# Commentationes Mathematicae Universitatis Caroline

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On subcompactness and countable subcompactness of metrizable spaces in ZF

# On subcompactness and countable subcompactness of metrizable spaces in ZF

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*Abstract.* We show in ZF that:

(i) Every subcompact metrizable space is completely metrizable, and every completely metrizable space is countably subcompact.

(ii) A metrizable space $\mathbf{X} = (X, T)$ is countably compact if and only if it is countably subcompact relative to $T$.

(iii) For every metrizable space $\mathbf{X} = (X, T)$, the following are equivalent:
(a) $\mathbf{X}$ is compact;
(b) for every open filter $\mathcal{F}$ of $\mathbf{X}$, $\bigcap\{\overline{F}\colon F \in \mathcal{F}\} \neq \emptyset$;
(c) $\mathbf{X}$ is subcompact relative to $T$.

We also show:

(iv) The negation of each of the statements,
(a) every countably subcompact metrizable space is completely metrizable,
(b) every countably subcompact metrizable space is subcompact,
(c) every completely metrizable space is subcompact,
is relatively consistent with ZF.

(v) AC if and only if for every family $\{\mathbf{X}_i\colon i \in I\}$ of metrizable subcompact spaces, for every family $\{\mathcal{B}_i\colon i \in I\}$ such that for every $i \in I$, $\mathcal{B}_i$ is a subcompact base for $\mathbf{X}_i$, the Tychonoff product $\mathbf{X} = \prod_{i \in I} \mathbf{X}_i$ is subcompact with respect to the standard base $\mathcal{B}$ of $\mathbf{X}$ generated by the family $\{\mathcal{B}_i\colon i \in I\}$.

*Keywords:* axiom of choice; compact; countably compact; subcompact; countably subcompact; lightly compact metric space

*Classification:* 03E25, 54D30, 54E35, 54E45, 54E50

## 1. Notation and terminology

In what follows, $\omega$ denotes the first infinite ordinal number and $\mathbb{N} = \omega \setminus \{0\}$.

Let $\mathbf{X} = (X, T)$ be a topological space and $\mathcal{H}$ be a nonempty subset of $\mathcal{P}(X) \setminus \{\emptyset\}$. Then $\mathcal{H}$ is called a *filterbase* of $X$ if and only if the intersection of any two members of $\mathcal{H}$ contains an element of $\mathcal{H}$. A filterbase $\mathcal{H}$ of $X$ closed under supersets, i.e., for all $A \in \mathcal{H}$ and $B \in \mathcal{P}(X)$, if $A \subset B$ then $B \in \mathcal{H}$, is called a *filter* of $X$. A filterbase (or filter) $\mathcal{H} \subseteq T$ of $X$ is called *open filterbase*

(*open filter*, respectively) of $\mathbf{X}$. An open filter $\mathcal{H}$ of $\mathbf{X}$ is called *total* if and only if for every $x \in X$, there exists a neighborhood $V$ of $x$ such that $X \backslash \overline{V} \in \mathcal{H}$.

Assume $\mathbf{X}$ is $T_3$ and $\mathcal{B}$ is an open base of $\mathbf{X}$. An open filterbase $\mathcal{F} \subseteq \mathcal{B}$ of $\mathbf{X}$ is called a *regular $\mathcal{B}$-filterbase* if and only if for every $F \in \mathcal{F}$ there exists $B \in \mathcal{F}$ with $\overline{B} \subseteq F$. If a regular $\mathcal{B}$-filterbase $\mathcal{F}$ is countable, then $\mathcal{F}$ is called a *countable regular $\mathcal{B}$-filterbase*. In particular, the countable regular $\mathcal{B}$-filterbase $\mathcal{F} = \{F_n \in \mathcal{B} \colon n \in \mathbb{N}, \overline{F_{n+1}} \subseteq F_n\}$ is called a *regular $\mathcal{B}$-sequence*.

Let $\mathcal{U}$ be a family of subsets of $X$. An element $x \in X$ is called a *cluster point* of $\mathcal{U}$ if and only if every neighborhood of $x$ meets infinitely many members of $\mathcal{U}$. Moreover, $\mathcal{U}$ is said to be *locally finite* if $\mathcal{U}$ has no cluster points.

A space $\mathbf{X}$ is said to be *compact* (or *countably compact*) if and only if every open cover $\mathcal{U}$ of $\mathbf{X}$ (countable open cover $\mathcal{U}$ of $\mathbf{X}$, respectively) has a finite subcover $\mathcal{V}$. Equivalently, $\mathbf{X}$ is compact (or countably compact) if and only if the intersection of every family (countable family, respectively) of closed sets of $\mathbf{X}$ with the finite intersection property (fip for abbreviation) is nonempty.

A space $\mathbf{X}$ is said to be *lightly compact* (or *countably lightly compact*) if and only if $\mathbf{X}$ has no infinite (no countably infinite, respectively) locally finite families of open subsets. Light compactness has been introduced in [14]. Countable light compactness is the condition $(B_3)$ (Every pairwise disjoint family $\mathcal{U} = \{U_n \colon n \in \mathbb{N}\}$ of nonempty open subsets of $\mathbf{X}$ has a cluster point in $\mathbf{X}$) in [3] and it is equivalent to light compactness in ZFC (= Zermelo–Fraenkel set theory ZF together with axiom of choice AC). Lightly compact spaces are also called *feebly compact*, see e.g. [15].

A $T_3$ space $\mathbf{X}$ is called *subcompact* (or *countably subcompact*) if there exists an open base $\mathcal{B}$ such that for every regular $\mathcal{B}$-filterbase (for every countable regular $\mathcal{B}$-filterbase, respectively) $\mathcal{F}$, $\bigcap\{F \colon F \in \mathcal{F}\} \neq \emptyset$. The base $\mathcal{B}$ is called *subcompact* (*countably subcompact*, respectively). Subcompact and countably subcompact spaces have been introduced and investigated in [5].

Let $\mathbf{X} = (X, d)$ be a metric space, $x \in X$ and $\varepsilon > 0$. Then $B_d(x, \varepsilon) = \{y \in X \colon d(x, y) < \varepsilon\}$ ($D_d(x, \varepsilon) = \{y \in X \colon d(x, y) \leq \varepsilon\}$, respectively) denotes the open (closed, respectively) ball in $\mathbf{X}$ with center $x$ and radius $\varepsilon$. If no confusion is likely to arise we shall omit the subscript $d$ from $B_d(x, \varepsilon)$ and $D_d(x, \varepsilon)$. Given $B \subseteq X$, $B \neq \emptyset$,

$$\delta(B) = \sup\{d(x, y) \colon x, y \in B\} \in \mathbb{R}_+ \cup \{\infty\}$$

will denote the *diameter* of $B$. In what follows, $T_d$ will denote the topology on $X$ produced by the family of all open balls of $\mathbf{X}$.

A space $\mathbf{X}$ is called *subcompact* (or *countably subcompact*) if and only if the topological space $(X, T_d)$ is subcompact (countably subcompact, respectively).

A sequence of points $(x_n)_{n \in \mathbb{N}}$ of $\mathbf{X}$ is called *Cauchy* if and only if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $d(x_n, x_m) < \varepsilon$.

A metric space $(X, d)$ is said to be *complete* if and only if every Cauchy sequence in $X$ converges to some point of $X$.

A *completion* of $\mathbf{X}$ is a complete metric space $(Y, \varrho)$ together with an isometric map $H \colon \mathbf{X} \to \mathbf{Y}$ such that $\overline{H(X)} = Y$. It is a well-known ZF result that for every $x_0 \in X$ the mapping:

$$H \colon (X, d) \to (C_b(X, \mathbb{R}), \varrho), \qquad H(x) = f_x,$$

where $C_b(X, \mathbb{R})$ is the family of all bounded continuous functions from $X$ to $\mathbb{R}$, $\varrho$ is the sup metric ($\varrho(f, g) = \sup\{|f(x) - g(x)| \colon x \in X\}$) and for every $x \in X$, $f_x \colon X \to \mathbb{R}$ is the function given by:

$$f_x(t) = d(x, t) - d(x_0, t),$$

is such an isometric map. Thus, $\mathbf{Y} = (\overline{H(X)}, \varrho)$ is a completion of $\mathbf{X}$.

A topological space $\mathbf{X} = (X, T)$ is said to be *completely metrizable*, or *topologically complete* if and only if there is a metric $d \colon X \times X \to [0, \infty)$ such that $T_d = T$ and $(X, d)$ is a complete metric space.

Let $X$ be an infinite set. We say that $X$ is *Dedekind infinite* (or *weakly Dedekind infinite*) if and only if $X$ ($\mathcal{P}(X)$, respectively) has a countably infinite subset. Otherwise, $X$ is called *Dedekind finite* (*weakly Dedekind finite*, respectively).

Below we list the weak forms of the axiom of choice we shall use in this paper.

- AC: For every family $\mathcal{A}$ of nonempty sets there exists a function $f$ such that for all $x \in \mathcal{A}$, $f(x) \in x$.
- CAC: AC restricted to countable families.
- IDI($\mathbb{R}$): Every infinite subset of $\mathbb{R}$ is Dedekind infinite.
- IWDI: Every infinite set is weakly Dedekind infinite.

For ZF models satisfying CAC, IDI($\mathbb{R}$), or their negations, we refer the reader to [7].

## 2.   Introduction and some preliminary results

In this paper, the intended context for reasoning will be ZF. If a statement is provable in ZF we will add (ZF) in the beginning of that statement. Otherwise, there will appear (ZFC).

Most mathematicians are aware if AC is used in a proof of a mathematical statement. However, deciding if use of AC in a specific proof is unnecessary, or determining the exact portion of AC needed to carry out the proof, is not so obvious. In our opinion, working in ZF, leads to a better understanding of the

various mathematical notions involved in a proof. We find the following quotation of H. Herrlich, expressed in [6], quite illuminating and corroborative to the opinion expressed earlier.
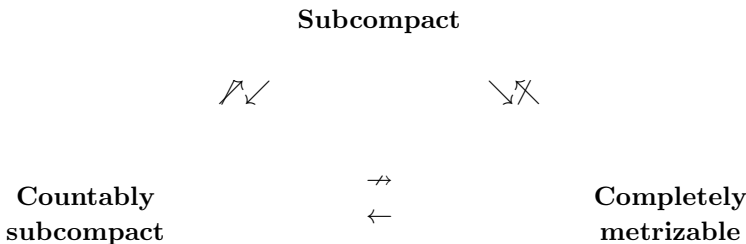
> Ordinarily topology is dealt with in the setting of ZFC. Although AC is neither evidently true nor evidently false, this adherence to AC seems to be based on a general belief that adoption of AC enables topologists to prove more and better theorems. Aside from the trivial observation that no theorem T in ZFC is lost in ZF, it simply turns into the implication AC → T, which often enough can be even improved to an equivalence WC ↔ T for a suitable weak form WC of AC.

Various topological completeness properties have been invented in order to generalize the definition of complete metric space to the context of topologies. J. De Groot introduced in [5] two such properties. Namely, subcompactness and countable subcompactness. The justification for the introduction of the aforementioned notions, as he points out, is the validity of the following theorem.

**Theorem 1** ([5], ZFC). *Let* $\mathbf{X} = (X, T)$ *be a metrizable space. The following properties are equivalent:*

   (i) $\mathbf{X}$ *is countably subcompact;*
  (ii) $\mathbf{X}$ *is subcompact;*
 (iii) $\mathbf{X}$ *is topologically complete.*

Regarding Theorem 1, it is straightforward to see that the implication (ii) → (i) holds true in ZF, but the status of the remaining implications is unknown. The proofs given in [5] require some weak forms of the axiom of choice, such as CAC, in several places. The following web of implications\nonimplications summarizes the established relations, in this project, between the notions sited in Theorem 1.

<div align="center">

**Subcompact**

$\nearrow\swarrow$ $\qquad\qquad\qquad$ $\searrow\nwarrow\!\!\!\times$

**Countably** $\qquad\qquad$ $\xrightarrow{\quad}$ $\qquad\qquad$ **Completely**
**subcompact** $\qquad\qquad$ $\xleftarrow{\quad}$ $\qquad\qquad$ **metrizable**

</div>

In particular, in Theorem 15 we show that (ii) → (iii) and (iii) → (i) are valid in ZF and, in Theorem 16, we show that each of the following nonimplications (i) $\nrightarrow$ (ii), (i) $\nrightarrow$ (iii) and (iii) $\nrightarrow$ (ii) is consistent with ZF.

J. De Groot has established in [5, page 762] the following ZFC characterization of compact Hausdorff spaces:

  ○ $(D)$ A Hausdorff space $\mathbf{X} = (X, T)$ is compact if and only if it is Tychonoff and subcompact relative to $T$.

As expected, $(D)$ is not a theorem of ZF. It is consistent with ZF the existence of non-Tychonoff compact Hausdorff spaces, see e.g., Example 2.4 page 81 in [4]. As a by-product of $(D)$, if we restrict to the class of metrizable spaces, we get the following characterization of compactness:

  ○ $(A)$ A metrizable space $\mathbf{X} = (X, T)$ is compact if and only if it is subcompact relative to $T$.

Since there are complete, noncompact metric spaces, it follows from Theorem 1 and $(A)$ that the notion "subcompact with respect to the base of all open sets" is strictly stronger than subcompactness. The most natural question which pops up at this point is

**Question 1.** Is $(A)$ a theorem of ZF?

Of course a compact metrizable space is subcompact with respect to any base of open sets. So, Question 1 actually concerns the converse of $(A)$. In the forthcoming Theorem 11 we show, in ZF, that a metrizable space is countably compact if and only if it is countably subcompact with respect to the base of all open sets. Since a countably compact metrizable space is compact in ZF + CAC, see e.g. [10], it follows that $(A)$ is a theorem of ZF + CAC. In Theorem 14 we answer Question 1 in the affirmative. So, subcompactness of metrizable spaces with respect to the base of all open sets is the strongest of all forms of compactness of metrizable spaces in ZF, see e.g. [9].

The next theorem is from [5] and concerns products of subcompact spaces. It shows, in ZFC, that subcompactness is an invariant for the forming of topological products.

**Theorem 2** ([5], ZFC)**.** *Let $\{\mathbf{X}_i = (X_i, T_i) : i \in I\}$ be a family of subcompact $T_3$ spaces, $\mathcal{B} = \{\mathcal{B}_i : i \in I\}$ a family of sets such that for every $i \in I$, $\mathcal{B}_i$ is a subcompact base for $\mathbf{X}_i$, and $\mathbf{X} = \prod_{i \in I} \mathbf{X}_i$ be their product. Then $\mathbf{X}$ is subcompact with respect to the standard base $\mathcal{C}$ generated by the family $\mathcal{B}$.*

The question which arises now is whether Theorem 2 holds in ZF. We show in the forthcoming Theorem 17 that the answer, as expected, is in the negative.

We list the following known results here for future reference.

**Theorem 3** ([12], ZF)**.** *Let $\mathbf{X} = (X, T)$ be a topological space. The following are equivalent:*

  (i) $\mathbf{X}$ *is countably lightly compact;*

(ii) *Every countable open filterbase $\mathcal{F}$ of $\mathbf{X}$ has a point of adherence $\left(\bigcap\{\overline{F}: F \in \mathcal{F}\} \neq \emptyset\right)$;*

(iii) *$\mathbf{X}$ has no countably infinite pairwise disjoint locally finite family of open sets.*

**Theorem 4** ([11], ZF). *A metrizable space is countably compact if and only if it is countably lightly compact.*

**Theorem 5** ([8], ZF). *Let $\mathbf{X} = (X, T)$ be a regular space and $\mathcal{B}$ be an open base of $\mathbf{X}$. Then, $\mathbf{X}$ is countably $\mathcal{B}$-subcompact if and only if every regular $\mathcal{B}$-sequence has a nonempty intersection.*

**Theorem 6** ([16], ZF). *A $G_\delta$ subspace of a completely metrizable space is completely metrizable.*

**Theorem 7** ([1]). *If there exists an infinite Dedekind finite subset of $\mathbb{R}$ then there exists a dense one also.*

The following result shows that for regular spaces their total filters coincide with those whose intersections of the closures of their members are empty.

**Proposition 8** (ZF). *An open filter $\mathcal{F}$ of a regular space $\mathbf{X} = (X, T)$ is total if and only if $\bigcap\{\overline{F}: F \in \mathcal{F}\} = \emptyset$.*

PROOF: ($\rightarrow$) This is straightforward.

($\leftarrow$) Let $\mathcal{F}$ be an open filter of $\mathbf{X}$ with $\bigcap\{\overline{F}: F \in \mathcal{F}\} = \emptyset$. Then, for every $x \in X$, there is $F \in \mathcal{F}$ with $x \in \overline{F}^c$. Since $\mathbf{X}$ is regular, there exists a neighborhood $V$ of $x$ with $\overline{V} \subseteq \overline{F}^c$. Therefore, $F \subseteq \overline{F} \subseteq \overline{V}^c$ and $\overline{V}^c \in \mathcal{F}$, meaning that $\mathcal{F}$ is total. $\square$

We point out here that for every topological space $\mathbf{X} = (X, T)$ if $\mathcal{F}$ is a total filter of $\mathbf{X}$ then $\bigcap\{\overline{F}: F \in \mathcal{F}\} = \emptyset$. The following example shows that the converse of Proposition 8 is not true in case $\mathbf{X}$ is not regular.

**Example 9** (ZF). Let $T_{\mathbb{Z}}$ be the topology on the set $\mathbb{Z}$ of all integers in which every point of $\mathbb{Z}\backslash\{0\}$ is isolated while neighborhoods of 0 are all cofinite subsets of $\mathbb{Z}$ including 0. Let $\mathbf{Y}$ be the product of the discrete space $\mathbb{N}$ with $(\mathbb{Z}, T_{\mathbb{Z}})$ and $X = \{\infty\} \cup \mathbb{N} \times \mathbb{Z}$. Topologize $X$ by declaring neighborhoods of $\mathbb{N} \times \mathbb{Z}$ to be the old ones whereas basic neighborhoods of $\infty$ are all sets of the form $\{\infty\} \cup O$ where, $O$ is a subset of $\mathbb{N} \times \{-i: i \in \mathbb{N}\}$ such that for all but finitely many $n \in \mathbb{N}$, $O$ contains all but finitely many members of the $n$th copy of the set of negative integers, i.e.

$$|\{n\} \times \{-i: i \in \mathbb{N}\}\backslash O| < \aleph_0.$$

We leave it as an easy exercise for the reader to verify that $\mathbf{X}$ is a (noncompact) Hausdorff, nonregular space (the closed set $\{(0, n) : n \in \mathbb{N}\}$ of $\mathbf{X}$ and the point $\infty \in X$ cannot be separated by open sets). Clearly, for every $k \in \mathbb{N}$

$$F_k = \bigcup\{\{n\} \times \omega : n \geq k\}$$

is a clopen subset of $\mathbf{X}$ with

$$\bigcap\{\overline{F_k} : k \in \mathbb{N}\} = \bigcap\{F_k : k \in \mathbb{N}\} = \emptyset.$$

Let $\mathcal{F}$ be the open filter of $\mathbf{X}$ generated by the open filterbase $\{F_k : k \in \mathbb{N}\}$ of $\mathbf{X}$, i.e.

$$\mathcal{F} = \{F \subseteq X : F \text{ is open in } \mathbf{X} \text{ and for some } k \in \mathbb{N}, \ F_k \subseteq F\}.$$

Clearly, $\bigcap\{\overline{F} : F \in \mathcal{F}\} = \emptyset$. We claim that for every neighborhood $V_\infty$ of $\infty$ and every $k \in \mathbb{N}$, $\overline{V_\infty}^c \not\supseteq F_k$. To this end, fix $t \in \mathbb{N}$ such that for all $n \geq t$, $|\{n\} \times \{-i : i \in \mathbb{N}\}\backslash V_\infty| < \aleph_0$. Then, for all $n \geq t$ and for all $k \in \mathbb{N}$, $(n, 0) \in \overline{V_\infty} \cap F_k \neq \emptyset$ meaning that $F_k \not\subseteq \overline{V_\infty}^c$. Hence, $\overline{V_\infty}^c \notin F$ and $\mathcal{F}$ is not a total filter.

Clearly, an infinite $T_1$ space has open filters with empty intersection, e.g., the open filter generated by the family of all cofinite sets. We show next that no compact regular space has total filters. In fact, a regular space is compact if and only if it has no total filters.

**Theorem 10** (ZF). *A regular space* $\mathbf{X} = (X, T)$ *is compact if and only if it has no total filters.*

PROOF: Fix a regular space $\mathbf{X} = (X, T)$.

($\rightarrow$) We show that $\mathbf{X}$ has no total filters. Assume the contrary and let $\mathcal{F}$ be a total filter of $\mathbf{X}$. By Proposition 8, $\bigcap\{\overline{F} : F \in \mathcal{F}\} = \emptyset$. Contradiction!

($\leftarrow$) We show that $\mathbf{X}$ is compact. To this end, we assume the contrary and fix an open cover $\mathcal{U}$ of $\mathbf{X}$ without a finite subcover. Let $\mathcal{V} = \{V \in T : \overline{V} \subseteq U \text{ for some } U \in \mathcal{U}\}$. By the regularity of $\mathbf{X}$, it follows easily that $\mathcal{V}$ is an open cover of $\mathbf{X}$ such that the closed cover $\{\overline{V} : V \in \mathcal{V}\}$ of $\mathbf{X}$ has no finite subcover. It is easy to see that $\{\overline{V}^c : V \in \mathcal{V}\}$ is a family of open sets of $\mathbf{X}$ with the fip. Let $\mathcal{F}$ be the open filter of $\mathbf{X}$ generated by $\{\overline{V}^c : V \in \mathcal{V}\}$. Since $\mathcal{V}$ covers $X$, it follows that for every $x \in X$, there is a $V \in \mathcal{V}$ with $x \in V$ and $\overline{V}^c \in \mathcal{F}$, meaning that $\mathcal{F}$ is total and contradicting our hypothesis. Therefore, $\mathbf{X}$ is compact as required. $\quad\square$

### 3. Main results

Our first result in this section shows, in ZF, that in the class of all metrizable spaces, countable subcompactness relative to the base of all open sets is equivalent to countable compactness, as well as to countable light compactness.

**Theorem 11** (ZF). *Let* $\mathbf{X} = (X, T)$ *be a metrizable space. The following are equivalent:*

(i) $\mathbf{X}$ *is countably subcompact with respect to* $T$;

(ii) $\mathbf{X}$ *is countably lightly compact;*

(iii) $\mathbf{X}$ *is countably compact.*

PROOF: Fix a metric $d$ on $X$ with $T = T_d$. It suffices, in view of Theorem 4, to show (i) $\leftrightarrow$ (ii).

(i) $\to$ (ii) Assume the contrary and, by Theorem 3, fix a countably infinite, pairwise disjoint, locally finite family $\mathcal{U}_0 = \{U_{0i} : i \in \mathbb{N}\}$ of open nonempty subsets of $\mathbf{X}$. We are going to construct inductively families $\mathcal{U}_n = \{U_{ni} : i \geq n\}$, $n \in \mathbb{N}$ of open sets of $\mathbf{X}$ such that for all $i \geq n$, $\overline{U_{ni}} \subseteq U_{(n-1)i}$. For every nonempty open set $U$ of $\mathbf{X}$ let

$$(1) \qquad t_U = \min \left\{ n \in \mathbb{N} : \left\{ x \in U : d(x, U^c) > \frac{1}{n} \right\} \neq \emptyset \right\}.$$

We begin the induction by letting for $n = 1$ and every $i \in \mathbb{N}$

$$U_{1i} = \left\{ x \in U_{0i} : d(x, U_{0i}^c) > \frac{1}{t_{U_{0i}}} \right\}.$$

Since $\mathcal{U}_0$ is locally finite, it follows that $\mathcal{U}_1 = \{U_{1i} : i \geq 1\}$ is a locally finite family of nonempty open sets of $\mathbf{X}$. Furthermore, for all $i \geq 1$, $\overline{U_{1i}} \subseteq U_{0i}$.

For $n = k + 1$, use the induction hypothesis on $\mathcal{U}_k = \{U_{ki} : i \geq k\}$ and define for every $i \geq n$

$$U_{ni} = \left\{ x \in U_{ki} : d(x, U_{ki}^c) > \frac{1}{t_{U_{ki}}} \right\}.$$

Clearly, $\mathcal{U}_n = \{U_{ni} : i \geq n\}$ is a locally finite family of open sets of $\mathbf{X}$, and for all $i \geq n$, $\overline{U_{ni}} \subseteq U_{ki}$ terminating the induction.

Let $F_0 = \bigcup \mathcal{U}_0$ and for every $n \in \mathbb{N}$ put $F_n = \bigcup \mathcal{U}_n$. Clearly,

$$\overline{F_1} = \overline{\bigcup \{U_{1i} : i \geq 1\}} = \bigcup \{\overline{U_{1i}} : i \geq 1\} \subseteq F_0,$$

and for all $n > 1$

$$\overline{F_n} = \overline{\bigcup \{U_{ni} : i \geq n\}} = \bigcup \{\overline{U_{ni}} : i \geq n\} \subseteq \bigcup \{U_{(n-1)i} : i \geq n - 1\} = F_{n-1}.$$

Hence, $\mathcal{F} = \{F_n \colon n \in \omega\}$ is a regular $T_d$-sequence. Thus, by the subcompactness of $\mathbf{X}$ with respect to $T_d$, and Theorem 5, it follows that $\bigcap \mathcal{F} \neq \emptyset$. Since $\mathcal{U}_0$ is pairwise disjoint and $F_0 \in \mathcal{F}$, it follows that for every $x \in \bigcap \mathcal{F}$, $x \in U_{0i}$ for some $i \in \mathbb{N}$. Fix $x \in \bigcap \mathcal{F}$. Since $F_0 \in \mathcal{F}$, it follows that $x \in U_{0i}$ for some $i \in \mathbb{N}$. Since $\mathcal{U}_0$ is pairwise disjoint, it follows that $x \notin F_{i+1}$ and thus $x \notin \bigcap \mathcal{F}$. Contradiction! Thus, $\mathbf{X}$ is countably lightly compact as required.

(ii) $\rightarrow$ (i) Fix a countable regular filterbase $\mathcal{F}$ of open subsets of $\mathbf{X}$. By our hypothesis and Theorem 3, it follows that $\bigcap \{\overline{F} \colon F \in \mathcal{F}\} \neq \emptyset$. Since $\mathcal{F}$ is regular it follows that $\bigcap \mathcal{F} \neq \emptyset$, and $\mathbf{X}$ is subcompact as required. $\qquad \square$

As a corollary to Theorem 11 we get:

**Corollary 12** (ZF). *A second countable metrizable space $\mathbf{X} = (X, T)$ is compact if and only if it is countably subcompact with respect to $T$.*

PROOF: It suffices, in view of Theorem 11, to show that a countably compact, second countable metrizable space $\mathbf{X}$ is compact. Fix a countable base $\mathcal{B}$ of $\mathbf{X}$. Since $\mathbf{X}$ is compact if and only if every open cover $\mathcal{U}$ of $\mathbf{X}$ by members of $\mathcal{B}$ has a finite subcover, it follows at once from the countable compactness of $\mathbf{X}$ that $\mathbf{X}$ is compact. $\qquad \square$

Taking into consideration Theorem 11, one may ask whether the statement "every countably compact metrizable space is subcompact with respect to the base of all open sets" is a theorem of ZF. We observe next that this is not the case by establishing that it implies IWDI, a statement whose negation is known to be consistent with ZF, see e.g., Form 82 in [7].

**Theorem 13.** *The proposition "every countably compact metrizable space is subcompact with respect to the base of all open sets" implies IWDI.*

PROOF: Assume the contrary and let $X$ be an infinite weakly Dedekind-finite set. Let $d$ be the discrete metric on $X$. By our hypothesis, $\mathbf{X}$ has no denumerable open covers. So, $\mathbf{X}$ is trivially countably compact. Therefore, by our hypothesis, $\mathbf{X}$ is subcompact with respect to $T_d$. Since the family $\mathcal{F}$ of all cofinite subsets of $\mathbf{X}$ is trivially a regular filter with $\bigcap \mathcal{F} = \emptyset$, we arrive at a contradiction. Hence, IWDI holds true as required. $\qquad \square$

Next we answer Question 1 in the affirmative.

**Theorem 14** (ZF). *Let $\mathbf{X} = (X, T)$ be a metrizable space. The following are equivalent:*

(i) $\mathbf{X}$ *is compact;*
(ii) $\mathbf{X}$ *has no total filters;*
(iii) $\mathbf{X}$ *is subcompact relative to $T$.*

PROOF: Let $\mathbf{X} = (X, T)$ be a metrizable space and fix a metric $d$ on $X$ with $T = T_d$.

(i) $\to$ (ii) This follows from Theorem 10 and the fact that metrizable spaces are regular.

(ii) $\to$ (iii) Fix a regular filterbase $\mathcal{F}$ of $\mathbf{X}$. For our convenience we may assume that $\mathcal{F}$ is also an open filter of $\mathbf{X}$. We show that $\bigcap \mathcal{F} \neq \emptyset$. Assume the contrary and let $\bigcap \mathcal{F} = \emptyset$. We claim that $\mathcal{F}$ is a total filter. Since $\mathcal{F}$ is regular,

$$(2) \qquad \bigcap \mathcal{F} = \bigcap \{\overline{F} : F \in \mathcal{F}\}.$$

Since $\mathbf{X}$ is regular, by Proposition 8 and (2), it follows that $\mathcal{F}$ is a total filter. Contradiction!

(iii) $\to$ (i) Assume the contrary and fix a family $\mathcal{G}$ of closed subsets of $\mathbf{X}$ with the fip such that $\bigcap \mathcal{G} = \emptyset$. Without loss of generality we may assume that $\mathcal{G}$ is closed under finite intersections. Let

$$\mathcal{F} = \{U \in T : \text{ for some } G \in \mathcal{G}, \ G \subseteq U\}.$$

We claim that $\mathcal{F}$ is a regular filterbase of $\mathbf{X}$. To see that $\mathcal{F}$ is a filterbase of $\mathbf{X}$, fix $F_1, F_2 \in \mathcal{F}$ and let $G_1, G_2 \in \mathcal{G}$ satisfy $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$. Since $G_1 \cap G_2 \in \mathcal{G}$ and $G_1 \cap G_2 \subseteq F_1 \cap F_2$, it follows that $F_1 \cap F_2 \in \mathcal{F}$. To see that $\mathcal{F}$ is a regular filterbase, fix $F \in \mathcal{F}$ and let $G \in \mathcal{G}$ satisfy $G \subseteq F$. Since $\mathbf{X}$ is $T_4$ in ZF, it follows that there exist disjoint open sets $U, V$ of $\mathbf{X}$ with $G \subseteq U$ and $F^c \subseteq V$. It follows that $U \in \mathcal{F}$ and $V^c$ is closed. Clearly, $G \subseteq U \subseteq \overline{U} \subseteq V^c \subseteq F$. Hence, $\mathcal{F}$ is a regular filterbase.

Fix, by our hypothesis, $x \in \bigcap \mathcal{F}$. If $x \notin \bigcap \mathcal{G}$ then for some $G \in \mathcal{G}$, $x \notin G$. Hence, $G \subseteq \{x\}^c$ and $\{x\}^c \in \mathcal{F}$, meaning that $x \notin \bigcap \mathcal{F}$. Contradiction! Thus, $x \in \bigcap \mathcal{G}$, contradicting that $\bigcap \mathcal{G} = \emptyset$, and hence $\mathbf{X}$ is compact as required.     □

In ZFC, a subcompact metrizable space need not be countably compact, hence by Theorem 11, not countably subcompact with respect to the base of all open sets. Indeed, $\mathbb{N}$ with the discrete metric is subcompact ($\mathcal{B} = \{\{n\} : n \in \mathbb{N}\}$ is a subcompact base) but $\mathbb{N}$ is not countably compact. Next we show, in ZF, that every subcompact metrizable space is completely metrizable and every completely metrizable space is countably subcompact.

**Theorem 15** (ZF). (i) *Every subcompact metric space is a $G_\delta$ set in any of its completions.*

*In particular, every subcompact metric space is completely metrizable.*

(ii) *Every completely metrizable space is countably subcompact.*

PROOF: (i) A straightforward modification of the proof given in [5] page 763 shows that we can dispense with any use of AC. For the reader's convenience we

supply all the details below. Fix a metric space $\mathbf{X} = (X, d)$ having a subcompact base $\mathcal{B}$ and let $\mathbf{Y} = (Y, \sigma)$ be a completion of $\mathbf{X}$. Clearly, every $B \in \mathcal{B}$ can be expressed as $X \cap O$ for some open set $O$ of $\mathbf{Y}$ such that $\delta(B) = \delta(O)$ ($B$ is dense in $O$). Let

$$\mathcal{U} = \{U \in T_\sigma : U \cap X = B \text{ for some } B \in \mathcal{B}\}.$$

For every $n \in \mathbb{N}$ let

$$\mathcal{U}_n = \left\{U \in \mathcal{U} : \delta(U) < \frac{1}{n}\right\},$$

and $U_n = \bigcup \mathcal{U}_n$. We show that

$$X = \bigcap \{U_n : n \in \mathbb{N}\}.$$

Since $\mathcal{B}$ is a base for $\mathbf{X}$, it follows that for every $x \in X$ and every $n \in \mathbb{N}$, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $\delta(B) < 1/n$. Hence, there exists $U \in \mathcal{U}$ such that $x \in U$ and $\delta(U) < 1/n$. Therefore, for every $n \in \mathbb{N}$, $x \in U_n$. Hence,

(3)                                         $X \subseteq \bigcap \{U_n : n \in \mathbb{N}\}.$

To see the other direction of the inclusion, fix $y \in \bigcap \{U_n : n \in \mathbb{N}\}$. Since for every $n \in \mathbb{N}$, $y \in U_n$ it follows that there exists $U \in \mathcal{U}_n$ with $y \in U$. Hence,

(4)                                         $\bigcap \{U \in \mathcal{U} : y \in U\} = \{y\}.$

It is easy to see that

(5)      for every $U \in \mathcal{U}$ with $y \in U$ there exists $V \in \mathcal{U}$ with $y \in V$, $\overline{V} \subseteq U$.

Let $\mathcal{F} = \{B \in \mathcal{B} : B = U \cap X \text{ for some } U \in \mathcal{U} \text{ with } y \in U\}$. We claim that $\mathcal{F}$ is a regular filterbase of $\mathcal{B}$. It is not hard to verify that $\mathcal{F}$ is a filterbase of $\mathcal{B}$. To see that $\mathcal{F}$ is regular, fix $B \in \mathcal{F}$ and let $U \in \mathcal{U}$ satisfy $y \in U$ and $B = U \cap X$. By (5), there exists $V \in \mathcal{U}$ with $y \in V$ and $\overline{V} \subseteq U$. Let $B_V = V \cap X \in \mathcal{B}$. We have:

$$\overline{B_V}^X \subseteq \overline{B_V} \subseteq \overline{V} \subseteq U.$$

Therefore, for every $x \in \overline{B_V}^X$, $x \in U \cap X = B$, meaning that $\overline{B_V}^X \subseteq B$. Hence, $\mathcal{F}$ is a regular filterbase of $\mathcal{B}$ as claimed. By the subcompactness of $\mathcal{B}$ and (4) we get

$$\emptyset \neq \bigcap \mathcal{F} \subseteq \bigcap \{U \in \mathcal{U} : y \in U\} = \{y\}.$$

Therefore, $\bigcap \mathcal{F} = \{y\}$ and consequently $\bigcap \{U_n : n \in \mathbb{N}\} \subseteq X$. Hence, $X$ is a $G_\delta$ set in the complete metric space $\mathbf{Y}$.

The second assertion follows at once from the first part and Theorem 6.

(ii) Fix a complete metric space $\mathbf{X} = (X, d)$ and let $X_1, X_2$ be the sets of all limit and isolated points of $\mathbf{X}$, respectively. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, where

$$\mathcal{B}_1 = \left\{ B\left(x, \frac{1}{n}\right) : n \in \mathbb{N}, \ x \in X_1 \text{ and } B\left(x, \frac{1}{n}\right) \text{ has no other center } y \neq x \right\},$$

$$\mathcal{B}_2 = \left\{ B\left(x, \frac{1}{n}\right) \setminus \{y\} : n \in \mathbb{N}, \ x, y \in X_1, \ x \neq y, \ B\left(x, \frac{1}{n}\right) = B\left(y, \frac{1}{n}\right) \right\}$$

and

$$\mathcal{B}_3 = \{\{x\} : x \in X_2\}.$$

It is straightforward to see that $\mathcal{B}$ is a base of $\mathbf{X}$.

We claim that $\mathcal{B}$ is countably subcompact. To this end fix, in view of Theorem 5, a regular $\mathcal{B}$-sequence $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$. If $F = \{x\}$ for some $F \in \mathcal{F}$, then $\bigcap \mathcal{F} = F$. Assume that $\mathcal{F} \subseteq \mathcal{B}_1 \cup \mathcal{B}_2$. For every $n \in \mathbb{N}$ define

$$x_n = \begin{cases} x, & \text{if } F_n = B\left(x, \dfrac{1}{k_n}\right) \in \mathcal{B}_1, \\[2mm] y, & \text{if } F_n = B\left(t, \dfrac{1}{k_n}\right) \setminus \{y\} \in \mathcal{B}_2, \end{cases}$$

where $k_n$ is the least $k \in \mathbb{N}$ such that $B(x, 1/k) \in \mathcal{B}_1$ or $B(x, 1/k) \setminus \{y\} \in \mathcal{B}_2$. Note that $x$ and $y$ are uniquely determined by $F_n$: for example, $x$ is the unique center of $F_n$ and $y$ is the unique element of $\overline{F_n} \setminus F_n$ such that $d(y, z) < 1/k_n$ for all $z \in F_n$.

By the well-known fact that every infinite sequence has a monotone subsequence, we can consider the following two cases:

(a) Let $(k_n)_{n \in \mathbb{N}}$ have a strictly increasing subsequence $(k_{i_n})_{n \in \mathbb{N}}$. In this case, it follows that $(x_{i_n})_{n \in \mathbb{N}}$ is a Cauchy sequence of $\mathbf{X}$. Hence, by the completeness of $\mathbf{X}$, $(x_{i_n})_{n \in \mathbb{N}}$ converges to some $x \in X$. It is straightforward to see that $x \in \bigcap \mathcal{F}$.

(b) Let $(k_n)_{n \in \mathbb{N}}$ have a decreasing subsequence $(k_{i_n})_{n \in \mathbb{N}}$. In this case, there must exist an $n_0 \in \mathbb{N}$ such that $k_{i_n} = k_{i_{n_0}}$ for all $n \geq n_0$. We claim that for all $n \geq n_0$, $x_{i_{n_0}} \in \overline{F_n}$. To this end, fix $n > n_0$. Since $\overline{F_{i_n}} \subseteq F_{i_{n_0}}$ it follows that $d(x_{i_n}, x_{i_{n_0}}) < 1/k_{i_n}$. Since $x_{i_{n_0}} \in X_1$, it follows that $x_{i_{n_0}} \in \overline{F_{i_n}}$. Therefore, $x_{i_{n_0}} \in \bigcap \mathcal{F}$.

From cases (a) and (b) it follows that $\bigcap \mathcal{F} \neq \emptyset$, and $\mathbf{X}$ is countably subcompact as required. □

In view of Theorem 11, a metrizable space which is countably subcompact with respect to the base of all open sets is completely metrizable in ZF. We show next that this is not the case if we only assume the space to be countably subcompact.

**Theorem 16.** *Each of the following statements*

(i) *every metrizable, countably subcompact topological space is completely metrizable;*

(ii) *every countably subcompact metrizable space is subcompact;*

(iii) *every completely metrizable space is subcompact;*

*implies* IDI($\mathbb{R}$). *In particular, none of* (i)–(iii) *is a theorem of* ZF.

PROOF: Assume the contrary and fix an infinite Dedekind finite subset $D$ of $\mathbb{R}$. Without loss of generality we may assume that $D \cap \mathbb{Q} = \emptyset$. By Theorem 7 we may assume that $D$ is dense in $\mathbb{R}$.

(i) Clearly,

$$\mathcal{B} = \{(x,y) \cap \mathbb{Q} \colon x, y \in D, \ x < y\}$$

is a base for $\mathbb{Q}$ endowed with the usual (Euclidean) metric $|\cdot|$. Since $D \times D$ is Dedekind finite, it follows that $\mathcal{B}$ has no denumerable subsets. Thus, $\mathbb{Q}$ is trivially countably subcompact with respect to $\mathcal{B}$. Hence, by our hypothesis $\mathbb{Q}$ is completely metrizable. However, $\mathbb{Q}$ is not completely metrizable (for every metric $d$ on $\mathbb{Q}$ with $T_d = T_{|\cdot|}$, $\mathbb{Q}$ can be expressed as a union of countably many closed nowhere dense sets. Therefore, $\mathbb{Q}$ is not Baire. Since separable completely metrizable spaces are Baire in ZF, see e.g. [2], it follows that $\mathbb{Q}$ is not completely metrizable). Contradiction!

(ii), (iii) It is easy to see that

$$\mathcal{B} = \{(x,y) \cap D \colon x, y \in D, \ x < y\}$$

is a base for the subspace $\mathbf{D}$ of $\mathbb{R}$ endowed with the usual metric. Since $D$ is Dedekind finite, $\mathbf{D}$ is countably subcompact (with respect to $\mathcal{B}$) and complete. Hence, by our hypotheses in (ii) and (iii), $\mathbf{D}$ is subcompact. By Theorem 15 (i), $D$ is $G_\delta$ in its completion $\mathbb{R}$. Fix a family $\{U_n \colon n \in \mathbb{N}\}$ of open sets of $\mathbb{R}$ with

$$D = \bigcap \{U_n \colon n \in \mathbb{N}\}$$

and define a map $f \colon D \to \mathbb{R}^\omega$ by requiring:

$$f(x) = \left(x, \frac{1}{d(x, U_1^c)}, \frac{1}{d(x, U_2^c)}, \dots\right).$$

Clearly, for every $n \in \mathbb{N}$, the function $h_n \colon \mathbb{R} \to \mathbb{R}$, $h_n(x) = d(x, U_n^c)$ is strictly positive on $U_n$ and continuous. Hence, $f$ is well defined, continuous and $1 : 1$. Since, $f^{-1} \colon f(D) \to D$ coincides with the restriction of the projection $\pi_0$ to $f(D)$, it follows that $f^{-1}$ is continuous. Therefore, $f \colon D \to f(D)$ is a homeomorphism.

We claim that $f(D)$ is closed in $\mathbb{R}^\omega$. To this end, fix $y = (a_0, a_1, \dots, a_n, \dots) \in \mathbb{R}^\omega \setminus f(D)$. We consider the following cases:

(a) Let $a_0 \in D$. In this case there exists $k \in \mathbb{N}$, such that $1/d(a_0, U_k^c) \neq a_k$. Since $1/h_k$ is continuous, it follows that for $\varepsilon = |1/d(a_0, U_k^c) - a_k|$ there is a $\delta > 0$ such that for all $x \in (a_0 - \delta, a_0 + \delta)$,

$$\left| \frac{1}{d(a_0, U_k^c)} - \frac{1}{d(x, U_k^c)} \right| < \frac{\varepsilon}{3}.$$

Therefore,

$$V = (a_0 - \delta, a_0 + \delta) \times \mathbb{R} \times \cdots \times \mathbb{R} \times \left( a_k - \frac{\varepsilon}{3}, a_k + \frac{\varepsilon}{3} \right) \times \mathbb{R} \times \cdots$$

is a neighborhood of $y$ avoiding $f(D)$.

(b) Let $a_0 \notin D$. In this case there exists $k \in \mathbb{N}$, such that $a_0 \in U_k^c$. Since, $\lim_{x \to a_0} d(x, U_k^c) = 0$, it follows that for $\varepsilon = 1/(|a_k| + 1)$, there exists a $\delta > 0$ such that for every $x \in (a_0 - \delta, a_0 + \delta)$, $d(x, U_k^c) < \varepsilon$. Then,

$$U = (a_0 - \delta, a_0 + \delta) \times \mathbb{R} \times \cdots \times \mathbb{R} \times \left( a_k - \frac{1}{2}, a_k + \frac{1}{2} \right) \times \mathbb{R} \times \cdots$$

is a neighborhood of $y$ included in $f(D)^c$.

From (a) and (b) it follows that $f(D)$ is a closed subset of $\mathbb{R}^\omega$ as claimed.

In [13] it has been shown, in ZF, that the family of all nonemtpy closed subsets of $\mathbb{R}^\omega$ has a choice set. Since $\mathbb{R}^\omega$ is separable, hence second countable, it follows that its subspace $f(D)$ is also second countable. Therefore $f(D)$, and consequently $\mathbf{D}$, is separable, contradicting the fact that $D$ is Dedekind finite.

The second assertion follows from the fact that in Cohen's basic model $\mathcal{M}1$ in [7], IDI($\mathbb{R}$) fails.                                    $\square$

**Remark 1**. One can easily adopt the proof of part (ii) of Theorem 16 to get a ZF proof of Theorem 6 ($\mathbb{R}^\omega$ is completely metrizable in ZF).

Next, we show that AC is equivalent to the assertion that subcompactness is an invariant for the forming of topological products.

**Theorem 17.** *The following are equivalent:*
- (i) *AC;*
- (ii) *for every family $\{\mathbf{X}_i : i \in I\}$ of subcompact $T_3$ spaces for every family $\{\mathcal{B}_i : i \in I\}$ such that for every $i \in I$, $\mathcal{B}_i$ is a subcompact base for $\mathbf{X}_i$, the Tychonoff product $\mathbf{X} = \prod_{i \in I} \mathbf{X}_i$ is subcompact with respect to the standard base $\mathcal{B}$ of $\mathbf{X}$ generated by the family $\{\mathcal{B}_i : i \in I\}$;*
- (iii) *for every family $\{\mathbf{X}_i : i \in I\}$ of metrizable subcompact spaces, for every family $\{\mathcal{B}_i : i \in I\}$ such that for every $i \in I$, $\mathcal{B}_i$ is a subcompact base for $\mathbf{X}_i$, the Tychonoff product $\mathbf{X} = \prod_{i \in I} \mathbf{X}_i$ is subcompact with respect to the standard base $\mathcal{B}$ of $\mathbf{X}$ generated by the family $\{\mathcal{B}_i : i \in I\}$.*

PROOF: (i) → (ii) follows from Theorem 2 and (ii) → (iii) is straightforward.

(iii) → (i) Fix a pairwise disjoint family $\mathcal{A} = \{A_i : i \in I\}$ of nonempty sets and let $\infty$ be a set not in $\bigcup \mathcal{A}$. For every $i \in I$, let $X_i = A_i \cup \{\infty\}$ carry the discrete metric. Clearly, for every $i \in I$, $\mathcal{B}_i = \{\{x\} : x \in X_i\} \cup \{A_i\}$ is a subcompact base for $\mathbf{X}_i$. By our hypothesis, the product $\mathbf{X} = \prod_{i \in I} \mathbf{X}_i$ is subcompact with respect to the standard base $\mathcal{B}$ of $\mathbf{X}$ generated by the family $\{\mathcal{B}_i : i \in I\}$. It is straightforward to verify that

(6) $\qquad \mathcal{F} = \left\{ \bigcap Q : Q \text{ is a finite nonempty subset of } \{\pi_i^{-1}(A_i) : i \in I\} \right\}$

is a regular $\mathcal{B}$-filterbase of $\mathbf{X}$. Hence, by our hypothesis, $\bigcap \mathcal{F} \neq \emptyset$. Clearly, any element $f \in \bigcap \mathcal{F}$ is a choice function of $\mathcal{A}$. $\qquad \square$

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