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Self-small products of abelian groups

JOSEF DVOŘÁK, JAN ŽEMLIČKA

Abstract. Let A and B be two abelian groups. The group A is called B -small if the covariant functor $\text{Hom}(A, -)$ commutes with all direct sums $B^{(\kappa)}$ and A is self-small provided it is A -small. The paper characterizes self-small products applying developed closure properties of the classes of relatively small groups. As a consequence, self-small products of finitely generated abelian groups are described.

Keywords: self-small abelian group; slender group

Classification: 20K40, 20K20, 20K21

Introduction

The study of modules whose covariant functor $\text{Hom}(M, -)$ commutes with all direct sums, which is a condition providing a categorial generalization of the notion of finitely generated module, started in 60's by the work of H. Bass [6, page 54] and R. Rentschler [19]. Such modules have appeared as a useful tool in diverse contexts and under various names (small, Σ -compact, U-compact, dually slender) in ring theory, module theory and in the study of abelian groups. In 1974, D.M. Arnold and C.E. Murley published their influential paper [5] dedicated to a weaker variant of the studied condition, namely commuting of the functor $\text{Hom}(A, -)$ with direct sums of the tested module itself. Groups and modules satisfying this restricted condition are usually called self-small in literature. Many interesting results concerning self-small modules over unital rings in general have appeared later [1], [10], [11], [16], [18], self-small abelian groups proving to be a particularly successful tool [2], [4], [3], [7], [8], [9].

The aim of this paper is to deepen the present knowledge about structure of self-small groups and about possibilities of testing abelian groups for self-smallness by adopting some ideas of the papers [2], [13], [17] and extending several results of [12], [20]. Namely, we deal with the notion of a relatively small abelian group (defined in [2], [16], cf. also relatively compact objects in [17]) which serves as a tool for characterization of those products of groups that are self-small.

Throughout the paper *module* means a right module over an associative ring with unit and an abelian group is a module over the ring of integers. Note that we will use the term *group* instead of abelian group frequently, as non-abelian groups are not considered here. If A and B are two abelian groups, then $\text{Hom}(A, B)$ denotes the abelian group of homomorphisms $A \rightarrow B$. A *family of groups* means a discrete diagram in the category of abelian groups, so a family may contain more than one copy of a group. The set of all prime numbers is denoted by \mathbb{P} and we identify cardinals with least ordinals of given cardinality.

For non-explained terminology we refer to [14], [15].

1. Relatively small groups

Let A be an abelian group and \mathcal{N} a family of abelian groups. It is well-known (and easy to see) that the functor $\text{Hom}(A, -)$ induces an injective homomorphism of abelian groups

$$\Psi_{\mathcal{N}}: \bigoplus_{N \in \mathcal{N}} \text{Hom}(A, N) \rightarrow \text{Hom}\left(A, \bigoplus \mathcal{N}\right)$$

given by the rule $\Psi_{\mathcal{N}}((f_N)_N) = \sum_N f_N$, cf. e.g. [17, Lemma 1.3], where $\sum_N f_N \in \text{Hom}(A, \bigoplus \mathcal{N})$ is defined by the rule $a \rightarrow \sum_N f_N(a)$ for f_N viewed as a homomorphism into $\bigoplus \mathcal{N}$. Suppose, then, that \mathcal{C} is a class of groups. We say that A is \mathcal{C} -small if $\Psi_{\mathcal{N}}$ is an isomorphism for any family \mathcal{N} of groups from the class \mathcal{C} . If B is an abelian group, A is said to be B -small provided it is a $\{B\}$ -small group, cf. [2], [13], [16], [17]. It is clear that A -small abelian groups A are exactly *self-small* ones as defined in [5].

Example 1.1. (1) Every finitely generated abelian group is small, so B -small for every group B . In, particular each finite group is self-small.

(2) Let A and B be two abelian groups such that $\text{Hom}(A, B) = 0$. Then it is easy to see that A is B -small.

In particular, if $p, q \in \mathbb{P}$ are different primes, A_p is an abelian p -group and A_q is an abelian q -group, then A_p is A_q -small and \mathbb{Z} -small.

Example 1.2. It is clear, \mathbb{Q} and \mathbb{Q}/\mathbb{Z} are \mathbb{Q} -small groups but neither \mathbb{Q} nor \mathbb{Q}/\mathbb{Z} is \mathbb{Q}/\mathbb{Z} -small. Furthermore, \mathbb{Q} -small groups are precisely groups of finite torsion-free rank by [2, Corollary 4.3.].

We start with an elementary observation which translates the definition of a relative small group to an easily tested condition, cf. [19, Section 1], [17, Lemma 1.4 (2)] and [13, Theorem 1.6 (2)]:

Lemma 1.3. *Let A be an abelian group and \mathcal{C} a class of abelian groups. Then A is \mathcal{C} -small if and only if for each family \mathcal{N} of groups contained in the class \mathcal{C} and every*

$f \in \text{Hom}(A, \bigoplus \mathcal{N})$ there exists a finite family $\mathcal{F} \subseteq \mathcal{N}$ such that $f(A) \subseteq \bigoplus \mathcal{F}$. In particular, for a group B , the group A is B -small if and only if for every index set I and every $f \in \text{Hom}(A, B^{(I)})$ there exists a finite subset $F \subseteq I$ such that $f(A) \subseteq B^{(F)}$.

PROOF: The argument of the proof is well known; if $\Psi_{\mathcal{N}}$ is onto and $f \in \text{Hom}(A, \bigoplus \mathcal{N})$, then f is of the form $\sum_{N_i} f_{N_i}$ for a finite family $(N_i : i = 1, \dots, n) \subseteq \mathcal{N}$ and $f_{N_i} \in \text{Hom}(A, N_i)$, hence $f(A) = \sum_{N_i} f_{N_i}(A) \subseteq \bigoplus_{i=1}^n N_i$. On the other hand, if $f(A) \subseteq \bigoplus_{i=1}^n N_i \subseteq \bigoplus \mathcal{N}$, then $\Psi_{\mathcal{N}}((\pi_{N_i} f)_{N_i}) = f$, where π_{N_i} denotes the projection onto the corresponding component. \square

The observation that the concept of relatively small groups is general enough if we consider relative smallness over a set of groups, which is a particular case of general [13, Lemma 2.1], presents a first application of the previous lemma. To that end, for a class of groups define

$$\text{Add}(\mathcal{C}) = \left\{ A : A \text{ is a direct summand of } \bigoplus_{\alpha < \kappa} C_{\alpha} \text{ for some cardinal } \kappa \text{ and } C_{\alpha} \in \mathcal{C} \right\}$$

and by $\text{Add}(A)$ denote $\text{Add}(\{A\})$.

Lemma 1.4. *Let A be an abelian group and \mathcal{C} be a set of abelian groups. Then the following conditions are equivalent:*

- (1) A is $\bigoplus \mathcal{C}$ -small;
- (2) A is \mathcal{C} -small;
- (3) A is $\text{Add}(\bigoplus \mathcal{C})$ -small.

PROOF: (1) \Rightarrow (3) Put $B = \bigoplus \mathcal{C}$, let \mathcal{N} be a family of groups contained in $\text{Add}(B)$, and $f \in \text{Hom}(A, \bigoplus \mathcal{N})$. Then for each $N \in \mathcal{N}$ there exists a cardinal κ_N for which $N \subseteq B^{(\kappa_N)}$ (N is also a direct summand of $B^{(\kappa_N)}$), and so $f(A) \subseteq \bigoplus \mathcal{N} \subseteq \bigoplus_{N \in \mathcal{N}} B^{(\kappa_N)}$. Since A is B -compact, there exists finite family $\mathcal{F} \subseteq \mathcal{N}$ such that $f(A) \subseteq \bigoplus_{N \in \mathcal{F}} B^{(\kappa_N)}$ which implies that $f(A) \subseteq \bigoplus \mathcal{F}$.

(3) \Rightarrow (2) It is obvious since $\mathcal{C} \subseteq \text{Add}(\bigoplus \mathcal{C})$.

(2) \Rightarrow (1) As any group $B \in \mathcal{C}$ is a direct summand of $\bigoplus \mathcal{C}$, the same argument as in the implication (1) \Rightarrow (3) proves the assertion. \square

Since $\text{Add}(B) = \text{Add}(B^{(\kappa)})$ for an arbitrary group B and a nonzero cardinal κ , we obtain the following useful criterion:

Corollary 1.5. *Let A and B be abelian groups and κ a nonzero cardinal. Then A is B -small if and only if A is $B^{(\kappa)}$ -small.*

As a consequence, we can formulate a well-known closure property of the class of all self-small groups.

Corollary 1.6. *Let κ be a cardinal and A an abelian group. Then $A^{(\kappa)}$ is self-small if and only if A is self-small and κ is finite.*

Let us formulate a variant of the assertion [2, Theorem 4.1.], which generalizes the classical criterion of self-small groups [5, Proposition 1.1] for the case of relatively small groups, cf. [13, Lemma 3.3]. Recall that the family $(A_i : i < \omega)$ is said to be ω -filtration of a group A , if it is a chain of subgroups of A , i.e. $A_i \subseteq A_{i+1}$ for each $i < \omega$, with $A = \bigcup_{n < \omega} A_n$.

Proposition 1.7. *The following conditions are equivalent for abelian groups A and B :*

- (1) A is not B -small;
- (2) there exists a homomorphism $f \in \text{Hom}(A, B^{(\omega)})$ such that $f(A) \not\subseteq B^{(n)}$ for all $n < \omega$;
- (3) there exists an ω -filtration $(A_i : i < \omega)$ of A such that for each $n < \omega$ there exists a nonzero $f_n \in \text{Hom}(A, B)$ satisfying $f_n(A_n) = 0$;
- (4) there exists an ω -filtration $(A_i : i < \omega)$ of A such that $\text{Hom}(A/A_n, B) \neq 0$ for each $n < \omega$.

PROOF: The proof works using similar arguments as in [5, Proposition 1.1].

(1) \Rightarrow (2) By Lemma 1.3 there exists a set I and $g \in \text{Hom}(A, B^{(I)})$ such that $g(A) \not\subseteq B^{(F)}$ for any finite $F \subset I$. Then we can construct by induction a sequence of finite sets $I_n \subset I$ such that $I_0 = \emptyset$, $|I_n \setminus I_{n-1}| = 1$ and $\ker \pi_{I_{n-1}} g \supsetneq \ker \pi_{I_n} g$ for all $n < \omega$ where $\pi_{I_n} \in \text{Hom}(B^{(I)}, B^{(I_n)})$ denotes the natural projection. If we put $I_\omega = \bigcup_{i < \omega} I_i$, then $\pi_{I_\omega} g \in \text{Hom}(A, B^{(I_\omega)})$ represents the desired homomorphism.

(2) \Rightarrow (3) Let $f \in \text{Hom}(A, B^{(\omega)})$ satisfy the condition (2) and define $A_n = f^{-1}(B^{(n, \omega)})$ where $B^{(n, \omega)} = \{b \in B^{(\omega)} : \pi_i(b) = 0 \ \forall i \leq n\}$ for natural projections $\pi_i : B^{(\omega)} \rightarrow B$ onto the i th coordinate. Then $A = \bigcup_{i < \omega} A_i$ and for each $i < \omega$ there exists $n_i > i$ such that $f_i = \pi_{n_i} f \neq 0$ with $f_i(A_i) = 0$.

(3) \Rightarrow (4) It is enough to observe that any nonzero $f_n \in \text{Hom}(A, B)$ that satisfies $f_n(A_n) = 0$ can be factorized through the natural projection $A \rightarrow A/A_n$, so there exists a nonzero $\tilde{f}_n \in \text{Hom}(A/A_n, B)$.

(4) \Rightarrow (1) Let $f_i \in \text{Hom}(A/A_i, B)$ denote a nonzero homomorphism and define a homomorphism $f \in \text{Hom}(A, B^\omega)$ by the rule $\pi_i(f(a)) = f_i(a + A_i)$ for each $a \in A$ and $i < \omega$. Then $f(A) \subseteq B^{(\omega)}$ since for each $a \in A$ there exists n such that $a \in A_i$ for all $i \geq n$, hence $f \in \text{Hom}(A, B^{(\omega)})$. On the other hand, $f(A) \not\subseteq B^{(n)}$ for any $n < \omega$ as $\pi_n f \neq 0$, $i < \omega$. Thus A is not B -small by Lemma 1.3. \square

The previous assertion applied on $A = B$ allows us to reformulate [12, Proposition 9].

Corollary 1.8. *The following conditions are equivalent for an abelian group A :*

- (1) A is not self-small;
- (2) there exists an ω -filtration $(A_i : i < \omega)$ of A such that $\text{Hom}(A/A_n, A) \neq 0$ for each $n < \omega$;
- (3) there exists an ω -filtration $(A_i : i < \omega)$ of A such that for each $n < \omega$ there exists a nonzero $\varphi_n \in \text{End}(A)$ satisfying $\varphi_n(A_n) = 0$.

Example 1.9. Put $P = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$. Then $\text{Hom}(\mathbb{Q}, P) = 0$ by [12, Example 4], hence \mathbb{Q} is P -small. On the other hand, if we put $B = P / \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p$, then there exists an ω -filtration $(B_i : i < \omega)$ of B such that $\text{Hom}(B/B_n, \mathbb{Q}) \neq 0$ for each n by [12, Example 3]. If we take preimages A_n of all B_n in canonical projection $P \rightarrow P / \bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p$, then $(A_i : i < \omega)$ forms an ω -filtration of A satisfying $\text{Hom}(A/A_n, \mathbb{Q}) \cong \text{Hom}(B/B_n, \mathbb{Q}) \neq 0$, hence P is not \mathbb{Q} -small by Proposition 1.3. Note that the claim follows also from [2, Corollary 4.3.].

2. Closure properties of relative smallness

First, let us formulate several elementary relations between classes of relatively small groups.

Lemma 2.1. *Let A, B and C be abelian groups and I be a set. Suppose that A is B -small.*

- (1) *If C is a subgroup of A , then A/C is B -small.*
- (2) *If C is embeddable into B^I , then A is C -small.*

PROOF: (1) Proving indirectly, we assume that $\bar{A} = A/C$ is not B -small. Then there exists an ω -filtration $(\bar{A}_i : i < \omega)$ of \bar{A} for which $\text{Hom}(\bar{A}/\bar{A}_n, B) \neq 0$ for all $n < \omega$ by Proposition 1.7. If we lift all the groups of the ω -filtration of \bar{A} to the ω -filtration $(A_i : i < \omega)$ of A satisfying the conditions $C \leq A_n$ and $A_n/C = \bar{A}_n$ for each n , then $\text{Hom}(A/A_n, B) \cong \text{Hom}(\bar{A}/\bar{A}_n, B) \neq 0$, hence A is not B -small by Proposition 1.7.

(2) We may suppose without loss of generality that $C \leq B^I$. Assume A is not C -small and consider the ω -filtration $(A_i : i < \omega)$ of A for which homeomorphism $\text{Hom}(A/A_n, C) \neq 0$ provided by Proposition 1.7. Then we have homeomorphism $\text{Hom}(A/A_n, B^I) \neq 0$ for each $n < \omega$ and since for each nonzero $f_n \in \text{Hom}(A/A_n, B^I)$ there exists $i \in I$ such that $\pi_i f_n \neq 0$, we conclude that $\text{Hom}(A/A_n, B) \neq 0$ for every $n < \omega$, a contradiction. \square

Proposition 2.2. *Let A be a self-small abelian group.*

- (1) *If $f \in \text{Hom}(A, A^I)$ for an index set I , then $f(A)$ is self-small.*
- (2) *If $I \subseteq \text{End}(A)$, then $A / \bigcap \{\ker \iota : \iota \in I\}$ is self-small.*

PROOF: (1) Since A is A -small, $f(A)$ is A -small by Lemma 2.1 (1). Thus $f(A)$ is $f(A)$ -small by Lemma 2.1 (2).

(2) If $\varphi: A \rightarrow A^I$ is defined by the rule $\pi_\iota \varphi = \iota$ for each $\iota \in I$, then $\ker \varphi = \bigcap \{\ker \iota: \iota \in I\}$, hence $A / \bigcap \{\ker \iota: \iota \in I\} \cong f(A)$ is self-small by (1), cf. also [13, Example 2.10]. \square

The next assertion describes closure properties concerning extensions.

Proposition 2.3. *Let A and C be abelian groups and $B \leq C$.*

- (1) *If both B and C/B are A -small, then C is A -small.*
- (2) *If A is B -small and C/B -small, then A is C -small.*

PROOF: Similarly as in Lemma 2.1, we will use throughout the whole proof the correspondence of relative nonsmallness and properties of ω -filtrations given by Proposition 1.7. Let us denote by $\pi_B: C \rightarrow C/B$ the natural projection and by $\iota_B: B \rightarrow C$ the inclusion homomorphism.

(1) Suppose that $(C_n: n < \omega)$ is an ω -filtration of C . Then $(C_n \cap B: n < \omega)$ is an ω -filtration of B and $(C_n + B/B: n < \omega)$ is an ω -filtration of C/B . Since B and C/B are A -small, there exists n such that $f(B) = 0$ whenever $f \in \text{Hom}(B, A)$ satisfies $f(B \cap C_n) = 0$, and $\tilde{f}(C/B) = 0$ whenever $\tilde{f} \in \text{Hom}(C/B, A)$ satisfies $\tilde{f}(C_n + B/B) = 0$.

Let $f \in \text{Hom}(C, A)$ be such that $f(C_n) = 0$, then $f(B) = 0$ as $f(C_n \cap B) = 0$ and there exists $\tilde{f} \in \text{Hom}(C/B, A)$ for which $\tilde{f}\pi_B = f$, where $\pi_B: C \rightarrow C/B$ denotes the natural projection. Now, $\tilde{f}(C/B) = 0$ since $\tilde{f}(C_n + B/B) = 0$, hence $f = \tilde{f}\pi_B = 0$. We have proved that C is an A -small group.

(2) Let $(A_n: n < \omega)$ be an ω -filtration of A . Since A is B -small, there exists n for which both $\text{Hom}(A/A_n, B)$ and $\text{Hom}(A/A_n, C/B)$ vanish. If $f \in \text{Hom}(A/A_n, C)$ is nonzero, then, we have $\pi_B f = 0$ since $\pi_B f \in \text{Hom}(A/A_n, C/B) = 0$. Hence f factorizes through B as $f = \iota_B \tilde{f}$ for some $\tilde{f} \in \text{Hom}(A/A_n, B)$. By the assumption on $\text{Hom}(A/A_n, B)$ we have $\tilde{f} = 0$ and therefore $f = 0$. Thus $\text{Hom}(A/A_n, C) = 0$ and so A is C -small. \square

Example 2.4. The implication of the previous claim cannot be reversed:

(1) $\prod_{p \in \mathbb{P}} \mathbb{Z}_p$ is self-small by [20, Theorem 2.5 and Example 2.7], but $\bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p$ is not $\prod_{p \in \mathbb{P}} \mathbb{Z}_p$ -small.

(2) Since $\text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$, the group \mathbb{Q}/\mathbb{Z} is \mathbb{Q} -small, but \mathbb{Q}/\mathbb{Z} is not \mathbb{Q}/\mathbb{Z} -small.

Lemma 2.5. *Let A be an abelian group and \mathcal{M} a finite family of abelian groups.*

- (1) *If N is A -small for each $N \in \mathcal{M}$, then $\bigoplus \mathcal{M}$ is A -small.*
- (2) *If A is N -small for each $N \in \mathcal{M}$, then A is $\bigoplus \mathcal{M}$ -small.*

PROOF: Put $M = \bigoplus \mathcal{M}$. Both of the proofs proceed by induction on the cardinality of \mathcal{M} .

(1) If $|\mathcal{M}| \leq 1$, there is nothing to prove. Let the assertion hold true for $|\mathcal{M}| - 1$ and put $M_N = \bigoplus \mathcal{M} \setminus \{N\}$ for arbitrary $N \in \mathcal{M}$. Since M_N is A -small by the induction hypothesis, N is A -small by the hypothesis and $M/N \cong M_N$, we get that M is A -small by Proposition 2.3 (1).

(2) The same induction argument as in (1) shows A is M -small by Lemma 2.3 (2), since A is N -small by the hypothesis and it is M_N -small for each $N \in \mathcal{M}$ by the induction hypothesis. \square

As the main result of the section we describe which finite sums of relatively small abelian groups are again relatively small.

Proposition 2.6. *Let \mathcal{M} and \mathcal{N} be finite families of abelian groups. The following conditions are equivalent:*

- (1) $\bigoplus \mathcal{M}$ is $\bigoplus \mathcal{N}$ -small;
- (2) M is $\bigoplus \mathcal{N}$ -small for each $M \in \mathcal{M}$;
- (3) $\bigoplus \mathcal{M}$ is N -small for each $N \in \mathcal{N}$;
- (4) M is N -small for each $M \in \mathcal{M}$ and $N \in \mathcal{N}$;
- (5) for each $M \in \mathcal{M}$, $N \in \mathcal{N}$; and ω -filtration $(M_i : i < \omega)$ of M , there exist $i < \omega$ with $\text{Hom}(M/M_i, N) = 0$.

PROOF: (1) \Rightarrow (2) Put $F_M := \bigoplus (\mathcal{M} \setminus \{M\}) \leq \bigoplus \mathcal{M}$ and since $(\bigoplus \mathcal{M})/F_M \cong M$, the claim follows from Lemma 2.1 (1).

(1) \Rightarrow (3) Since $N \leq \bigoplus \mathcal{N}$ the assertion is clear by Lemma 2.1 (2).

(2) \Rightarrow (4), (3) \Rightarrow (4) It follows from Lemma 2.1 again.

The implication (4) \Rightarrow (3) is a consequence of Lemma 2.5 (1), while the implication (3) \Rightarrow (1) is shown in Lemma 2.5 (2).

(4) \Leftrightarrow (5) It is an immediate consequence of Proposition 1.7. \square

As a consequence we reformulate [12, Proposition 5]:

Corollary 2.7. *The following conditions are equivalent for a finite family of abelian groups \mathcal{M} and $M = \bigoplus \mathcal{M}$:*

- (1) M is self-small;
- (2) N_1 is N_2 -small for each $N_1, N_2 \in \mathcal{M}$;
- (3) for every $N_1, N_2 \in \mathcal{M}$ and ω -filtration $(M_i : i < \omega)$ of N_1 there exist $i < \omega$ with $\text{Hom}(N_1/M_i, N_2) = 0$.

Example 2.8. Since $\text{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$ and \mathbb{Q} is self-small and \mathbb{Z} is small so \mathbb{Z} -small and \mathbb{Q} -small, the group $\mathbb{Z} \oplus \mathbb{Q}$ is self-small by Corollary 2.7.

3. Self-small products

We start the section by a criterion of self-smallness of a general product, cf. [13, Theorem 5.4].

Theorem 3.1. *Let \mathcal{M} be a family of abelian groups and put $M = \prod \mathcal{M}$ and $S = \bigoplus \mathcal{M}$. Then the following conditions are equivalent:*

- (1) M is self-small;
- (2) M is S -small;
- (3) M is $\bigoplus \mathcal{C}$ -small for each countable family $\mathcal{C} \subseteq \mathcal{M}$.

PROOF: The implications (1) \Rightarrow (2) \Rightarrow (3) follow from Lemma 2.1 (2), since S is embeddable into M and $\bigoplus \mathcal{C}$ is embeddable into S .

(3) \Rightarrow (1) Proving indirectly, assume that M is not self-small. Then there exists an ω -filtration $(M_i : i < \omega)$ of M for which $\text{Hom}(M/M_n, M) \neq 0$ for all $n < \omega$ by Proposition 1.7. Note that if $0 \neq f: M/M_n \rightarrow M = \prod \mathcal{M}$, then there must exist an $N \in \mathcal{M}$ with $0 \neq \pi_N f: M/M_n \rightarrow N$, where $\pi_N: M \rightarrow N$ denotes the canonical projection of the product. Hence for each $n < \omega$ there exists $A_n \in \mathcal{M}$ such that $\text{Hom}(M/M_n, A_n) \neq 0$. If we put $\mathcal{C} = \{A_i : i < \omega\}$, then all A_i 's are embeddable into $\bigoplus \mathcal{C}$, hence $\text{Hom}(M/M_n, \bigoplus \mathcal{C}) \neq 0$ for each $n < \omega$, which implies that M is not $\bigoplus \mathcal{C}$ -small by Proposition 1.7. \square

As A^κ is $A^{(\kappa)}$ -small if and only if it is A -small by Corollary 1.5 we obtain the following consequence of Theorem 3.1.

Corollary 3.2. *Let A be an abelian group and I a set. Then A^I is self-small if and only if it is A -small.*

Example 3.3. (1) \mathbb{Q}^ω is not self-small, since it is an infinitely generated \mathbb{Q} -vector space, hence it is not \mathbb{Q} -small.

(2) We have recalled in Example 2.4 that $\prod_{p \in \mathbb{P}} \mathbb{Z}_p$ is self-small, so it is $\bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p$ -small group by Theorem 3.1.

Let us denote by $T_A = \bigoplus_{p \in \mathbb{P}} A_{(p)}$ the torsion part of an abelian group A where $A_{(p)}$ denotes the p -component of the torsion part.

Lemma 3.4. *Let $p \in \mathbb{P}$, P be a nonzero p -group, R a nonzero torsion group, \mathcal{T} a family of finite torsion groups, and κ be a cardinal. Then:*

- (1) \mathbb{Z}_p^κ is P -small if and only if κ is finite;
- (2) \mathbb{Z}^κ is R -small if and only if κ is finite;
- (3) if $\prod \mathcal{T}$ is P -small, then $\{T \in \mathcal{T} : T_{(p)} \neq 0\}$ is finite;
- (4) if $\prod \mathcal{T}$ is R -small, then $\{T \in \mathcal{T} : T_{(p)} \neq 0\}$ is finite for each $p \in \mathbb{P}$ satisfying $R_{(p)} \neq 0$.

PROOF: (1) If κ is finite, then \mathbb{Z}_p^κ is finite, and so P -small (it is, in fact, small). If κ is infinite, then \mathbb{Z}_p^κ is an infinitely generated vector space over \mathbb{Z}_p . Hence infinite direct sum of groups \mathbb{Z}_p , which is not \mathbb{Z}_p -small, so it is not P -small by Lemma 2.1 (2), since there exists $Q \leq P$ with $Q \simeq \mathbb{Z}_p$.

(2) It is enough to prove the direct implication. Suppose that κ is infinite. Since there exists $p \in \mathbb{P}$ such that $R_{(p)} \neq 0$ and $\mathbb{Z}^\kappa / (p\mathbb{Z}^\kappa) \cong \mathbb{Z}_p^\kappa$ is not $R_{(p)}$ -small by (1). Then \mathbb{Z}^κ is not R -small by Lemma 2.1 (1).

(3) Put $\mathcal{T}_p = \{T_{(p)} : T \in \mathcal{T}, T_{(p)} \neq 0\}$, $\mathcal{S} = \{pS : S \in \mathcal{T}_p\}$ and suppose that $\kappa = |\mathcal{T}_p| = |\{T \in \mathcal{T} : T_{(p)} \neq 0\}|$ is infinite. Then $(\prod \mathcal{T}_p) / \prod \mathcal{S} \cong \mathbb{Z}_p^\kappa$ which is not P -small by (1), and so $\prod \mathcal{T}_p$ is not P -small by Lemma 2.1 (1). Now $\prod \mathcal{T}$ is not P -small by Lemma 2.1 (1) again, as $\prod \mathcal{T}_p$ is a direct summand of $\prod \mathcal{T}$.

(4) It follows from (3) and Lemma 2.1 (2). □

Lemma 3.5. *Let A_p be a finite p -group for each $p \in \mathbb{P}$. Then $\prod_{p \in \mathbb{P}} A_p$ is self-small.*

PROOF: From [14, Section 20, Exercise 5] we have that $A = \prod_{p \in \mathbb{P}} A_p / \bigoplus_{p \in \mathbb{P}} A_p$ is divisible, since $pA_q = A_q$ for all $q \neq p$. Repeating the argument of [20, Lemma 1.7], cf. also Example 3.3 (2), we get that if $f \in \text{Hom}(\prod_{p \neq q} A_p, A_q)$ where $q \in \mathbb{P}$, then $\bigoplus_{p \neq q} A_p \subseteq \ker f$, hence $\text{im } f$ is isomorphic to some factor of the divisible group A . Therefore $\text{im } f$ is divisible and at the same time a subgroup of a finite group, hence $\text{im } f = 0$. In consequence, $\text{Hom}(\prod_{p \neq q} A_p, A_q) = 0$ and the fact that A_q is self-small for each $q \in \mathbb{P}$ implies that $\prod_{p \in \mathbb{P}} A_p$ is self-small by applying [20, Proposition 1.6]. □

Now we are ready to describe self-small products of finitely generated groups.

Theorem 3.6. *Let \mathcal{M} be a family of nonzero finitely generated abelian groups such that at least one $N \in \mathcal{M}$ has nonzero torsion part and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:*

- (1) M is self-small;
- (2) S is \mathbb{Z} -small and $S_{(p)}$ -small for all $p \in \mathbb{P}$;
- (3) $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated;
- (4) there are only finitely many $A \in \mathcal{M}$ which are infinite and for each $p \in \mathbb{P}$ there are only finitely many $A \in \mathcal{M}$ with $A_{(p)} \neq 0$;
- (5) the family $\{B \in \mathcal{M} : \text{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$;
- (6) there are only finitely many $A \in \mathcal{M}$ which are infinite and the family $\{B \in \mathcal{M} : \text{Hom}(C, B) \neq 0\}$ is finite for each finite $C \in \mathcal{M}$;
- (7) $M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p -groups M_p for each $p \in \mathbb{P}$.

PROOF: Any finitely generated group A is by [14, Theorem 15.5] isomorphic to a direct sum of a finite number of cyclic groups, so let $A \cong \bigoplus_{i=1}^n C_i$, where C_i is

cyclic. Let F_A be the direct sum of those C_i 's that are infinite and similarly T_A a direct sum of those that are finite. Then F_A is free, T_A is the (finite) torsion part of A and $A \cong F_A \oplus T_A$. Put $F = \bigoplus_{A \in \mathcal{M}} F_A$ and $T = \bigoplus_{A \in \mathcal{M}} T_A$ and note that $S \cong F \oplus T$ where F is a free abelian group and T is the torsion part of S . Furthermore $M \cong \prod_{A \in \mathcal{M}} F_A \oplus \prod_{A \in \mathcal{M}} T_A$.

(2) \Leftrightarrow (3) \Leftrightarrow (4) Note that $\text{Hom}(T, \mathbb{Z}) = 0$, hence S is \mathbb{Z} -small if and only if F is \mathbb{Z} -small. Thus S is \mathbb{Z} -small if and only if $Q \cong F$ is finitely generated which holds true if and only if there are only finitely many $A \in \mathcal{M}$ with nonzero F_A , i.e. which are infinite. Furthermore, it is easy to see that S is $S_{(p)}$ -small if and only if $S_{(p)}$ is finite if and only if there exists only finitely many $A \in \mathcal{M}$ such that $A_{(p)} \neq 0$.

(4) \Rightarrow (5) Take $A \in \mathcal{M}$. Then $\text{Hom}(B, A) \neq 0$ if and only if $F_B \neq 0$ or there exists $p \in \mathbb{P}$ satisfying $(T_A)_{(p)} \neq 0 \neq (T_B)_{(p)}$. Since A is finitely generated, there exist $p_i \in \mathbb{P}$, $i = 1, \dots, k$ such that $T_A = \bigoplus_{i=1}^k (T_A)_{(p_i)}$. In total, we get

$$\{B \in \mathcal{M} : \text{Hom}(B, A) \neq 0\} \subseteq \{B : F_B \neq 0\} \cup \bigcup_{i=1}^k \{B : (T_B)_{(p_i)} \neq 0\},$$

where both sets on the right-hand side are finite.

(5) \Rightarrow (4) Let $A \in \mathcal{M}$ be infinite, i.e. $F_A \neq 0$. If B is infinite, then $\text{Hom}(B, A) \neq 0$, hence there exist only finitely many infinite groups $B \in \mathcal{M}$. Similarly, if $A, B \in \mathcal{M}$ are such that $A_{(p)} \neq 0 \neq B_{(p)}$, then $\text{Hom}(B, A) \neq 0$, so for each $p \in \mathbb{P}$ there are only finitely many $B \in \mathcal{M}$ such that $B_{(p)} \neq 0$.

(1) \Rightarrow (4) Since M is self-small, it is S -small by Theorem 3.1. Furthermore, $\prod_{A \in \mathcal{M}} F_A$ being a direct summand, hence a factor of M , it is M -small and in consequence T -small by Lemma 2.1 (2), so $\prod_{A \in \mathcal{M}} F_A$ is finitely generated by Lemma 3.4 (2). Therefore there exist only finitely many A with $F_A \neq 0$. Similarly, since $\prod_{A \in \mathcal{M}} T_A$ is T -small, there exist only finitely many $A \in \mathcal{M}$ such that $A_{(p)} = (T_A)_{(p)} \neq 0$ for each $p \in \mathbb{P}$ by Lemma 3.4 (4).

(3) \Rightarrow (7) Note that by (3) $F = \bigoplus_{A \in \mathcal{M}} F_A$ is finitely generated. Moreover,

$$\begin{aligned} \prod_{A \in \mathcal{M}} T_A &= \prod_{A \in \mathcal{M}} \bigoplus_{p \in \mathbb{P}} (T_A)_{(p)} \cong \prod_{A \in \mathcal{M}} \prod_{p \in \mathbb{P}} (T_A)_{(p)} \\ &\cong \prod_{p \in \mathbb{P}} \prod_{A \in \mathcal{M}} (T_A)_{(p)} \cong \prod_{p \in \mathbb{P}} \bigoplus_{A \in \mathcal{M}} (T_A)_{(p)}, \end{aligned}$$

because T_A is finite for all $A \in \mathcal{M}$ and for each $p \in \mathbb{P}$ there exist only finitely many A with $(T_A)_{(p)} \neq 0$. Then $M_p = \bigoplus_{A \in \mathcal{M}} (T_A)_{(p)}$ is a finite p -group for all $p \in \mathbb{P}$ and $M \cong F \oplus \prod_{A \in \mathcal{M}} T_A \cong F \oplus \prod_{p \in \mathbb{P}} M_p$.

(7) \Rightarrow (1) By Theorem 3.1 it is enough to prove that M is $F \oplus \bigoplus_{p \in \mathbb{P}} M_p$ -small. Since F is finitely generated, it is $F \oplus \bigoplus_{p \in \mathbb{P}} M_p$ -small. As $\text{Hom}(\prod_{p \in \mathbb{P}} M_p, F) = 0$,

it remains to show that $\prod_{p \in \mathbb{P}} M_p$ is $\bigoplus_{p \in \mathbb{P}} M_p$ -small by Proposition 2.6, which holds true by Lemma 3.5 and Theorem 3.1.

(5) \Leftrightarrow (6) The assertion concerning infinite groups follows from the equivalence of (4) and (5). The rest is a consequence of the fact that $\text{Hom}(C, B) \neq 0$ if and only if $\text{Hom}(B, C) \neq 0$ for each pair of finitely generated torsion abelian groups B, C . \square

An uncountable cardinal κ is *measurable* if it admits a κ -additive measure $\mu: \kappa \rightarrow \{0; 1\}$ such that $\mu(\kappa) = 1$ and $\mu(x) = 0$ for $x \in \kappa$. A group G is called *slender*, if for any homomorphism $f: \mathbb{Z}^\omega \rightarrow G$, $f(\mathbf{e}_i) = 0$ for almost all $i \in \omega$, where \mathbf{e}_i denotes the element of \mathbb{Z}^ω with $\pi_j(\mathbf{e}_i) = \delta_{i,j}$. Recall that \mathbb{Z} is slender by [15, Theorem 94.2] and that for a nonmeasurable cardinal κ we have $\text{Hom}(\mathbb{Z}^\kappa, \mathbb{Z}) \cong \mathbb{Z}^{(\kappa)}$ by [15, Corollary 94.5], cf. also [2, Theorem 3.6].

Lemma 3.7. \mathbb{Z}^κ is \mathbb{Z} -small for each cardinal κ .

PROOF: For finite κ there is nothing to prove, so let us suppose that κ is infinite and assume that \mathbb{Z}^κ is not \mathbb{Z} -small. Then there exists a homomorphism $g \in \text{Hom}(\mathbb{Z}^\kappa, \mathbb{Z}^{(\omega)})$ such that $\text{im } g$ is infinitely generated by Proposition 1.7, hence $\text{im } g$ is a free abelian group of infinite rank. Since $\text{im } g \cong \mathbb{Z}^{(\omega)}$ is projective, $\mathbb{Z}^{(\omega)}$ is a direct summand of \mathbb{Z}^κ , i.e. there exists a group A for which $\mathbb{Z}^\kappa \cong \mathbb{Z}^{(\omega)} \oplus A$.

First, assume that $\kappa = \omega$. Then $\text{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \mathbb{Z}^{(\omega)}$ by [15, Corollary 94.5] as \mathbb{Z} is slender by [15, Theorem 94.2]. Hence

$$\begin{aligned} \mathbb{Z}^{(\omega)} &\cong \text{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}^{(\omega)} \oplus A, \mathbb{Z}) \\ &\cong \text{Hom}(\mathbb{Z}^{(\omega)}, \mathbb{Z}) \oplus \text{Hom}(A, \mathbb{Z}) \cong \mathbb{Z}^\omega \oplus \text{Hom}(A, \mathbb{Z}) \end{aligned}$$

which is impossible for cardinality reasons (i.e. $|\mathbb{Z}^{(\omega)}| < |\mathbb{Z}^\omega|$).

We have proved that \mathbb{Z}^ω is \mathbb{Z} -small, so $\kappa > \omega$. Let $\lambda \geq \kappa$ be a nonmeasurable cardinal (it exists, as for instance each singular cardinal is nonmeasurable). Then $\text{Hom}(\mathbb{Z}^\lambda, \mathbb{Z}) \cong \mathbb{Z}^{(\lambda)}$ by [15, Corollary 94.5] and $\mathbb{Z}^\lambda \cong \mathbb{Z}^\lambda \oplus \mathbb{Z}^\kappa$ as $\lambda + \kappa = \lambda$, hence $\mathbb{Z}^\lambda \cong \mathbb{Z}^{(\omega)} \oplus B$ for $B = \mathbb{Z}^\lambda \oplus A$. We get

$$\mathbb{Z}^{(\lambda)} \cong \text{Hom}(\mathbb{Z}^\lambda, \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}^{(\omega)} \oplus B, \mathbb{Z}) \cong \mathbb{Z}^\omega \oplus \text{Hom}(A, \mathbb{Z}),$$

which implies that \mathbb{Z}^ω is embeddable into $\mathbb{Z}^{(\lambda)}$, so it is an infinitely generated free group. This contradicts the fact that \mathbb{Z}^ω is \mathbb{Z} -small. \square

Example 3.8. Expressing Proposition 2.3 (1) via the language of short exact sequences, we can say that relative smallness is transferred from the outer members to the middle one. The other direction, however, is more complicated: While Lemma 2.1 (1) implies the transfer from the middle member to the right, the

previous example shows that the transfer to the left does not occur generally. Indeed, we have $\mathbb{Z}^{(\omega)} \hookrightarrow \mathbb{Z}^\omega$, but $\mathbb{Z}^{(\omega)}$ is not \mathbb{Z} -small.

Using Corollary 3.2 we can formulate an important consequence:

Corollary 3.9. \mathbb{Z}^κ is self-small for each cardinal κ .

We finish the paper by a general criterion of self-small products of finitely generated groups.

Theorem 3.10. Let \mathcal{M} be a family of nonzero finitely generated abelian groups and put $M = \prod \mathcal{M}$, $S = \bigoplus \mathcal{M}$ and $Q = S/T_S$. Then the following conditions are equivalent:

- (1) M is self-small;
- (2) either $T_S = 0$, or $S_{(p)}$ is finite for each $p \in \mathbb{P}$ and Q is finitely generated;
- (3) either all $A \in \mathcal{M}$ are free, or the family $\{B \in \mathcal{M} : \text{Hom}(B, A) \neq 0\}$ is finite for each $A \in \mathcal{M}$;
- (4) either $M \cong \mathbb{Z}^\kappa$ for some cardinal κ , or $M \cong F \oplus \prod_{p \in \mathbb{P}} M_p$ for a finitely generated free group F and finite abelian p -groups M_p for each $p \in \mathbb{P}$.

PROOF: The torsion part of M is zero if and only if all groups $A \in \mathcal{M}$ are free which means that $M \cong \mathbb{Z}^\kappa$ for some cardinal κ is self-small by Corollary 3.9. The case when the torsion part of M is nonzero follows directly from Theorem 3.6. \square

Example 3.11. The assumption in condition (4) of the previous theorem that F is finitely generated cannot be omitted without additional conditions, since, e.g., the group $\mathbb{Q} \times \prod_{p \in \mathbb{P}} \mathbb{Z}_p$ is not self-small by [12, Example 3].

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REFERENCES

- [1] Albrecht U., Breaz S., *A note on self-small modules over RM-domains*, J. Algebra Appl. **13** (2014), no. 1, 1350073, 8 pages.
- [2] Albrecht U., Breaz S., Schultz P., *Functorial properties of Hom and Ext*, in: Groups and Model Theory, Contemp. Math., 576, Amer. Math. Soc., Providence, 2012, pages 1–15.
- [3] Albrecht U., Breaz S., Wickless W., *Purity and self-small groups*, Comm. Algebra **35** (2007), no. 11, 3789–3807.
- [4] Albrecht U., Breaz S., Wickless W., *Self-small abelian groups*, Bull. Aust. Math. Soc. **80** (2009), no. 2, 205–216.
- [5] Arnold D. M., Murley C. E., *Abelian groups, A, such that $\text{Hom}(A, - - -)$ preserves direct sums of copies of A*, Pacific J. Math. **56** (1975), no. 1, 7–20.

- [6] Bass H., *Algebraic K-theory*, Mathematics Lecture Note Series, W. A. Benjamin, New York, 1968.
- [7] Breaz S., *Self-small abelian groups as modules over their endomorphism rings*, *Comm. Algebra* **31** (2003), no. 10, 4911–4924.
- [8] Breaz S., *A mixed version for a Fuchs' lemma*, *Rend. Semin. Mat. Univ. Padova* **144** (2020), 61–71.
- [9] Breaz S., Schultz P., *Dualities for self-small groups*, *Proc. Amer. Math. Soc.* **140** (2012), no. 1, 69–82.
- [10] Breaz S., Žemlička J., *When every self-small module is finitely generated*, *J. Algebra* **315** (2007), no. 2, 885–893.
- [11] Colpi R., Menini C., *On the structure of \star -modules*, *J. Algebra* **158** (1993), no. 2, 400–419.
- [12] Dvořák J., *On products of self-small abelian groups*, *Stud. Univ. Babeş-Bolyai Math.* **60** (2015), no. 1, 13–17.
- [13] Dvořák J., Žemlička J., *Autocompact objects of Ab_5 categories*, *Theory Appl. Categ.* **37** (2021), Paper No. 30, 979–995.
- [14] Fuchs L., *Infinite Abelian Groups. Vol. I*, Pure and Applied Mathematics, 36, Academic Press, New York, 1970.
- [15] Fuchs L., *Infinite Abelian Groups. Vol. II*, Pure and Applied Mathematics, 36-II, Academic Press, New York, 1973.
- [16] Gómez Pardo J.L., Militaru G., Năstăsescu C., *When is $\text{HOM}_R(M, -)$ equal to $\text{Hom}_R(M, -)$ in the category $R\text{-gr}$?*, *Comm. Algebra* **22** (1994), no. 8, 3171–3181.
- [17] Kálnai P., Žemlička J., *Compactness in abelian categories*, *J. Algebra* **534** (2019), 273–288.
- [18] Modoi G.C., *Constructing large self-small modules*, *Stud. Univ. Babeş-Bolyai Math.* **64** (2019), no. 1, 3–10.
- [19] Rentschler R., *Sur les modules M tels que $\text{Hom}(M, -)$ commute avec les sommes directes*, *C. R. Acad. Sci. Paris Sér. A-B* **268** (1969), A930–A933 (French).
- [20] Žemlička J., *When products of self-small modules are self-small*, *Comm. Algebra* **36** (2008), no. 7, 2570–2576.

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