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Abstract. In a Tychonoff space X, the point $p \in X$ is called a C^*-point if every real-valued continuous function on $C \smallsetminus \{p\}$ can be extended continuously to p. Every point in an extremally disconnected space is a C^*-point. A classic example is the space $\mathbf{W}^* = \omega_1 + 1$ consisting of the countable ordinals together with ω_1. The point ω_1 is known to be a C^*-point as well as a P-point. We supply a characterization of C^*-points in totally ordered spaces. The remainder of our time is aimed at studying when a point in a product space is a C^*-point. This process leads to many interesting new discoveries.

Keywords: ring of continuous functions; C^*-embedded; P-point

Classification: 54G10, 54D15, 54F05

1. Introduction

The work for this article began more than 20 years ago. As it happens sometimes, the work was stored away in some drawer and lost to the ravages of time. I found the paper about two years ago and with the decline of my co-author's health I felt it prudent to work on it. I had hoped to finish the project as I was given permission by Jorge to work on it. Now, with the recent passing of J. Martinez (1945–2020) I felt even a stronger desire to complete the project.

One of the well-known theorems of ordered spaces is that a point is a P-point precisely when it is not the limit of an (nontrivial) ascending or descending sequence of points. Disciples of the text [5] are familiar with the example $\mathbf{W}^* = \omega_1 + 1$ consisting of the countable ordinals together with the first uncountable ordinal. In this case, the point ω_1 is in fact a P-point of $\mathbf{W}^*$. The proof of this involves a demonstration that every continuous function on ω_1 is eventually constant and therefore has a continuous extension to the point ω_1. In other words, the Čech–Stone compactification of ω_1 is $\omega_1 + 1$. Specifically, this means that ω_1 is also a C^*-point of $\mathbf{W}^*$. It is this fact that drives this article.

All spaces here are assumed to be Tychonoff, that is completely regular and Hausdorff. For a given space X, $C(X)$ denotes the ring of continuous real-valued functions defined on X and βX is its Čech–Stone compactification. For

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$f \in C(X)$, recall that the cozeroset of f is

$$\operatorname{coz}(f) = \{x \in X : f(x) \neq 0\},$$

and that $Z(f)$, the zeroset of f, is its complement. The subring of $C(X)$ consisting of those bounded functions shall be denoted by $C^*(X)$. A subspace Y of X is said to be C^*-*embedded* (or C-*embedded*) if every $f \in C^*(Y)$ ($f \in C(Y)$, respectively) can be extended to a continuous function on all of X. For any unexplained topological terminology we refer the reader to either [5] or [4].

Definition 1.1. Let X be a space and $p \in X$. If $X \smallsetminus \{p\}$ is C^*-embedded in X, then p is called a C^*-*point* of X. If each point of X is a C^*-point of X, we say that X is a C^*-space. If we change the phrase C^*-embedded in the above definition to C-embedded, then we have a definition of a C-point and C-space. Since a C-embedded subset is C^*-embedded it follows that a C-point is a C^*-point.

Recall the notion of an e.d. point, invented by E. van Douwen, see [2]. Thus $p \in X$ is an *e.d. point* of X if for each pair of disjoint opens sets, p fails to be in the closure of one of them. Van Douwen showed that an e.d. point is a C^*-point. The converse is false as witnessed by the point $\omega_1 \in \mathbf{W}^*$.

The difference between a C-point and C^*-point is seen in the next result.

Proposition 1.2 ([6, Theorem 4.2])**.** *Let X be a space. The point $p \in X$ is a C-point if and only if $p \in X$ is a C^*-point and not a G_δ-point of X.*

Example 1.3. The example $\Sigma = \mathbb{N} \cup \{\sigma\}$ for some $\sigma \in \beta\mathbb{N} \smallsetminus \mathbb{N}$ is extremally disconnected and hence a C^*-space. The point σ is a G_δ-point and hence Σ is not a C-space.

On the other hand, every compact extremally disconnected space has the property that the only G_δ-points are isolated and hence are C-spaces.

In general, for any point $p \in \beta X \smallsetminus X$ the point p is a C-point of βX and a C^*-point of $X \cup \{p\}$.

The following is not surprising.

Lemma 1.4. *Suppose that X is a Tychonoff space. Then $p \in X$ is a C^*-point in X if and only if it is a C^*-point in any neighborhood U of p.*

2. C^*-points in totally ordered spaces

We begin by recalling a familiar definition.

Definition 2.1. Let X be a space. The point $p \in X$ is a P-*point* if any countable intersection of neighborhoods of p is also a neighborhood of p.

Not every P-point is a C^*-point. For example, consider an uncountable discrete space D, and let λD denote the space $D \cup \{\lambda\}$ obtained by adjoining the point λ, such that each $d \in D$ remains isolated, and the neighborhoods of λ are the sets S for which $D \setminus S$ is countable. Then it is easy to see that λ is not a C^*-point, but it is a P-point.

The main theorem (Theorem 2.3) of this section will characterize the points in an ordered space which are C^*-points.

First, let us recall the basics about ordered spaces.

Definition & Remarks 2.2. By a totally ordered space $(X, \leq)$ we mean a totally ordered set with the interval topology. For $x, y \in X$ the notation $(-\infty, x)$, $(-\infty, x]$, (x, y), etc. should be understood.

To simplify the discussion leading up to Theorem 2.3 it is useful to introduce some special terminology. First, denote by $(\widehat{X}, \leq)$ the Dedekind cut completion of $(X, \leq)$. A point $p \in \widehat{X} \setminus X$ will be called a *hole in* X. Suggestively, for each $p \in \widehat{X}$ we denote by $(X < p)$ (or $(p < X)$) the set of points of X strictly less (greater, respectively) than p. Note that if p is a hole in X, then it is not isolated in $\widehat{X}$, and both $(X < p)$ and $(p < X)$ are clopen sets of X; moreover, $(X < p)$ has no largest element (in X), nor $(p < X)$ a least element.

A point $x \in X$ is an *embedded point* if it is neither an endpoint – the least or largest element of X – nor a successor, nor a predecessor. Clearly, an embedded point cannot be a C^*-point since the characteristic function on (x, ∞) defined on $X \setminus \{x\}$ has no continuous extension.

One more convention: if $(X, \leq)$ is an ordered space and $f \in C(X)$, we say that f is *constant on a tail of* X, if there is some $x \in X$, x not being the largest element, such that f is constant on the interval (x, ∞).

Theorem 2.3. *Let $(X, \leq)$ be a totally ordered space and $p \in X$. Then p is a C^*-point of X if and only if it is a P-point which is not an embedded point and not the supremum or infimum of holes in X.*

Proof: We first suppose that $p \in X$ is a C^*-point. As we pointed out above p is not an embedded point, and therefore it is a successor or predecessor. Without loss of generality, we assume that p is a predecessor of X, i.e. the subset $(-\infty, p]$ is a clopen subset of X. The point p must also be a P-point, or else it is a G_δ-point of X, which in this case means there is an ascending sequence, say $\{x_n\}$, whose limit is p. Since ordered spaces are normal, the closed subset $\{x_n\}$ of $(-\infty, p)$ is C-embedded in $(-\infty, p)$ and hence also in $X \setminus \{p\}$. Therefore, there is a continuous function $f \in C^*(X \setminus \{p\})$ such that $f(x_n) = 0$ whenever n is even and $f(x_n) = 1$ whenever n is odd. This function cannot be continuously extended to p, contradicting that x is a C^*-point of X. Therefore, p is also a P-point.

Next, suppose that p is a supremum of holes; then it must be a predecessor or else the largest element of X, by the previous paragraph. We may then just as well assume that it is the largest, and also transfinitely choose a well-ordered κ-sequence $(p_\sigma)_{\sigma<\kappa}$ of holes whose limit is p, for some cardinal κ. Let T_i denote the trace on X of the interval $(p_\sigma, p_{\sigma+1})$. Then $(X < p)$ is a topological sum of the subspaces T_i. Let k be the characteristic function of the T_σ for all odd $i \in I$. This is continuous on $(X < p)$, but cannot be extended to p, a contradiction.

Conversely, suppose that p is a P-point which is not an embedded point, and also neither the supremum or infimum of holes. If p is isolated there's nothing to prove. Otherwise, we may suppose without loss of generality that p is the largest point of X. Note that the cofinality of p is uncountable; that is to say, if $S \subseteq X$ is well-ordered and cofinal, then $|S| \geq \omega_1$.

We claim that any continuous function on $X \smallsetminus \{p\}$ is constant on a tail of this space; that is, if $f \in C(X \smallsetminus \{p\})$ there is a $q < p$ in X such that f is constant on the interval (q, p). Let us prove that; a proof that mimics the case for $\mathbf{W}^*$. Since p is not a supremum of holes, there is an $x \in X$ such that there are no holes of X above x. Suppose that the claim is false; let g be a continuous function on Y which is not constant on any tail. We will construct a cofinal, well-ordered sequence S in $Y \equiv X \smallsetminus \{p\}$, such that

(a) S with the relative topology is a totally ordered space (to underscore: with respect to the interval topology!);
(b) $h = g|_S$ is not constant on any tail of Y.

If we can do this we will have reached a contradiction, because such an S, in the relative topology, is an ordered space, homeomorphic to the ordinal μ, where μ is the cofinality of p. And it is well known that any continuous real valued function on a well ordered space is constant on a tail; see 5.12 of [5], for the proof of this for $\mu = \omega_1$.

Let $p_1 = x$. Suppose that for an ordinal ν we have $\{\, p_\mu\colon \mu < \nu \,\}$, such that

(i) for each $\mu < \nu$, the restriction of g to $\{\, p_\alpha\colon \alpha < \mu \,\}$ is not constant on any tail; and
(ii) for each $\mu < \nu$, $\{\, p_\alpha\colon \alpha < \mu \,\}$ is a totally ordered space in the relative topology.

Now let us select p_ν: there are two cases. If ν has a predecessor μ, pick p_ν to be any element of Y larger than p_μ, with $g(p_\nu) \neq g(p_\mu)$. By assumption this can be done. If ν is a limit ordinal, let

$$p_\nu = \sup \{\, p_\mu\colon \mu < \nu \,\},$$

which exists by the choice of p_1. Now in both cases, the enlarged set $\{\, p_\mu\colon \mu \le \nu \,\}$ satisfies (i) and (ii) above. By transfinite induction then, we have constructed $S = \{\, p_\mu\colon \mu < \tau \,\}$, for suitable τ, with the stated properties. □

Armed with Theorem 2.3 it is easy to conclude the following.

Corollary 2.4. *Any totally ordered space $(X, \le)$ which is also a C^*-space is discrete.*

PROOF: Assume the hypotheses of the corollary. If X is not discrete, then let $p \in X$ be a nonisolated point. By Theorem 2.3 we may assume that p is the largest element, and that there is a $q < p$ in X, such that the interval $[q, p]$ is a Dedekind complete, and hence a compact, P-space. Then it must be finite, contradicting that p is not isolated. □

3. C^*-points in product spaces

We begin with a general result, which pits the density of one factor in a product space against, the degree to which the system of neighborhoods of a point in the other is closed under intersections.

Let us first recall some definitions. In what follows κ denotes an infinite cardinal number.

Definition 3.1. Suppose that X is a space and $p \in X$. We say that p is a P^κ-*point* if the intersection of fewer than κ neighborhoods of p is a neighborhood of p. The *density character* $d(Y)$ of a space Y is the least cardinality of a dense subset of Y.

Proposition 3.2. *Let κ be an uncountable cardinal. Suppose that X and Y are spaces, and that $p \in X$ is a nonisolated P^κ-point. If $q \in Y$ is nonisolated as well, and $d(Y) < \kappa$, then (p, q) is a C^*-point of $X \times Y$.*

PROOF: Suppose that S is a dense subset of Y with $q \notin S$ and $|S| < \kappa$. Let $f \in C^*(X \times \{(p, q)\})$. For each $y \in Y \setminus \{q\}$ put $f_y(x) = f(x, y)$ with $x \in X$; then $f_y \in C^*(X)$. Let $r_y = f_y(p)$, and set

$$U = \bigcap_{s \in S} f_s^{-1}(\{r_s\}).$$

Since $p \in X$ is a P^κ-point, we have that U is a neighborhood of p. Thus, for any $x, x' \in U$, $f(x, \cdot) = f(x', \cdot)$ on the dense set S, and therefore everywhere on $Y \setminus \{q\}$. Since $q \in Y$ is not isolated, vertical continuity implies that $f(x, q) = f(x', q)$, as long as $x, x' \neq p$. Moreover, since p is not isolated, there are at least two distinct x and x' in U that witness these coincidences.

Now define $f(p,q) = f(x,q)$ for any $x \in U$ different than p; this is unambiguous. It is easy to see now that f is continuously extended to (p,q); we leave the verification to the reader. □

From the above we have, immediately:

Corollary 3.3. *If X is a P-space without isolated points and Y is a separable space without isolated points, then $X \times Y$ is a C^*-space.*

Remark 3.4. In Proposition 3.2, one cannot weaken the assumption on $p \in X$ and suppose that it is only an almost P-point. First, we remind the reader: $x \in X$ is an *almost P-point* if any zeroset W containing x has nontrivial interior. Equivalently, in any Tychonoff space X, $x \in X$ is an almost P-point if and only if for any $f \in C(X)$ with $x \in Z(f)$ we have $x \in \mathrm{cl}_X \mathrm{int}_X Z(f)$.

Consider now the following totally ordered space. Space X is formed by taking the union of $\omega_1 + 1$, the space of ordinals not exceeding ω_1, and the closed unit interval I, by glueing the points ω_1 and $0 \in I$. Call the new point p. Thus in X, $a < b$ if they both lie in the copy of $\omega_1 + 1$ or the copy of I, and $a < b$ obtains in either of those chains, or $a < \omega_1$ and $0 < b$ in I. In the interval topology, this makes p an almost P-point which is not a P-point.

We claim that $(p,0) \in X \times I$ is not a C^*-point. Consider the square in $X \times I$, homeomorphic to $I \times I$, defined by

$$Y = \{(a,b) \colon a \geq p\}.$$

Since $Y \cong I \times I$, it is a metric space, and thus hereditarily normal. This means that there is a continuous bounded function $g \in C(Y)$, so that the horizontal edge $\{(s,0) \colon s > p\}$ maps to 1 and the vertical edge $\{(p,t) \colon t > 0\}$ maps to 0. Extend g to $(X \times I) \smallsetminus \{(p,0)\}$ by defining it to be zero everywhere else. Clearly, g cannot be extended to $(p,0)$.

Remark 3.5. On the other hand, one may be able to improve on the separability implicit in the assumption of Proposition 3.2 when $\kappa = \omega_1$.

Consider $X = \omega_1 + 1 \times \omega_1 + 1$; it is known that (ω_1, ω_1) is a C^*-point.

We would like to characterize the spaces X for which $X \times I$ is a C^*-space. Theorem 3.8 does this for ordered spaces. Unfortunately, we do not have a complete answer.

The next proposition is a preliminary to that theorem.

Proposition 3.6. *Suppose that $(p,q) \in X \times I$ is a C^*-point. Then $p \in X$ is neither isolated nor a G_δ-point.*

The proof of Proposition 3.6 is facilitated by observing the following first.

Lemma 3.7. *For any space X, any $p \in X$ and $t \in I$, $X \times [0,t] \setminus \{(p,t)\}$ is C^*-embedded in $X \times I \setminus \{(p,t)\}$.*

PROOF: Suppose that $f \in C^*(X \times [0,t] \setminus \{(p,t)\})$. Let

$$\bar{f}(x,t) = \begin{cases} f(x,r), & \text{if } 0 \le r \le t, \\ f\big(x, t\big(\frac{r-1}{t-1}\big)\big), & \text{if } t \le r \le 1, \end{cases}$$

wherever the above makes sense. Then $\bar{f}$ extends f continuously to $X \times I \setminus \{(p,t)\}$. □

The point of Lemma 3.7 is that in Proposition 3.6 it is enough to prove the claim for $t = 1$.

PROOF OF PROPOSITION 3.6: It makes the presentation easier to identify I with the interval $[0,\infty]$, and assume that (p,∞) is a C^*-point. Evidently, p cannot be isolated, otherwise ∞ is forced to be a C^*-point of $[0,\infty]$, which is absurd. Now suppose, by way of contradiction, that there is a function $f \in C^*(X)$ vanishing only at p. We may assume that $0 \le f \le 1$ without loss of generality. Now define $h\colon X \times [0,\infty] \setminus \{(p,\infty)\} \to \mathbb{R}$ by

$$h(x,t) = \begin{cases} tf(x) \wedge 1, & \text{if } t \in \mathbb{R}, \\ 1, & \text{if } t = \infty \text{ and } x \ne p. \end{cases}$$

It is easy to verify the continuity of h; h obviously does not extend to (p,∞). □

By massaging the proof of Proposition 3.6 one can improve it when $(X,\le)$ is an ordered space, and get a converse for Corollary 3.3.

Theorem 3.8. *Suppose that $(X,\le)$ is an ordered space. Then $(p,t) \in X \times I$ is a C^*-point if and only if p is a nonisolated P-point. Furthermore, $X \times I$ is a C^*-space if and only if X is a P-space with no isolated points.*

PROOF: The sufficiency is a consequence of Proposition 3.2.

For the reverse, suppose that (p,t) is a C^*-point. We have already ruled out that p is a G_δ-point, and it cannot be isolated (Proposition 3.6). (And, incidentally, p is not an endpoint either.) Suppose, by way of contradiction, that p is not a P-point. We employ the argument in the proof of Proposition 3.6, regarding the point t, identifying I with the interval $[0,\infty]$ and taking $t = \infty$.

Now, one of the following must occur, since p is not a P-point: either its cofinality is uncountable and its co-initiality is countable, or the reverse. Also note that if a point x in a totally ordered space has countable cofinality then x is a G_δ-point in the interval $(-\infty, x]$. This means, in turn, that there is a function $f \in C(X)$ which vanishes at p, such that, either f vanishes on a half-interval below p, and is strictly positive on a half-interval above p, or else does the order-reverse. In describing the situation we also factor in Lemma 1.4, in that we trace to an open interval in X around p. Now, either

$$Z(f) = \{x \in X\colon\ x \leq p\}, \qquad \text{and} \qquad \mathrm{pos}(f) = \{x \in X\colon\ x > p\},$$

with $p \in \mathrm{cl}_X \, \mathrm{pos}(f)$, or the order-reverse holds.

Now we modify the definition of h in the proof of Proposition 3.6 as follows. Of the two events, let us take the first of these; the one displayed above. Then in the definition of h in the preceding proof, remove the second line, and replace it with two new ones. Thus:

$$h(x,t) = \begin{cases} tf(x) \wedge 1, & \text{if } \ t \in \mathbb{R}, \\ 1, & \text{if } \ t = \infty \ \text{ and } \ x > p, \\ 0, & \text{if } \ t = \infty \ \text{ and } \ x < p. \end{cases}$$

Again, h is continuous everywhere it is defined, and not extendible to (p, ∞), which is the desired contradiction. □

Corollary 3.9. *The product $X \times I$ is a C^*-space if and only if $X \times I$ is a C-space.*

Proof: A nonisolated P-point is not a G_δ-point. Apply Proposition 1.2. □

Remark 3.10. Observe that $p \in X$ not being a G_δ-point is not sufficient to make $(p,t) \in X \times I$ a C^*-point.

Let αD stand for the one-point compactification of a discrete uncountable set D. Note that α, the point at infinity, is not a G_δ-point of αD. However, $(\alpha, 1) \in \alpha D \times I$ is not a C^*-point. To see this, observe that if $d_1, d_2, \ldots$ is any infinite sequence in D, then $(d_n)_n$ converges to α. The sequence and its limit together form a copy of $\alpha\mathbb{N}$, the one-point compactification of the discrete natural numbers. Now as $\alpha\mathbb{N} \times I$ is a metric space, there is a bounded continuous function $g \in C(\alpha\mathbb{N} \times I \setminus \{(\alpha,1)\})$ which maps the horizontal edge $\{(d_n, 1)\colon n \in \mathbb{N}\}$ to 1 and the vertical edge $\{(\alpha, t)\colon t < 1\}$ to 0. Extend g to all of $\alpha D \times I \setminus \{(\alpha,1)\}$ by mapping all other vertical edges to 0. Then the reader can easily check that g is continuous where defined and not extendible to $(\alpha, 1)$.

It is tempting to generalize this example in a number of ways. However, what was easy here, namely the extension of g to all points except $(\alpha, 1)$ becomes problematic in more general situations. What seems to make the extendibility tick here is that the remaining points of αD are isolated.

Theorem 3.8 seem to insinuate a class of points to us. In the next section we explore that formally.

4. $P^\flat$-points

The import of Proposition 3.6 is this: if $X \times I$ is a C^*-space then X has no G_δ-points. As to the converse, Example 3.10 shows that we can have a non G_δ-point $p \in X$ such that (p, q) is not a C^*-point. We are interested in points that are either isolated or not a G_δ-point. To be precise such a point shall be called a *ngd-point.* It then follows that any almost P-point is a ngd-point; the converse is not true. Our proof of Proposition 3.6 leads us to consider the following definition.

Definition & Remarks 4.1. (a) A point $p \in X$ is called a $P^\flat$*-point* if for each zeroset Z containing p, either $p \in \text{int}_X Z$ or p is an accumulation point of $Z \setminus \text{int}_X Z$. The space X is a $P^\flat$*-space* if every point of X is a $P^\flat$-point.

The definitions are worded in this way to guarantee that any P-point, including an isolated one, is $P^\flat$, and that should be clear. Likewise, it is evident that any $P^\flat$-point is a ngd-point.

(b) The proof of Theorem 3.8 reveals that, if $(X, \leq)$ is an ordered space, then any $P^\flat$-point is a P-point.

Here is a basic characterization of $P^\flat$-spaces. It follows straight from the definition.

Proposition 4.2. *Suppose that X is a space. Then X is a $P^\flat$-space if and only if for each zeroset Z, $Z \setminus \text{int}_X Z$ contains no isolated points of $Z \setminus \text{int}_X Z$.*

Example 4.3. We give an example of a compact, connected totally ordered almost P-space which is not $P^\flat$.

We take $\widetilde{H}$, the Dedekind completion of an η_1-set, with top and bottom elements adjoined. For an account of η_1-sets the reader is referred to [5], Chapter 13 and several of its exercises; in particular, it is mentioned there that the η_1-set H itself is a P-space, see [5, 13 P.1]. Recall also that in a totally ordered space $(X, \leq)$, a point $p \in X$ is an almost P-point if and only if it has either uncountable coinitiality or else uncountable cofinality. The statement 13 J.4 of [5] says precisely that of $\widetilde{H}$; thus, $\widetilde{H}$ is an almost P-space. Space $\widetilde{H}$ is compact and connected because it is complete and has no gaps or successor pairs.

As we are in an ordered space, any $P^\flat$-point is a P-point. Thus, since $\widetilde{H}$ is not a P-space we are done.

We point out that one need not go to the lengths of the preceding example to produce a totally ordered space containing a ngd-point which is not $P^\flat$; that is, not a P-point. Simply use the space X in the remarks of 3.4. Indeed, the following should be noted at this point; the proof should now be straightforward, in view of the various comments made in the proof of Theorem 3.8 and elsewhere about totally ordered spaces. We leave it to the reader.

Proposition 4.4. *In any totally ordered space a ngd-point is necessarily an almost P-point. Thus, for totally ordered spaces, almost P-points coincide with ngd-points.*

Example 4.5. An example of a $P^\flat$-point which is not an almost P-point.

Consider the product $X = (\omega_1 + 1) \times \alpha\mathbb{N}$. We denote by ∞ the point at infinity of the second factor. Now the point $p = (\omega_1, \infty)$ is not an almost P-point, as the function f defined by $f(\mu, n) = 1/n$ is continuous, vanishes at p, while $Z(f)$ is nowhere dense.

On the other hand, suppose that $g \in C(X)$ vanishes at p. As explained in [5, 8.20], there is a countable ordinal μ such that, for each $\nu \geq \mu$, $g(\mu, \cdot) = g(\nu, \cdot)$. In particular, for all such ν, $g(\nu, \infty) = 0$. Thus, $\infty \in \mathrm{int}_{\alpha\mathbb{N}} Z(g(\omega_1, \cdot))$ if and only if $\infty \in \mathrm{int}_{\alpha\mathbb{N}} Z(g(\nu, \cdot))$ for each $\nu \geq \mu$. If this is the case $(\omega_1, \infty) \in \mathrm{int}_X Z(g)$. Otherwise, (ω_1, ∞) is an accumulation point of $Z(g) \setminus \mathrm{int}_X Z(g)$. This shows that (ω_1, ∞) is a strong $P^\flat$-point.

What is true, and easy to see after a little thought, is the following; we leave the verification to the reader.

Proposition 4.6. *A ngd-point is either almost P or else $P^\flat$.*

The following proposition should be compared to Proposition 3.2. It will lead us to $P^\flat$-spaces which are not P-spaces. We will improve upon the final assertion of Proposition 4.7 in Theorem 4.9 ahead, albeit with an added assumption.

Proposition 4.7. *Let κ be an uncountable cardinal. Suppose that $p \in X$ is a nonisolated P^κ-point. Let Y be any space satisfying $d(Y) < \kappa$. Then for each $y \in Y$, (p, y) is a $P^\flat$-point of $X \times Y$. Thus, if X is a P-space without isolated points and Y is separable, then $X \times Y$ is a $P^\flat$-space.*

Proof: The second claim is immediate from the first, letting $\kappa = \omega_1$.

Now let S be a dense subset of Y, with $|S| < \kappa$. Suppose that $f \in C^*(X \times Y)$ and $(p, y) \in Z(f)$. Let

$$W \equiv \{x \in X \colon f(p, s) = f(x, s),\ \forall\, s \in S\}.$$

Note that W is an intersection of fewer than κ zerosets containing p, and therefore a neighborhood of p. Since p is not isolated, at least one $x \neq p$ lies in W. Observe as well that, since S is dense the functions $f(p, \cdot)$ and $f(x, \cdot)$ are identical for each $x \in W$.

Let us assume now, that $Z(f)$ is not a neighborhood of (p, y). Suppose that $U \times U'$ is a box neighborhood of (x, y), where $x \in W$. Without loss of generality we may take $U \subseteq W$. Then there is a point $(x_o, t) \in (U \times U') \cap \mathrm{coz}(f)$, because of the assumption on (p, y). But $f(x_o, t) = f(x, t)$, and so $(x, t) \in (U \times U') \cap \mathrm{coz}(f)$, proving that $(x,y) \in Z(f) \smallsetminus \mathrm{int}_{X\times Y} Z(f)$ for each $x \in V$. This makes it clear that (p,s) is not isolated in $Z(f) \smallsetminus \mathrm{int}_{X\times Y} Z(f)$, and so (p,s) is a $P^\flat$-point. □

Remark 4.8. As promised, Proposition 4.7 delivers examples of spaces which are $P^\flat$, but not P-spaces. In fact, if X is any P-space without isolated points, then $X \times I$ is a $P^\flat$-space without any P-points or almost P-points!

Theorem 3.8 tells us that a totally ordered space $(X,\leq)$ without isolated points is a P-space precisely when $X \times I$ is a C^*-space. It seems reasonable to ask then for which totally ordered spaces X without isolated points $X \times I$ is a $P^\flat$-space. The next proposition gives a partial answer.

Theorem 4.9. *Suppose that $(X,\leq)$ is a totally ordered, connected almost P-space. Then $X \times Y$ is a $P^\flat$-space for each separable space Y.*

Proof: Suppose that $(p,y) \in X \times Y$ and $f \in C^*(X\times Y)$, with $(p,y) \in Z(f)$. We suppose, without loss of generality, that p has uncountable coinitiality in X. Let S be a countable dense subset of Y. Once again, let

$$W \equiv \{x \in X \colon f(p,s) = f(x,s),\ \forall\, s \in S\}.$$

The set W is a zeroset of X, containing p, and in view of the uncountable coinitiality of p, there is a $q > p$ such that $[p,q) \subseteq W$; that is, for each $t \in S$ and each $p \leq x < q$, $f(x,t) = f(p,t)$. By the density of S in Y, it follows, as in the previous proof, that, identically, $f(x,\cdot) = f(p,\cdot)$ for each such x. In particular, $(x,y) \in Z(f)$ for each $x \in [p,q)$.

To proceed, there are a number of cases to consider:

(a) *For each $x \in [p,q)$, $y \in \mathrm{cl}_Y \,\mathrm{coz}_Y(f(x,\cdot))$.* Evidently, $(x,y) \in \mathrm{cl}_{X\times Y} \times \mathrm{coz}_{X\times Y}(f)$, and therefore, since p is not isolated, it follows that $(p,y) \in Z(f) \smallsetminus \mathrm{int}_{X\times Y} Z(f)$.

(b) *For each $x \in [p,q)$, $y \in \mathrm{int}_Y Z_Y(f(x,\cdot))$.* Then f vanishes on the box $[p,q) \times B_o$ for a suitable open set $B_o \subseteq Y$ containing y. Two possibilities remain:

(b-1) Next, suppose that for each neighborhood V of p in X and each neighborhood B of $y \in Y$ there is a $t \in B \cap B_o$, $t \neq y$, such that V contains a point of the boundary of $Z(f(\cdot,t))$. Then each box neighborhood $V \times B$ of (p,y) contains a point of the boundary of $Z(f)$, which is necessarily distinct from (p,y) and which proves that $(p,y) \in Z(f) \smallsetminus \mathrm{int}_{X\times Y} Z(f)$ and an accumulation point of it.

(b-2) Left to consider: there exist neighborhoods W of p and $O \subseteq B_o$ of y such that for each $t \in O$ with $t \neq y$, W does not intersect the boundary of $Z(f(\cdot,t))$. Now, W does intersect $Z(f(\cdot,t))$ for each such t, because it intersects $[p,q)$. Since X is connected we may shrink W a bit and assume that it is an interval and connected. As we have the partition

$$W = W \cap \operatorname{coz}(f(\cdot,t)) \cup W \cap \operatorname{int}_X Z(f(\cdot,t))$$

for each $t \in O$. Since the second component of the above partition is nonempty, connectedness implies that $O \subseteq Z(f(\cdot,y))$. Putting it another way, f vanishes on the neighborhood $W \times O$ of (p,y).

The upshot of this is that (p,y) is a $P^\flat$-point of $X \times Y$, and the proof is complete. $\square$

It is worth making a couple of remarks about Theorem 4.9 and its proof.

Remark 4.10. (a) Recall that $\widetilde{H}$ stands for the two-point compactification of the Dedekind completion of the η_1-set H. First, observe that Theorem 4.9 and Theorem 3.8 imply that $\widetilde{H} \times I$ is a compact, connected $P^\flat$-space which is not a C^*-space. Unlike, compact P-spaces, which are necessarily finite, as is well known, here we have a continuum which is $P^\flat$.

(b) The following example shows that the connectedness assumption in Theorem 4.9 cannot be removed; not even if Y is a compact metric space, and not even if it is the interval I. To simplify the presentation we use $Y = \alpha\mathbb{N}$, but the reader will readily appreciate how to adapt things for I.

Let S be any totally ordered P-space without isolated points. Then X is the totally ordered space constructed as the following disjoint union:

$$X = \left(\bigcup_{n=1}^{\infty} S_n\right) \cup \{p\} \cup S_\infty,$$

where $S_n = S = S_\infty$ for each natural number n and the ordering on X is defined by $a < b$ if

- both $a, b \in S_m$ for some $m \in \mathbb{N} \cup \{\infty\}$ and $a < b$ in S; or
- $a = p$ and $b \in S_\infty$; or
- $b = p$ and $a \in S_m$ for some $m \in \mathbb{N}$; or
- $a \in S_m$ and $b \in S_n$ for some $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$ with $m < n$.

It is easy to verify that X is an almost P-space without isolated points; p is the only non P-point, and it has countable cofinality.

Now let us consider $X \times \alpha\mathbb{N}$, and we identify $\alpha\mathbb{N}$ with the sequence $(1/n)_{n\in\mathbb{N}}$ in I together with the limit 0, endowed with the ordinary metric topology. What we will show is that the point $(p,0)$ is not a $P^\flat$-point.

To that end, consider the following function $f \in C(X \times \alpha\mathbb{N})$

$$f(x,t) = \begin{cases} 0, & \text{if } \ t = 0 \ \text{ or } \ x \geq p, \\ 0, & \text{if } \ t = \frac{1}{n} \ \text{ and } \ x \in S_m \ \text{ with } \ m \neq n, \\ \frac{1}{n}, & \text{if } \ t = \frac{1}{n} \ \text{ and } \ x \in S_n. \end{cases}$$

We leave it to the reader to check that f is continuous, and that it satisfies all the conditions of case (b-2) in the proof of Theorem 4.9. However, observe as well that $(p,0)$ is the only point on the boundary of $Z(f)$. This shows that $(p,0)$ is not a $P^\flat$-point.

(c) Looking ahead to Proposition 4.11, notice that the example in (b) also has the following feature: the space X is a $P^\flat$-space (Proposition), but $X \times \alpha\mathbb{N}$ is not.

Here is a property which distinguishes the behavior of ngd-spaces from the $P^\flat$ ones.

Proposition 4.11. *Suppose that $X \times Y$ is an ngd-space; then either X or Y is an ngd-space.*

Proof: By way of contradiction suppose that there exist $a \in X$ and $b \in Y$ and zerosets Z_a and Z_b of X and Y, respectively, containing a and b, respectively, as isolated points. Thus we may find neighborhoods V_a and V_b of a and b, respectively, such that $V_a \cap Z_a = \{a\}$ and $V_b \cap Z_b = \{b\}$.

Now note that $Z_a \times Z_b$ is a zeroset of $X \times Y$ containing (a,b), and that

$$(V_a \times V_b) \cap (Z_a \times Z_b) = \{(a,b)\},$$

which contradicts that (a,b) is an ngd-point of the product. $\square$

Let us conclude this section with a brief observation.

Remark 4.12. In the conclusion of Proposition 4.11 we may add: but not necessarily both. Once again, use the example $\widetilde{H} \times I$, of 4.10 (a); we have that $\widetilde{H} \times I$ is $P^\flat$, while the interval is not an ngd-space. (Indeed, note that in any second countable space, and hence in any metric space, the only ngd-points are the isolated ones.)

This example also shows that Proposition 4.11 fails for $P^\flat$-spaces.

5. Open questions

We summarize the discussion in this article around a few focussed open questions.

Remark 5.1. We have characterized for an ordered space $(X, \leq)$, when $X \times I$ is a C^*-space. By Theorem 3.8, this happens precisely when X is a P-space without any isolated points.

In fact, for spaces without isolated points we have the following implications:

$$X \text{ is a } P\text{-space} \;\Rightarrow\; X \times I \text{ is a } C^*\text{-space} \;\Rightarrow X \text{ is a } P^\flat\text{-space.}$$

We note, furthermore (keeping to spaces without isolated points):

(i) The second arrow does not reverse (Remark 3.10).

(ii) There is more to the first implication above; according to Proposition 4.7 we also have

$$X \text{ is a } P\text{-space} \;\Rightarrow\; X \times I \text{ is a strong } P^\flat\text{-space.}$$

(iii) The implication in (ii) does not reverse; recall the comments in 4.10 (a).

(iv) We do not know whether the first arrow in the display above reverses, nor whether one can conclude that X is a P-space from the assumption that $X \times I$ is *both* a C^*-space and a strongly $P^\flat$-space.

(v) We do not know either if the conclusion of the second arrow above can be strengthened to "strong $P^\flat$-space". It can for totally ordered spaces, where the strongly P-flat points are precisely the P-points, and one can invoke Theorem 3.8 again.

Remark 5.2. We have some information regarding the question: when is $X \times I$ a strong $P^\flat$-space? Most of what we know is in the form of counterexamples to possible conjectures.

(a) For spaces without isolated points one has the implication

$$X \text{ is a } P\text{-space} \;\Rightarrow X \times I \text{ is strongly } P^\flat,$$

as observed in Remark 5.1 (ii); and, as already noted, this implication does not reverse. In fact one cannot conclude that X is a P-space from the assumption that $X \times I$ is strongly $P^\flat$ (Theorem 4.9); not even for totally ordered spaces.

(b) Unlike for $P^\flat$-spaces the assumption that $X \times Y$ is strongly $P^\flat$ does not imply that either factor has that property. The example in 4.10 (a) witnesses this, and, in particular, the following failure for totally ordered spaces to boot:

$$X \times I \text{ is a strong } P^\flat\text{-space} \;\not\Rightarrow X \text{ is strongly } P^\flat.$$

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