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UNIFORMLY CONVEX SPIRAL FUNCTIONS AND UNIFORMLY SPIRALLIKE FUNCTIONS ASSOCIATED WITH PASCAL DISTRIBUTION SERIES

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Abstract. The aim of this paper is to find the necessary and sufficient conditions and inclusion relations for Pascal distribution series to be in the classes $\mathcal{SP}_p(\alpha,\beta)$ and $\mathcal{UCV}_p(\alpha,\beta)$ of uniformly spirallike functions. Further, we consider an integral operator related to Pascal distribution series. Several corollaries and consequences of the main results are also considered.

Keywords: analytic function; Hadamard product; uniformly spirallike function; Pascal distribution series

MSC 2020: 30C45

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}\colon\ |z|<1\}$ and normalized by the conditions $f(0)=0=f'(0)-1$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n,\quad z\in\mathbb{U}. \tag{1.2}$$

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A function $f \in \mathcal{A}$ is spirallike if

$$\Re\Big(\mathrm{e}^{-\mathrm{i}\alpha}\frac{zf'(z)}{f(z)}\Big) > 0$$

for some α with $|\alpha| < \frac{1}{2}\pi$ and for all $z \in \mathbb{U}$. Also $f(z)$ is convex spirallike if $zf'(z)$ is spirallike.

In [28], Selvaraj and Geetha introduced the following subclasses of unifromly spirallike and convex spirallike functions.

Definition 1.1. A function f of the form (1.1) is said to be in the class $\mathcal{SP}_p(\alpha,\beta)$ if it satisfies the following condition:

$$\Re\Big(\mathrm{e}^{-\mathrm{i}\alpha}\frac{zf'(z)}{f(z)}\Big) > \Big|\frac{zf'(z)}{f(z)} - 1\Big| + \beta, \quad |\alpha| < \frac{\pi}{2},\ 0 \leqslant \beta < 1$$

and $f \in \mathcal{UCV}_p(\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{SP}_p(\alpha,\beta)$.

We write

$$\mathcal{TSP}_p(\alpha,\beta) = \mathcal{SP}_p(\alpha,\beta) \cap \mathcal{T} \quad \text{and} \quad \mathcal{UCT}_p(\alpha,\beta) = \mathcal{UCV}_p(\alpha,\beta) \cap \mathcal{T}.$$

In particular, we note that $\mathcal{SP}_p(\alpha,0) = \mathcal{SP}_p(\alpha)$ and $\mathcal{UCV}_p(\alpha,0) = \mathcal{UCV}_p(\alpha)$, the classes of uniformly spirallike and uniformly convex spirallike functions were introduced by Ravichandran et al. (see [25]).

For $\alpha = 0$, the classes $\mathcal{UCV}_p(\alpha)$ and $\mathcal{SP}_p(\alpha)$, respectively, reduces to the classes $\mathcal{UCV}$ and $\mathcal{SP}$ introduced and studied by Ronning (see [27]). For more interesting developments of some related subclasses of uniformly spirallike and uniformly convex spirallike functions, the readers may be referred to the works of Frasin (see [7], [8]), Goodman (see [15], [16]), Al-Hawary and Frasin (see [1]), Kanas and Wiśniowska (see [17], [18]) and Ronning (see [26], [27]).

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(A,B)$, $\tau \in \mathbb{C}\setminus\{0\}$, $-1 \leqslant B < A \leqslant 1$, if it satisfies the inequality

$$\Big|\frac{f'(z)-1}{(A-B)\tau - B[f'(z)-1]}\Big| < 1, \quad z \in \mathbb{U}.$$

This class was introduced by Dixit and Pal, see [3].

A variable X is said to be *Pascal distribution* if it takes the values $0,1,2,3,\ldots$ with probabilities $(1-q)^m$, $\frac{1}{1!}qm(1-q)^m$, $\frac{1}{2!}q^2m(m+1)(1-q)^m$, $\frac{1}{3!}q^3m(m+1)\times(m+2)(1-q)^m,\ldots$, respectively, where q and m are called the parameters, and thus

$$P(X=k) = \binom{k+m-1}{m-1}q^k(1-q)^m, \quad k = 0,1,2,3,\ldots$$

Very recently, El-Deeb et al. (see [6]) introduced a power series whose coefficients are probabilities of Pascal distribution

$$\Psi_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n, \quad z \in \mathbb{U}, \tag{1.3}$$

where $m \geqslant 1$, $0 \leqslant q \leqslant 1$ and we note that by ratio test, the radius of convergence of above series is infinity. We also define the series

$$\Phi_q^m(z) = 2z - \Psi_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n, \quad z \in \mathbb{U}. \tag{1.4}$$

Now, we consider the linear operator

$$\mathcal{I}_q^m(z)\colon\ \mathcal{A} \to \mathcal{A}$$

defined by the convolution or Hadamard product

$$\mathcal{I}_q^m f(z) = \Psi_q^m(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m a_n z^n, \quad z \in \mathbb{U}. \tag{1.5}$$

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions by using hypergeometric functions (see for example [2], [12], [19], [29], [30]) and by the recent investigations (see for example [4], [5], [9]–[11], [13], [14], [20]–[24]), in the present paper we determine the necessary and sufficient conditions for $\Phi_q^m(z)$ to be in our classes $\mathcal{TSP}_p(\alpha,\beta)$ and $\mathcal{UCT}_p(\alpha,\beta)$ and connections of these subclasses with $\mathcal{R}^\tau(A,B)$. Finally, we give conditions for the integral operator $\mathcal{G}_q^m f(z) = \int_0^z \Psi_q^m(t)t^{-1}\,\mathrm{d}t$ belonging to the above classes.

To establish our main results, we need the following lemmas.

Lemma 1.2 ([28]). *A function f of the form* (1.2) *is in $\mathcal{TSP}_p(\alpha,\beta)$ if and only if it satisfies*

$$\sum_{n=2}^{\infty} (2n - \cos\alpha - \beta)|a_n| \leqslant \cos\alpha - \beta, \quad |\alpha| < \frac{\pi}{2},\ 0 \leqslant \beta < 1. \tag{1.6}$$

In particular, when $\beta = 0$, we obtain a necessary and sufficient condition for a function f of the form (1.2) to be in the class $\mathcal{TSP}_p(\alpha)$ as

$$\sum_{n=2}^{\infty} (2n - \cos\alpha)|a_n| \leqslant \cos\alpha, \quad |\alpha| < \frac{\pi}{2}. \tag{1.7}$$

Lemma 1.3 ([28]). *A function f of the form* (1.2) *is in* $\mathcal{UCT}_p(\alpha,\beta)$ *if and only if it satisfies*

$$\sum_{n=2}^{\infty} n(2n-\cos\alpha-\beta)|a_n| \leqslant \cos\alpha-\beta, \quad |\alpha|<\frac{\pi}{2},\ 0\leqslant\beta<1. \tag{1.8}$$

In particular, when $\beta=0$, we obtain a necessary and sufficient condition for a function f of the form (1.2) to be in the class $\mathcal{UCT}_p(\alpha)$ as

$$\sum_{n=2}^{\infty} n(2n-\cos\alpha)|a_n| \leqslant \cos\alpha, \quad |\alpha|<\frac{\pi}{2}. \tag{1.9}$$

Lemma 1.4 ([3]). *If $f\in\mathcal{R}^\tau(A,B)$ is of the form* (1.1), *then*

$$|a_n| \leqslant (A-B)\frac{|\tau|}{n}, \quad n\in\mathbb{N}\setminus\{1\}.$$

The result is sharp.

2. The necessary and sufficient conditions

For convenience, in the sequel we use the following identities for $m\geqslant 1$ and $0\leqslant q<1$:

$$\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n=\frac{1}{(1-q)^m}, \qquad \sum_{n=0}^{\infty}\binom{n+m-2}{m-2}q^n=\frac{1}{(1-q)^{m-1}},$$
$$\sum_{n=0}^{\infty}\binom{n+m}{m}q^n=\frac{1}{(1-q)^{m+1}}, \qquad \sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n=\frac{1}{(1-q)^{m+2}}.$$

By simple calculations we derive the following relations:

$$\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}=\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n-1=\frac{1}{(1-q)^m}-1,$$
$$\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}=qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^n=\frac{qm}{(1-q)^{m+1}},$$

and

$$\begin{aligned}\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}&=q^2m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n\\&=\frac{q^2m(m+1)}{(1-q)^{m+2}}.\end{aligned}$$

Unless otherwise mentioned, we shall assume in this paper that $|\alpha| < \frac{1}{2}\pi$, $0 \leqslant \beta < 1$ and $0 \leqslant q < 1$. First we obtain the necessary and sufficient conditions for $\Phi_q^m(z)$ to be in $\mathcal{TSP}_p(\alpha, \beta)$.

Theorem 2.1. *If $m \geqslant 1$, then $\Phi_q^m(z) \in \mathcal{TSP}_p(\alpha, \beta)$ if and only if*

$$\frac{2qm}{1-q} + (2 - \cos\alpha - \beta)(1 - (1-q)^m) \leqslant \cos\alpha - \beta. \tag{2.1}$$

Proof. Since

$$\Phi_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n \tag{2.2}$$

in view of Lemma 1.2, it suffices to show that

$$\sum_{n=2}^{\infty} (2n - \cos\alpha - \beta) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \leqslant \cos\alpha - \beta. \tag{2.3}$$

Writing

$$n = (n-1) + 1$$

in (2.3) we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} (2n - \cos\alpha - \beta) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \\
&\quad= \sum_{n=2}^{\infty} (2(n-1) + 2 - \cos\alpha - \beta) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \\
&\quad= 2\sum_{n=2}^{\infty} (n-1) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \\
&\qquad+ (2 - \cos\alpha - \beta) \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \\
&\quad= 2qm(1-q)^m \sum_{n=0}^{\infty} \binom{n+m}{m} q^n \\
&\qquad+ (2 - \cos\alpha - \beta)(1-q)^m \left(\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n - 1 \right) \\
&\quad= \frac{2qm}{1-q} + (2 - \cos\alpha - \beta)(1 - (1-q)^m).
\end{aligned}$$

But this last expression is bounded above by $\cos\alpha - \beta$ if and only if (2.1) holds. $\square$

Theorem 2.2. *If $m \geqslant 1$, then $\Phi_q^m(z) \in \mathcal{UCT}_p(\alpha,\beta)$ if and only if*

$$\frac{2q^2m(m+1)}{(1-q)^2} + (6-\cos\alpha-\beta)\frac{qm}{1-q} + (2-\cos\alpha-\beta)(1-(1-q)^m) \leqslant \cos\alpha-\beta. \tag{2.4}$$

P r o o f. In view of Lemma 1.3, we must show that

$$\sum_{n=2}^{\infty} n(2n-\cos\alpha-\beta)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leqslant \cos\alpha-\beta. \tag{2.5}$$

Writing

$$n = (n-1)+1$$

and

$$n^2 = (n-1)(n-2)+3(n-1)+1$$

in (2.5) we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(2n-\cos\alpha-\beta)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&\quad= \sum_{n=2}^{\infty}(2(n-1)(n-2)+6(n-1))\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&\qquad+ \sum_{n=2}^{\infty}(n-1)(-\cos\alpha-\beta)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&\qquad+ \sum_{n=2}^{\infty}(2-\cos\alpha-\beta)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&\quad= 2q^2m(m+1)(1-q)^m\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n \\
&\qquad+ (6-\cos\alpha-\beta)qm(1-q)^m\sum_{n=0}^{\infty}\binom{n+m}{m}q^n \\
&\qquad+ (2-\cos\alpha-\beta)(1-q)^m\Big(\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n - 1\Big) \\
&\quad= 2q^2m(m+1)(1-q)^m\frac{1}{(1-q)^{m+2}} + (6-\cos\alpha-\beta)qm(1-q)^m\frac{1}{(1-q)^{m+1}} \\
&\qquad+ (2-\cos\alpha-\beta)(1-q)^m\Big(\frac{1}{(1-q)^m}-1\Big) \\
&\quad= \frac{2q^2m(m+1)}{(1-q)^2} + (6-\cos\alpha-\beta)\frac{qm}{1-q} + (2-\cos\alpha-\beta)(1-(1-q)^m).
\end{aligned}$$

Therefore we see that the last expression is bounded above by $\cos\alpha-\beta$ if (2.4) is satisfied. □

3. Inclusion properties

Making use of Lemma 1.4, we will study the action of the Pascal distribution series on the class $\mathcal{TSP}_p(\alpha,\beta)$.

Theorem 3.1. *Let $m>1$ and $f\in\mathcal{R}^\tau(A,B)$. Then $\mathcal{I}_q^m f(z)\in\mathcal{TSP}_p(\alpha,\beta)$ if*

(3.1)
$$(A-B)|\tau|\Big(2(1-(1-q)^m)-\frac{\cos\alpha+\beta}{q(m-1)}((1-q)-(1-q)^m-q(m-1)(1-q)^m)\Big)\leqslant\cos\alpha-\beta.$$

P r o o f. In view of Lemma 1.2, it suffices to show that

$$\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m|a_n|\leqslant\cos\alpha-\beta.$$

Since $f\in\mathcal{R}^\tau(A,B)$, then by Lemma 1.4, we have

$$|a_n|\leqslant\frac{(A-B)|\tau|}{n}. \tag{3.2}$$

Thus, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m|a_n|\\
&\quad\leqslant(A-B)|\tau|\Big(\sum_{n=2}^{\infty}\frac{1}{n}(2n-\cos\alpha-\beta)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\Big)\\
&\quad=(A-B)|\tau|(1-q)^m\\
&\qquad\times\Big(2\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}-(\cos\alpha+\beta)\sum_{n=2}^{\infty}\frac{1}{n}\binom{n+m-2}{m-1}q^{n-1}\Big)\\
&\quad=(A-B)|\tau|(1-q)^m\Big(2\Big(\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n-1\Big)\\
&\qquad-\frac{\cos\alpha+\beta}{q(m-1)}\Big(\sum_{n=0}^{\infty}\binom{n+m-2}{m-2}q^n-1-(m-1)q\Big)\Big)\\
&\quad=(A-B)|\tau|\\
&\qquad\times\Big(2(1-(1-q)^m)-\frac{\cos\alpha+\beta}{q(m-1)}((1-q)-(1-q)^m-q(m-1)(1-q)^m)\Big).
\end{aligned}$$

But this last expression is bounded by $\cos\alpha-\beta$ if (3.1) holds. This completes the proof of Theorem 3.1. □

Applying Lemma 1.3 and using the same technique as in the proof of Theorem 3.1, we have the following result.

Theorem 3.2. *Let* $m \geqslant 1$ *and* $f \in \mathcal{R}^{\tau}(A,B)$. *Then* $\mathcal{I}_q^m f(z) \in \mathcal{UCT}_p(\alpha,\beta)$ *if*

$$(A-B)|\tau|\Big(\frac{2qm}{1-q} + (2-\cos\alpha-\beta)(1-(1-q)^m)\Big) \leqslant \cos\alpha - \beta. \tag{3.3}$$

4. An integral operator

Theorem 4.1. *If* $m \geqslant 1$, *then the integral operator*

$$\mathcal{G}_q^m f(z) = \int_0^z \frac{\Psi_q^m(t)}{t}\,\mathrm{d}t \tag{4.1}$$

is in the class $\mathcal{UCT}_p(\alpha,\beta)$ *if and only if inequality* (2.1) *is satisfied.*

P r o o f. Since

$$\mathcal{G}_q^m f(z) = z + \sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\frac{z^n}{n},$$

by Lemma 1.3 we only need to show that

$$\sum_{n=2}^{\infty} n(2n-\cos\alpha-\beta)\frac{1}{n}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leqslant \cos\alpha-\beta,$$

or equivalently,

$$\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leqslant \cos\alpha-\beta. \tag{4.2}$$

The remaining part of the proof of Theorem 4.1 is similar to that of Theorem 2.1, and so we omit the details. □

Theorem 4.2. *If* $m > 1$, *then the integral operator* $\mathcal{G}_q^m f(z)$ *given by* (4.1) *is in the class* $\mathcal{TSP}_p(\alpha,\beta)$ *if and only if*

$$2(1-(1-q)^m) - \frac{\cos\alpha+\beta}{q(m-1)}((1-q)-(1-q)^m - q(m-1)(1-q)^m) \leqslant \cos\alpha-\beta.$$

P r o o f. The proof of Theorem 4.2 is similar to the proof of Theorem 4.1, so we omitted the proof of Theorem 4.2. □

5. Corollaries and consequences

By specializing the parameter $\beta = 0$ in Theorems 2.1–4.2, we obtain the following corollaries.

Corollary 5.1. *If* $m \geqslant 1$, *then* $\Phi_q^m(z) \in \mathcal{TSP}_p(\alpha)$ *if and only if*

$$\frac{2qm}{1-q} + (2-\cos\alpha)(1-(1-q)^m) \leqslant \cos\alpha. \tag{5.1}$$

Corollary 5.2. *If* $m \geqslant 1$, *then* $\Phi_q^m(z) \in \mathcal{UCT}_p(\alpha)$ *if and only if*

$$\frac{2q^2m(m+1)}{(1-q)^2} + (6-\cos\alpha)\frac{qm}{1-q} + (2-\cos\alpha)(1-(1-q)^m) \leqslant \cos\alpha. \tag{5.2}$$

Corollary 5.3. *Let* $m > 1$ *and* $f \in \mathcal{R}^\tau(A,B)$. *Then* $\mathcal{I}_q^m f(z) \in \mathcal{TSP}_p(\alpha)$ *if*

$$(A-B)|\tau|\Big(2(1-(1-q)^m) - \frac{\cos\alpha}{q(m-1)}((1-q)-(1-q)^m - q(m-1)(1-q)^m)\Big) \leqslant \cos\alpha. \tag{5.3}$$

Corollary 5.4. *Let* $m \geqslant 1$ *and* $f \in \mathcal{R}^\tau(A,B)$. *Then* $\mathcal{I}_q^m f(z) \in \mathcal{UCT}_p(\alpha)$ *if*

$$(A-B)|\tau|\Big(\frac{2qm}{1-q} + (2-\cos\alpha)(1-(1-q)^m)\Big) \leqslant \cos\alpha. \tag{5.4}$$

Corollary 5.5. *If* $m \geqslant 1$, *then the integral operator* $\mathcal{G}_q^m f(z)$ *given by* (4.1) *is in the class* $\mathcal{UCT}_p(\alpha)$ *if and only if inequality* (5.1) *is satisfied.*

Corollary 5.6. *If* $m > 1$, *then the integral operator* $\mathcal{G}_q^m f(z)$ *given by* (4.1) *is in the class* $\mathcal{TSP}_p(\alpha)$ *if and only if*

$$2(1-(1-q)^m) - \frac{\cos\alpha}{q(m-1)}((1-q)-(1-q)^m - q(m-1)(1-q)^m) \leqslant \cos\alpha.$$

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