize some recent ones reported by Han and Li (see [9]). In [23], Polat studied a 1D problem and he established a blow-up result for the solution with vanishing initial energy of the equation

$$u_t - \Delta u_{xx} + |u|^{m-2} u_t = |u|^{r-2} u. \tag{1.6}$$

Ouaoua and Maouni, in [22], considered the following nonlinear parabolic equation with the $p(x)$-Laplacian

$$u_t - \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) + \omega |u|^{m(x)-2} u_t = b|u|^{r(x)-2} u. \tag{1.7}$$

They proved a finite blow-up result for the solutions in the case $\omega = 0$ and exponential growth in the case $\omega > 0$ with negative initial energy. Many authors have studied the existence and nonexistence of solutions for the problem with variable exponents or constants, see [1]–[3], [6], [7], [13], [14], [18]–[21], [24], [25].

In Section 2 of the paper, we recall the definitions of the variable exponent Lebesgue space $L^{q(\cdot)}(\Omega)$, the Sobolev space and $W^{1,q(\cdot)}(\Omega)$ as well as some of their properties. In Section 3, we prove that the local solution is global in time. In Section 4, we state and prove our main result.

2. Preliminaries

We begin this section with some notations and definitions. Denote the $L^p(\Omega)$ norm of a Lebesgue function $u \in L^p(\Omega)$ by $\|\cdot\|_p$. We use the well-known Sobolev space $W_0^{1,p}(\Omega)$ such that u and $|\nabla u|$ are in $L^p(\Omega)$ and are equipped with the norm $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p$.

Let $q\colon \Omega \to [1, \infty]$ be a measurable function, where Ω is a domain in $\mathbb{R}^n$. We define the Lebesgue space with a variable exponent $q(\cdot)$ by

$$L^{q(\cdot)}(\Omega) := \{v\colon \Omega \to \mathbb{R} \text{ measurable in } \Omega,\ \varrho_{q(\cdot)}(\lambda v) < \infty \text{ for some } \lambda > 0\},$$

where $\varrho_{q(\cdot)}(v) = \int_\Omega |v(x)|^{q(x)}\,\mathrm{d}x$. The set $L^{q(\cdot)}(\Omega)$ is equipped with the norm (Luxemburg's norm)

$$\|v\|_{q(\cdot)} := \inf\biggl\{\lambda > 0\colon \int_\Omega \Bigl|\frac{v(x)}{\lambda}\Bigr|^{q(x)}\,\mathrm{d}x \leqslant 1\biggr\},$$

thus $L^{q(\cdot)}(\Omega)$ is a Banach space (see [4]).

Next we define the variable-exponent Sobolev space $W^{1,q(\cdot)}(\Omega)$ as

$$W^{1,q(\cdot)}(\Omega) := \{v \in L^{q(\cdot)}(\Omega) \text{ such that } \nabla v \text{ exists and } |\nabla v| \in L^{q(\cdot)}(\Omega)\}.$$

This is a Banach space with respect to the norm $\|v\|_{W^{1,q(\cdot)}(\Omega)} = \|v\|_{q(\cdot)} + \|\nabla v\|_{q(\cdot)}$.

Furthermore, we set $W_0^{1,q(\cdot)}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in the space $W^{1,q(\cdot)}(\Omega)$. Let us note that the space $W_0^{1,q(\cdot)}(\Omega)$ has a different definition in the case of variable exponents.

However, under the log-Hölder continuity condition, both the definitions are equivalent, see [4]. The space $W^{-1,q'(\cdot)}(\Omega)$, dual of $W_0^{1,q(\cdot)}(\Omega)$, is defined in the same way as the classical Sobolev spaces, where $1/q(\cdot) + 1/q'(\cdot) = 1$.

Lemma 1 ([4])**.** *If*

$$1 \leqslant q_1 := \operatorname*{ess\,inf}_{x\in\Omega} q(x) \leqslant q(x) \leqslant q_2 := \operatorname*{ess\,sup}_{x\in\Omega} q(x) < \infty,$$

then we have

$$\min\{\|u\|_{q(\cdot)}^{q_1}, \|u\|_{q(\cdot)}^{q_2}\} \leqslant \varrho_{q(\cdot)}(u) \leqslant \max\{\|u\|_{q(\cdot)}^{q_1}, \|u\|_{q(\cdot)}^{q_2}\}$$

for any $u \in L^{q(\cdot)}(\Omega)$.

Lemma 2 (Hölder's inequality, see [4])**.** *Suppose that* $p, q, s \geqslant 1$ *are measurable functions defined on* Ω *such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \quad \text{for a.e. } y \in \Omega.$$

If $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{q(\cdot)}(\Omega)$, *then* $uv \in L^{s(\cdot)}(\Omega)$ *with*

$$\|uv\|_{s(\cdot)} \leqslant c\|u\|_{p(\cdot)}\|v\|_{q(\cdot)}.$$

Lemma 3 ([4])**.** *If* $q\colon \Omega \to [1,\infty)$ *is a measurable function satisfying* (1.2) *then the embedding* $H_0^1(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ *is continuous and compact.*

Now, we state the key lemma for our problem.

Lemma 4 ([12])**.** *Let* $G\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be a non-increasing function and assume that there are two constants* $\alpha > 0$ *and* $C > 0$ *such that*

$$\int_t^\infty G^{\alpha+1}(s)\,\mathrm{d}s \leqslant CG^\alpha(0)G(s) \quad \forall\, t \in \mathbb{R}_+.$$

Then we have

$$G(t) \leqslant G(0)\Big(\frac{C+\alpha t}{C+\alpha C}\Big)^{-1/\alpha} \quad \forall\, t \geqslant C.$$

3. Global existence of solution

In order to state and prove our result, we define the potential energy and Nehari's functionals as

$$E(t) = E(u(t)) = \frac{a}{2}\|\nabla u(t)\|_2^2 + \frac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)}\,\mathrm{d}x, \tag{3.1}$$

$$I(t) = I(u(t)) = a\|\nabla u(t)\|_2^2 + b\|\nabla u(t)\|_2^{2(\gamma+1)} - \int_\Omega |u(t)|^{r(x)}\,\mathrm{d}x. \tag{3.2}$$

In the following, we consider $a = b = 1$ and this does not change the general result.

Lemma 5. *Under assumptions* (1.2), *we have*

$$E'(t) = -\|u_t(t)\|_2^2 - \int_\Omega |u(t)|^{m(x)-2}|u_t(t)|^2\,\mathrm{d}x \leqslant 0, \quad t \in [0,T], \tag{3.3}$$

and

$$E(t) \leqslant E(0).$$

P r o o f. We multiply the first equation of (1.1) by u_t and integrating over the domain Ω, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)}\,\mathrm{d}x\Big)\\ = -\|u_t(t)\|_2^2 - \int_\Omega |u(t)|^{m(x)-2}|u_t(t)|^2\,\mathrm{d}x,\end{aligned}$$

thus

$$E'(t) = -\|u_t(t)\|_2^2 - \int_\Omega |u(t)|^{m(x)-2}|u_t(t)|^2\,\mathrm{d}x \leqslant 0.$$

Integrating (3.3) over $(0,T)$, we obtain $E(t) \leqslant E(0)$. □

Lemma 6. *Let assumptions* (1.2) *hold, and* $r_1 > 2(\gamma+1)$, $I(0) > 0$ *and*

$$\beta_1 + \beta_2 < 1, \tag{3.4}$$

where

$$\begin{aligned}\beta_1 &:= \max\Big\{\alpha c_*^{r_1}\Big(\frac{2r_1}{r_1-2}E(0)\Big)^{(r_1-2)/2}, \alpha c_*^{r_2}\Big(\frac{pr_1}{r_1-p}E(0)\Big)^{(r_2-2)/2}\Big\},\\ \beta_2 &:= \max\Big\{(1-\alpha)c_*^{r_1}\Big(\frac{2(\gamma+1)r_1}{r_1-2(\gamma+1)}E(0)\Big)^{(r_1-2(\gamma+1))/(2(\gamma+1))},\\ &\qquad (1-\alpha)c_*^{r_2}\Big(\frac{2(\gamma+1)r_1}{r_1-2(\gamma+1)}E(0)\Big)^{(r_2-2(\gamma+1))/(2(\gamma+1))}\Big\}\end{aligned}$$

with $0 < \alpha < 1$ *and* c_* *is the best embedding constant of* $H_0^1(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$. *Then* $I(t) > 0$ *for all* $t \in [0,T]$.

P r o o f. Since $I(0) > 0$, then by continuity there exists T_* such that

$$I(t) \geqslant 0 \quad \forall\, t \in [0, T_*]. \tag{3.5}$$

Now, we have for all $t \in [0, T]$ that

$$\begin{aligned} E(t) = E(u(t)) &= \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)}\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \frac{1}{r_1}(\|\nabla u(t)\|_2^2 + \|\nabla u(t)\|_2^{2(\gamma+1)} - I(t)) \\ &\geqslant \frac{r_1-2}{2r_1}\|\nabla u(t)\|_2^2 + \frac{r_1 - 2(\gamma+1)}{2(\gamma+1)r_1}\|\nabla u(t)\|_2^{2(\gamma+1)} + \frac{1}{r_1}I(t). \end{aligned}$$

Using (3.5), we obtain

$$\frac{r_1-2}{2r_1}\|\nabla u(t)\|_2^2 + \frac{r_1 - 2(\gamma+1)}{2(\gamma+1)r_1}\|\nabla u(t)\|_2^{2(\gamma+1)} \leqslant E(t) \quad \forall\, t \in [0, T_*]. \tag{3.6}$$

By the definition of E, we get

$$\|\nabla u(t)\|_2^2 \leqslant \frac{2r_1}{r_1-2}E(t) \leqslant \frac{2r_1}{r_1-2}E(0) \tag{3.7}$$

and

$$\|\nabla u(t)\|_2^{2(\gamma+1)} \leqslant \frac{2(\gamma+1)r_1}{r_1 - 2(\gamma+1)}E(t) \leqslant \frac{2(\gamma+1)r_1}{r_1 - 2(\gamma+1)}E(0). \tag{3.8}$$

On the other hand, by Lemma 1, we have

$$\begin{aligned} \int_\Omega |u(t)|^{r(x)}\,\mathrm{d}x &\leqslant \max\{\|u(t)\|_{r(\cdot)}^{r_1}, \|u(t)\|_{r(\cdot)}^{r_2}\} \\ &= \alpha \max\{\|u(t)\|_{r(\cdot)}^{r_1}, \|u(t)\|_{r(\cdot)}^{r_2}\} + (1-\alpha)\max\{\|u(t)\|_{r(\cdot)}^{r_1}, \|u(t)\|_{r(\cdot)}^{r_2}\}. \end{aligned}$$

By the embedding of $H_0^1(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$, we obtain

$$\begin{aligned} &\int_\Omega |u(t)|^{r(x)}\,\mathrm{d}x \\ &\quad\leqslant \alpha \max\{c_*^{r_1}\|\nabla u(t)\|_2^{r_1}, c_*^{r_2}\|\nabla u(t)\|_2^{r_2}\} \\ &\qquad + (1-\alpha)\max\{c_*^{r_1}\|\nabla u(t)\|_2^{r_1}, c_*^{r_2}\|\nabla u(t)\|_2^{r_2}\} \\ &\quad\leqslant \alpha \max\{c_*^{r_1}\|\nabla u(t)\|_2^{r_1-2}, c_*^{r_2}\|\nabla u(t)\|_2^{r_2-2}\}\|\nabla u(t)\|_2^2 \\ &\qquad + (1-\alpha)\max\{c_*^{r_1}\|\nabla u(t)\|_2^{r_1-2(\gamma+1)}, c_*^{r_2}\|\nabla u(t)\|_2^{r_2-2(\gamma+1)}\}\|\nabla u(t)\|_2^{2(\gamma+1)}. \end{aligned}$$

By (3.7) and (3.8), we get

$$\int_\Omega |u(t)|^{r(x)}\,\mathrm{d}x \leqslant \beta_1\|\nabla u(t)\|_2^2 + \beta_2\|\nabla u(t)\|_2^{2(\gamma+1)} \quad \forall\, t\in[0,T_*]. \tag{3.9}$$

Since $\beta_1+\beta_2<1$, then

$$\int_\Omega |u(t)|^{r(x)}\,\mathrm{d}x < \|\nabla u(t)\|_2^2 + \|\nabla u(t)\|_2^{2(\gamma+1)} \quad \forall\, t\in[0,T_*]. \tag{3.10}$$

This implies that

$$I(t)>0 \quad \forall\, t\in[0,T_*].$$

Repeating the above procedure, we can extend T_* to T. □

Theorem 7 (Existence of weak solution)**.** *Assume that* (1.2) *holds. Let* $u_0\in L^2(\Omega)$ *be given. We also assume that* $m(\cdot)$ *and* $r(\cdot)$ *satisfy the log-Hölder continuity condition. Then problem* (1.1) *has a weak local solution*

$$u\in L^\infty((0,T),H_0^1(\Omega)), \quad u_t\in L^2((0,T),L^2(\Omega)).$$

P r o o f. We will use the Faedo-Galerkin method of approximation. Let $\{v_l\}_{l=1}^\infty$ be a basis of $H_0^1(\Omega)$ which forms a complete orthonormal system in $L^2(\Omega)$. Denote by $V_k=\mathrm{span}\{v_1,v_2,\ldots,v_k\}$ the subspace generated by the first k vectors of the basis $\{v_l\}_{l=1}^\infty$. After normalization, we have $\|v_l\|=1$ and for any given integer k, we consider the approximate solution

$$u_k(t)=\sum_{l=1}^k u_{lk}(t)v_l,$$

where u_k are the solutions to the Cauchy problem

$$\begin{aligned} &(u_k'(t),v_l)+\Big(M\Big(\int_\Omega|\nabla u_k(t)|^2\,\mathrm{d}x\Big)\Delta u_k(t),v_l\Big)+(|u_k(t)|^{m(x)-2}u_k'(t),v_l)\\ &\qquad=(|u_k(t)|^{r(x)-2}u_k(t),v_l), \quad l=1,2,\ldots,k, \end{aligned} \tag{3.11}$$

$$u_k(0)=u_{0k}=\sum_{i=1}^k(u_k(0),v_l)v_l\to u_0 \quad \text{in } L^2(\Omega). \tag{3.12}$$

Note that we can solve the system (3.11) and (3.12) by Picard's iterative method for ordinary differential equations. Hence, there exists a solution in $[0,T_*)$ for some $T_*>0$ and we can extend this solution to the whole interval $[0,T]$ for any given $T>0$ by making use of the a priori estimates below.

Multiplying equation (3.11) by $u'_{lk}(t)$ and summing over l from 1 to k, we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}\|\nabla u_k(t)\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u_k(t)\|_2^{2(\gamma+1)}-\int_\Omega \frac{1}{r(x)}|u_k(t)|^{r(x)}\,\mathrm{d}x\right)\mathrm{d}x = -\|u_{t,k}(t)\|_2^2-\int_\Omega |u_k(t)|^{m(x)-2}|u_{t,k}(t)|^2\,\mathrm{d}x. \tag{3.13}$$

Then

$$E'(u_k(t)) = -\|u_{t,k}(t)\|_2^2-\int_\Omega |u_k(t)|^{m(x)-2}|u_{t,k}(t)|^2\,\mathrm{d}x \leqslant 0.$$

Integrating (3.13) over $(0,T)$, we obtain the estimate

$$\frac{1}{2}\|\nabla u_k(t)\|_2^2+\frac{1}{2(\gamma+1)}\|\nabla u_k(t)\|_2^{2(\gamma+1)}-\int_\Omega \frac{1}{r(x)}|u_k(t)|^{r(x)}\,\mathrm{d}x + \int_0^t \|u_{t,k}(s)\|_2^2\,\mathrm{d}s+\int_0^t\int_\Omega |u_k(s)|^{m(x)-2}|u_{t,k}(s)|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant E(0). \tag{3.14}$$

Then, from (3.10), inequality (3.14) becomes

$$\frac{r_1-2}{2r_1}\sup_{t\in(0,T)}\|\nabla u_k(t)\|_2^2+\frac{r_1-2(\gamma+1)}{2r_1(\gamma+1)}\sup_{t\in(0,T)}\|\nabla u_k(t)\|_2^{2(\gamma+1)}\int_0^t \|u_{t,k}(s)\|_2^2\,\mathrm{d}s + \int_0^t\int_\Omega |u_k(s)|^{m(x)-2}|u_{t,k}(s)|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant E(0). \tag{3.15}$$

From (3.15), we conclude that

$$\begin{cases} \{u_k\} \text{ is uniformly bounded in } L^\infty([0,T],H_0^1(\Omega)),\\ \{u'_k\} \text{ is uniformly bounded in } L^2([0,T],L^2(\Omega)). \end{cases} \tag{3.16}$$

Furthemore, we have from Lemma 3 and (3.16) that

$$\begin{cases} \{|u_k|^{r(x)-2}u_k\} \text{ is uniformly bounded in } L^\infty([0,T],L^2(\Omega)),\\ \{|u_k|^{m(x)-2}u'_k\} \text{ is uniformly bounded in } L^\infty([0,T],L^2(\Omega)). \end{cases} \tag{3.17}$$

By (3.16) and (3.17), we infer that there exist a subsequence of u_k (denoted by the same symbol) and a function u such that

$$\begin{cases} u_k \rightharpoonup u \text{ weakly star in } L^\infty([0,T],H_0^1(\Omega)),\\ u'_k \rightharpoonup u' \text{ weakly star in } L^2([0,T],L^2(\Omega)),\\ |u_k|^{r(x)-2}u_k \rightharpoonup \varphi \text{ weakly in } L^\infty([0,T],L^2(\Omega)),\\ |u_k|^{m(x)-2}u'_k \rightharpoonup \psi \text{ weakly in } L^\infty([0,T],L^2(\Omega)). \end{cases} \tag{3.18}$$

By the Aubin-Lions compactness lemma (see [17]), we conclude from (3.18) that

$$u_k \rightharpoonup u \text{ strongly in } C([0,T], H_0^1(\Omega)),$$

which implies

$$u_k \rightharpoonup u \text{ everywhere in } \Omega \times [0,T]. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\begin{cases} |u_k|^{r(x)-2}u_k \rightharpoonup |u|^{r(x)-2}u \text{ weakly in } L^\infty([0,T], L^2(\Omega)), \\ |u_k|^{m(x)-2}u_k \rightharpoonup |u|^{m(x)-2}u' \text{ weakly in } L^\infty([0,T], L^2(\Omega)). \end{cases} \tag{3.20}$$

Letting $k \to \infty$ and passing to the limit in (3.11), we obtain

$$\begin{aligned} (u'(t), v_l) + \biggl(M\biggl(\int_\Omega |\nabla u(t)|^2 \,\mathrm{d}x\biggr)\Delta u(t), v_l\biggr) + (|u(t)|^{m(x)-2}u'(t), v_l) \\ = (|u(t)|^{r(x)-2}u(t), v_l), \quad l = 1, 2, \ldots, k. \end{aligned} \tag{3.21}$$

Since $\{v_l\}_{l=1}^\infty$ is a basis of $H_0^1(\Omega)$, we deduce that u satisfies equation (1.1). From (3.18) and Lemma 3.1.7 of [26] with $B = L^2(\Omega)$, we infer that

$$u_k(0) \rightharpoonup u(0) \text{ weakly in } L^2(\Omega). \tag{3.22}$$

We get from (3.12) and (3.22) that $u(0) = u_0$. Thus, the proof is complete. □

Theorem 8. *Under the assumptions of Lemma* 6, *the local solution of* (1.1) *is global.*

Proof. We have

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \int_\Omega \frac{1}{r(x)}|u(t)|^{r(x)} \,\mathrm{d}x \\ &\geqslant \frac{r_1 - 2}{2r_1}\|\nabla u(t)\|_2^2 + \frac{r_1 - 2(\gamma+1)}{2(\gamma+1)r_1}\|\nabla u(t)\|_2^{2(\gamma+1)}. \end{aligned}$$

Then

$$\|\nabla u(t)\|_2^2 \leqslant CE(t). \tag{3.23}$$

By Lemma 5, we obtain

$$\|\nabla u(t)\|_2^2 \leqslant CE(0). \tag{3.24}$$

This implies that the local solution is global in time. □

4. Stability of solution

In this section our main result is established based on Komornik's inequality (Theorem 9.1 of [12]). For this, we need the following lemma.

Lemma 9. *Suppose that the assumptions of Lemma 6 hold, then there exists a positive constant c such that*

$$\int_\Omega |u(t)|^{m(x)}\,\mathrm{d}x \leqslant cE(t). \tag{4.1}$$

Proof.

$$\begin{aligned}\int_\Omega |u(t)|^{m(x)}\,\mathrm{d}x &= \max\{\|u(t)\|_{m(\cdot)}^{m_1}, \|u(t)\|_{m(\cdot)}^{m_2}\}\\ &\leqslant \max\{c_*^{m_1}\|\nabla u(t)\|_2^{m_1}, c_*^{m_2}\|\nabla u(t)\|_2^{m_2}\}\\ &\leqslant \max\{c_*^{m_1}\|\nabla u(t)\|_2^{m_1-2}, c_*^{m_2}\|\nabla u(t)\|_2^{m_2-2}\}\|\nabla u(t)\|_2^2.\end{aligned}$$

Using (3.7), we obtain

$$\int_\Omega |u(t)|^{m(x)}\,\mathrm{d}x \leqslant cE(t).$$

□

Now, we state our main result

Theorem 10. *Let the assumptions of Lemma 6 hold, then there exists a constant $C > 0$ such that*

$$E(t) \leqslant E(0)\Big(\frac{C+qt}{C+qC}\Big)^{-1/q} \quad \forall\, t \geqslant C.$$

Proof. Multiplying the first equation of (1.1) by $u(t)E^q(t)$ $(q > 0)$ and integrating over $\Omega \times (S,T)$, we obtain

$$\int_S^T \int_\Omega E^q(t)\Big(u(t)u_t(t) - u(t)\Big(M\Big(\int_\Omega |\nabla u|^2\,\mathrm{d}x\Big)\Delta u + |u|^{m(x)-2}u_t\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_S^T E^q(t)\int_\Omega |u(t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t.$$

Then

$$\int_S^T \int_\Omega E^q(t)(u(t)u_t(t) + |\nabla u(t)|^2 + \|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2 + u(t)|u|^{m(x)-2}u_t)\,\mathrm{d}x\,\mathrm{d}t = \int_S^T E^q(t)\int_\Omega |u(t)|^{r(x)}\,\mathrm{d}x\,\mathrm{d}t.$$

We add and substract the term

$$\int_S^T E^q(t)\int_\Omega(\beta_1|\nabla u(t)|^2+\beta_2\|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2)\,\mathrm{d}x\,\mathrm{d}t$$

and use (3.9) to get

$$\begin{aligned}(4.2)\quad (1-\beta_1)&\int_S^T E^q(t)\int_\Omega(|\nabla u(t)|^2)\,\mathrm{d}x\,\mathrm{d}t\\ &+(1-\beta_2)\int_S^T E^q(t)\int_\Omega(\|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2)\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_S^T E^q(t)\int_\Omega(u(t)u_t(t))\,\mathrm{d}x\,\mathrm{d}t\\ &+\int_S^T E^q(t)\int_\Omega u(t)u_t(t)|u(t)|^{m(x)-2}\,\mathrm{d}x\,\mathrm{d}t\\ &=-\int_S^T E^q(t)\int_\Omega(\beta_1|\nabla u(t)|^2+\beta_2\|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2-|u(t)|^{r(x)})\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant 0.\end{aligned}$$

It is clear that

$$\begin{aligned}(4.3)\quad \xi\int_S^T E^q(t)\int_\Omega\Big(\frac{1}{2}|\nabla u(t)|^2+\frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2-\frac{|u(t)|^{r(x)}}{r(x)}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ \leqslant(1-\beta_1)\int_S^T E^q(t)\int_\Omega\frac{1}{2}|\nabla u(t)|^2\,\mathrm{d}x\,\mathrm{d}t\\ +(1-\beta_2)\int_S^T E^q(t)\int_\Omega\frac{1}{2(\gamma+1)}\|\nabla u(t)\|_2^{2\gamma}|\nabla u(t)|^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $\xi=\min((1-\beta_1),(1-\beta_2))$. By (4.2), (4.3) and the definition of $E(t)$, we get

$$\begin{aligned}(4.4)\qquad \xi\int_S^T E^{q+1}(t)\,\mathrm{d}t\leqslant&-\int_S^T E^q(t)\int_\Omega u(t)u_t(t)\,\mathrm{d}x\,\mathrm{d}t\\ &-\int_S^T E^q(t)\int_\Omega u(t)u_t(t)|u(t)|^{m(x)-2}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We estimate the terms on the right-hand side of (4.4). For the first term, we use the Young inequality

$$XY\leqslant\frac{\varepsilon}{\lambda_1}X^{\lambda_1}+\frac{1}{\lambda_2\varepsilon^{\lambda_2/\lambda_1}}Y^{\lambda_2},\quad X,Y\geqslant 0,\quad \varepsilon>0\quad\text{and}\quad\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1,$$

and get

$$(4.5)\qquad -\int_S^T E^q(t)\int_\Omega u(t)u_t(t)\,\mathrm{d}x\,\mathrm{d}t\leqslant\int_S^T E^q(t)\int_\Omega(\varepsilon c|u(t)|^2+c_\varepsilon|u_t(t)|^2)\,\mathrm{d}x\,\mathrm{d}t.$$

We use again the above Young inequality to obtain

$$(4.6)\quad -\int_S^T E^q(t)\int_\Omega u(t)u_t(t)|u(t)|^{m(x)-2}\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_S^T E^q(t)\int_\Omega |u(t)|^{(m(x)-2)/2}u_t(t)|u(t)|^{(m(x)-2)/2}u(t)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \int_S^T E^q(t)\int_\Omega (\varepsilon c|u(t)|^{m(x)} + c_\varepsilon |u(t)|^{m(x)-2}u_t^2(t))\,\mathrm{d}x\,\mathrm{d}t.$$

By (4.5) and (4.6), inequality (4.4) becomes

$$(4.7)\quad \xi\int_S^T E^{q+1}(t)\,\mathrm{d}t \leqslant \int_S^T E^q(t)\int_\Omega (\varepsilon c|u(t)|^2 + c_\varepsilon|u_t(t)|^2)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_S^T E^q(t)\int_\Omega (\varepsilon c|u(t)|^{m(x)} + c_\varepsilon |u(t)|^{m(x)-2}u_t^2(t))\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \varepsilon c\int_S^T E^q(t)\int_\Omega (|u(t)|^2 + |u(t)|^{m(x)})\,\mathrm{d}x\,\mathrm{d}t$$
$$+ c_\varepsilon\int_S^T E^q(t)\int_\Omega (|u_t(t)|^2 + |u(t)|^{m(x)-2}u_t^2(t))\,\mathrm{d}x\,\mathrm{d}t.$$

We use (3.23), Lemma 9 and definition of $E'(t)$ to obtain

$$(4.8)\quad \xi\int_S^T E^{q+1}(t)\,\mathrm{d}t \leqslant \varepsilon c\int_S^T E^{q+1}(t)\,\mathrm{d}t + c_\varepsilon\int_S^T E^q(t)(-E'(t))\,\mathrm{d}t.$$

This implies

$$(4.9)\quad \xi\int_S^T E^{q+1}(t)\,\mathrm{d}t \leqslant \varepsilon c\int_S^T E^{q+1}(t)\,\mathrm{d}t + c_\varepsilon(E^{q+1}(s) - E^{q+1}(T))$$
$$\leqslant \varepsilon c\int_S^T E^{q+1}(t)\,\mathrm{d}t + c_\varepsilon E^q(0)E(s).$$

Choosing ε so small that $\xi > \varepsilon c$, we arrive at

$$\int_S^T E^{q+1}(t)\,\mathrm{d}t \leqslant cE^q(0)E(s).$$

Letting $T\to\infty$, we get

$$\int_S^\infty E^{q+1}(t)\,\mathrm{d}t \leqslant cE^q(0)E(s).$$

Komornik's inequality yields the result. □

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