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ADDITIVE DECOMPOSITION OF MATRICES UNDER RANK CONDITIONS AND ZERO PATTERN CONSTRAINTS

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Dedicated to our friend and colleague Bernd Silbermann on the occasion of his 80-th birthday, with admiration

Abstract. This paper deals with additive decompositions $A = A_1 + \ldots + A_p$ of a given matrix A, where the ranks of the summands $A_1, \ldots, A_p$ are prescribed and meet certain zero pattern requirements. The latter are formulated in terms of directed bipartite graphs.

Keywords: additive decomposition; rank constraint; zero pattern constraint; directed bipartite graph; L-free directed bipartite graph; permutation L-free directed bipartite graph; Bell number; Stirling partition number

MSC 2020: 15A21, 05C50, 15A03, 05C20

1. Introduction

Let m, n be positive integers. The linear space of all $m \times n$ (complex) matrices will be denoted by $\mathbb{C}^{m\times n}$. We will consider subspaces of $\mathbb{C}^{m\times n}$ determined by a pattern of zeros. Here is how these are defined.

Let $\mathcal{Z}$ be a binary relation between the sets $\mathbf{M} = \{1, \ldots, m\}$ and $\mathbf{N} = \{1, \ldots, n\}$ (later to be identified with a directed bipartite graph from $\mathbf{M}$ to $\mathbf{N}$). With $\mathcal{Z}$ we associate the subset $\mathbb{C}^{m\times n}[\mathcal{Z}]$ of the set $\mathbb{C}^{m\times n}$ of all $m \times n$ (complex) matrices consisting of all $A = [a_{i,j}]_{i=1,j=1}^{m,n} \in \mathbb{C}^{m\times n}$ such that $a_{i,j} = 0$ whenever $(i,j) \notin \mathcal{Z}$. Evidently, $\mathbb{C}^{m\times n}[\mathcal{Z}]$ is closed under scalar multiplication and addition. So, regardless of additional properties of $\mathcal{Z}$, the set $\mathbb{C}^{m\times n}[\mathcal{Z}]$ is a linear subspace of $\mathbb{C}^{m\times n}$.

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The question that we will consider is the following: *Given $A \in \mathbb{C}^{m\times n}[\mathcal{Z}]$, a positive integer p, and integers $r_1, \ldots, r_p$ satisfying*

$$(1.1) \qquad 1 \leqslant r_j \leqslant \operatorname{rank} A, \quad j = 1, \ldots, p, \quad \operatorname{rank} A \leqslant r_1 + \ldots + r_p,$$

does there exist a decomposition $A = A_1 + \ldots + A_p$ such that

$$(1.2) \qquad A_j \in \mathbb{C}^{m\times n}[\mathcal{Z}], \quad \operatorname{rank} A_j = r_j, \quad j = 1, \ldots, p?$$

Note here that the second part of (1.1) is a necessary condition for the rank part of (1.2) to be fulfilled.

Generally, the answer to the above question is negative. Section 3 contains an example demonstrating this. In the same section, a positive answer is formulated in the main theorem of this paper featuring the condition that $\mathcal{Z}$ is L-free. This notion is modeled after a concept that has been introduced in [3] for directed graphs. It (also) plays a crucial role in [4] which is concerned with additive decompositions of a type different from the one considered in the present setting. References to related concepts featuring in the literature will be given in Section 2, where the necessary terminological framework is developed and auxiliary observations are presented. Section 4 is devoted to the proof of the main theorem referred to above. The section also contains a couple of illustrative examples. A requirement leading to additional conclusions is that in the second part of (1.1) equality instead of inequality is required. Such decompositions, named minimal, are discussed in Section 5. Attention is also payed to the issue of how many minimal decompositions can exist. The smallest number that can occur is described in terms of the so called *Bell number*, but there are also situations, where it is infinite. Section 6, the final section of the paper, consists of two subsections. In the first one, a norm optimization issue for the decompositions considered here is raised. The second subsection has as its background the fact that in the definition of L-freeness, the natural orders of the underlying sets of nodes $\mathbf{M} = \{1, \ldots, m\}$ and $\mathbf{N} = \{1, \ldots, n\}$ play a role. It is explained that, without giving up the main results of the paper, it is possible to relax the definition so that it becomes order independent. In this context the challenge to give a graph theoretical characterization presents itself.

2. Graph theoretical preparations and background

Let m and n be positive integers. The linear space of all $m \times n$ (complex) matrices will be denoted by $\mathbb{C}^{m\times n}$. We will consider subspaces of $\mathbb{C}^{m\times n}$ determined by a pattern of zeros. Here is how these are defined.

Write $\mathbf{M} = \{1, \ldots, m\}$ and $\mathbf{N} = \{1, \ldots, n\}$. The integers $1, \ldots, m$ constituting $\mathbf{M}$ correspond to the row numbers of matrices in $\mathbb{C}^{m\times n}$, the integers $1, \ldots, n$ constituting $\mathbf{N}$ to the column numbers. In this sense we may assume $\mathbf{M}$ and $\mathbf{N}$ to be disjoint. In line with this, a binary relation $\mathcal{Z}$ between $\mathbf{M}$ and $\mathbf{N}$, i.e., a subset of $\mathbf{M} \times \mathbf{N}$, will be viewed as a directed bipartite graph from $\mathbf{M}$ to $\mathbf{N}$. The directedness emphasizes the fact that no reflexivity is assumed (in accordance with $\mathcal{Z}$ consisting of ordered pairs). The notation $k \to^{\mathcal{Z}} l$ is employed to indicate that $(k, l) \in \mathcal{Z}$. In the same vein, $k \not\to^{\mathcal{Z}} l$ signals that $(k, l) \notin \mathcal{Z}$. If this happens to be convenient, $l \leftarrow^{\mathcal{Z}} k$ is used instead of $k \to^{\mathcal{Z}} l$.

With a directed bipartite graph $\mathcal{Z}$ as above, we associate the subset $\mathbb{C}^{m\times n}[\mathcal{Z}]$ of $\mathbb{C}^{m\times n}$ of all $A = [a_{k,l}]_{k=1,l=1}^{m,n} \in \mathbb{C}^{m\times n}$ such that $a_{k,l} = 0$ whenever $k \not\to^{\mathcal{Z}} l$. Evidently, $\mathbb{C}^{m\times n}[\mathcal{Z}]$ is closed under scalar multiplication and addition. So, regardless of additional properties of $\mathcal{Z}$, the set $\mathbb{C}^{m\times n}[\mathcal{Z}]$ is a linear subspace of $\mathbb{C}^{m\times n}$.

In line with [3] and [4], a quadruple (p, q, r, s) with $p, q \in \mathbf{M}$ and $r, s \in \mathbf{N}$ is called an L for $\mathcal{Z}$ if

$$p \to^{\mathcal{Z}} r \leftarrow^{\mathcal{Z}} q \to^{\mathcal{Z}} s, \quad p < q,\ r < s, \quad p \not\to^{\mathcal{Z}} s. \tag{2.1}$$

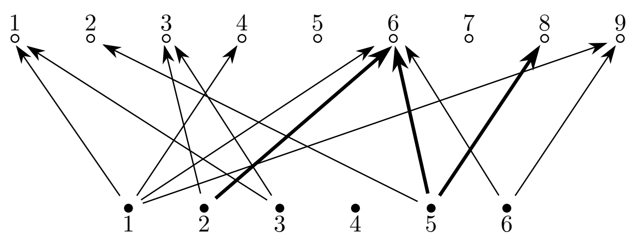
(Caveat: in expressions of this type, attention should be paid to the direction of the arrows.) Here is an example illustrating the definition.

Example 2.1. Take $m = 6$ and $n = 9$, and let the directed bipartite graph $\mathcal{Z}$ be given by the matrix diagram

$$\mathcal{Z} = \left(\begin{array}{c|ccccccccc} & \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} & \mathit{8} & \mathit{9} \\ \mathit{1} & * & 0 & 0 & * & 0 & * & 0 & 0 & * \\ \mathit{2} & 0 & 0 & * & 0 & 0 & \bigstar & 0 & \mathbf{0} & 0 \\ \mathit{3} & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathit{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathit{5} & 0 & * & 0 & 0 & 0 & \bigstar & 0 & \bigstar & 0 \\ \mathit{6} & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & * \end{array}\right).$$

Here a star or a zero at position (k, l) means that $k \to^{\mathcal{Z}} l$ or $k \not\to^{\mathcal{Z}} l$, respectively. Also, row and column numbers (corresponding to the nodes in the graph) are written in italics. The quadruple $(2, 5, 6, 8)$ is an L for $\mathcal{Z}$ (cf. the emphasized stars and zero). Indeed, $2 \to 6 \leftarrow 5 \to 8$ and $2 \not\to 8$.

An arrow diagram for the directed bipartite graph $\mathcal{Z}$ is



where the solid dots and the circles correspond respectively to the rows and the columns in the matrix diagram (symbols also to be used later in similar contexts). Note that in the matrix diagram, the L shape stands out prominently, whereas in arrow diagram this is not the case. There it looks more like an N. As will become apparent in the next paragraph, this is not by coincidence.

Circling back to the general situation, we say that $\mathcal{Z}$ is L-*free* if $\mathcal{Z}$ does not feature any L. Thus, $\mathcal{Z}$ is L-free if and only if

$$\left.\begin{array}{l} p, q \in \mathbf{M},\ r, s \in \mathbf{N} \\ p \to^{\mathcal{Z}} r \leftarrow^{\mathcal{Z}} q \to^{\mathcal{Z}} s \\ p \leqslant q,\ r \leqslant s \end{array}\right\} \Rightarrow p \to^{\mathcal{Z}} s.$$

This definition is modeled after a notion that has appeared in the literature earlier, actually in the context of working with directed graphs. The concept in question is that of being N-free, cf. [8]. In the present situation it directly translates into requiring that $\mathcal{Z}$ has no N's, an N being a quadruple (p, q, r, s) with $p, q \in \mathbf{M}$ and $r, s \in \mathbf{N}$ satisfying

$$p \to^{\mathcal{Z}} r \leftarrow^{\mathcal{Z}} q \to^{\mathcal{Z}} s, \quad p \neq q,\ s \neq r, \quad p \nrightarrow_{\mathcal{Z}} s. \tag{2.2}$$

In other words, $\mathcal{Z}$ is N-free if and only if

$$\left.\begin{array}{l} p \to^{\mathcal{Z}} r \leftarrow^{\mathcal{Z}} q \to^{\mathcal{Z}} s \\ p \neq q,\ r \neq s \end{array}\right\} \Rightarrow p \to^{\mathcal{Z}} s.$$

Clearly, each L for $\mathcal{Z}$ is an N. The converse in not true, however. As a matter of fact, the quadruple $(2, 3, 3, 1)$ is an N but not an L for the directed bipartite graph $\mathcal{Z}$ considered in Example 2.1.

Obviously, the property of being N-free implies that of being L-free. But, as is easily seen, the converse is not true, cf. Example 3 in [4].

The difference between (2.1) and (2.2) is that in (2.1), reference is made to the (standard) linear orders on the ground sets $\mathbf{M}$ and $\mathbf{N}$, whereas this is not the case in (2.2). Thus, for N-freeness, the specific form of the ground set is irrelevant.

As was mentioned, N-freeness was delineated as a property of directed graphs, or equivalently, as a property on a binary relation on a single ground set $\mathbf{K} = \{1, \ldots, k\}$. For a characterization of N-free partial orders in terms of the so called *Hasse diagram* (cf. [6]), see Theorem 2 in [8]. Other notions that are closely related to the concept of being L-free are in-ultra transitivity and out-ultra transitivity, pertaining again to directed graphs, and introduced in [1] and [2]. The first of these also contains a characterization in terms of the Hasse diagram. A characterization of a similar type seems to be out of reach for L-free partial order, cf. the last two paragraphs of Subsection 3.2 in [4].

3. Main theorem: formulation and relevance of the L-freeness assumption

If A is a matrix, the expressions $\operatorname{rank} A$, $\operatorname{Im} A$ and $\operatorname{Ker} A$ denote the rank, image and null space of A, respectively. The main result of this paper now reads as follows.

Theorem 3.1. *Let m, n be positive integers, let $\mathcal{Z}$ be a directed bipartite graph from $\mathbf{M} = \{1, \ldots, m\}$ to $\mathbf{N} = \{1, \ldots, n\}$, and suppose $\mathcal{Z}$ is L-free. Then, given a nonzero matrix A in $\mathbb{C}^{m\times n}[\mathcal{Z}]$ and positive integers $p, r_1, \ldots, r_p$ satisfying*

$$r_j \leqslant \operatorname{rank} A, \quad j = 1, \ldots, p, \quad \operatorname{rank} A \leqslant r_1 + \ldots + r_p,$$

there exists a decomposition $A = A_1 + \ldots + A_p$ such that

$$A_l \in \mathbb{C}^{m\times n}[\mathcal{Z}], \quad \operatorname{rank} A_l = r_l, \quad l = 1, \ldots, p, \tag{3.1}$$

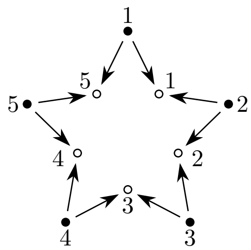
$$\operatorname{Im} A = \operatorname{Im} A_1 + \ldots + \operatorname{Im} A_p, \quad \operatorname{Ker} A = \operatorname{Ker} A_1 \cap \ldots \cap \operatorname{Ker} A_p. \tag{3.2}$$

The condition that the given matrix A is nonzero is imposed in order to avoid trivialities. The proof of the theorem will be given in the next section. Here we will make clear, by means of an example, that the requirement that $\mathcal{Z}$ is L-free, cannot simply be missed. The example is also of significance for Theorem 5.1 in Section 5, which is concerned with what we shall call minimal decompositions.

Example 3.1. For $n = 3, 4, 5, \ldots$, let the directed bipartite graph $\mathcal{Z}$ be given by the matrix diagram

$$\mathcal{Z} = \left(\begin{array}{c|cccccc} & \mathit{1} & \mathit{2} & \mathit{3} & \ldots & \ldots & n \\ \mathit{1} & * & 0 & 0 & \ldots & 0 & * \\ \mathit{2} & * & * & 0 & & & 0 \\ \mathit{3} & 0 & * & * & 0 & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & 0 & * & * & 0 \\ n & 0 & \ldots & \ldots & 0 & * & * \end{array}\right), \tag{3.3}$$

which can also be depicted by an arrow diagram, e.g. for the case $n = 5$,



in which, as before, the solid dots and the circles correspond respectively to the rows and the columns in the matrix diagram.

Note that $\mathcal{Z}$ is not L-free. Indeed, each of the $n-1$ quadruples

$$(1,2,1,2), (2,3,2,3), \ldots, ((n-1), n, (n-1), n)$$

is an L for $\mathcal{Z}$.

Introduce the (circulant) matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & -1 \\ -1 & 1 & 0 & & & 0 \\ 0 & -1 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & -1 & 1 & 0 \\ 0 & \ldots & \ldots & 0 & -1 & 1 \end{bmatrix}.$$

Clearly $A \in \mathbb{C}^{n\times n}[\mathcal{Z}]$, the null space of A has dimension one and correspondingly, $\operatorname{rank} A = n-1$. We claim that A cannot be written as a sum of $n-1$ rank one matrices $A_1, \ldots, A_{n-1}$ having the required zero pattern, i.e., belonging to $\mathbb{C}^{n\times n}[\mathcal{Z}]$. Here is the argument.

Suppose $A = A_1 + \ldots + A_{n-1}$ and the matrices $A_k = [a_{ij}^{(k)}]_{i,j=1}^{n}$ belong to $\mathbb{C}^{n\times n}[\mathcal{Z}]$ for $k = 1, \ldots, n-1$. Considering the n diagonal entries of A, we obtain the n equations,

$$a_{ii}^{(1)} + a_{ii}^{(2)} + \ldots + a_{ii}^{(n-1)} = 1, \quad 1 \leqslant i \leqslant n.$$

In each of these n equations at least one of the $n-1$ summands must be nonzero. Therefore, by the pigeonhole principle, there exists an integer k among $1, \ldots, n-1$ such that at least two of the diagonal entries of A_k are nonzero, say

$$a_{ii}^{(k)} \neq 0, \quad a_{jj}^{(k)} \neq 0,$$

with $1 \leqslant i < j \leqslant n$.

Since A_k is assumed to have the zero pattern determined by $\mathcal{Z}$ in (3.3) and $n \geqslant 3$, it follows that at least one of $a_{ij}^{(k)}$ or $a_{ji}^{(k)}$ is zero. On the other hand, A_k having rank one implies that the submatrix of A_k obtained by deleting all but the ith and jth rows and columns, that is

$$\begin{bmatrix} a_{ii}^{(k)} & a_{ij}^{(k)} \\ a_{ji}^{(k)} & a_{jj}^{(k)} \end{bmatrix},$$

must have rank at most one, too. However, the above statements imply that its determinant $a_{ii}^{(k)}a_{jj}^{(k)} - a_{ji}^{(k)}a_{ij}^{(k)} = a_{ii}^{(k)}a_{jj}^{(k)} \neq 0$ is nonzero. And with this we have reached a contradiction.

4. Main theorem: proof and illustrative example

This section is devoted to the proof of Theorem 3.1, the main result in this paper. For the purpose of adequate presentation it is divided into a couple of subsections. In the final one of these, the material will be illustrated with an example.

4.1. Decomposition ensembles and distribution schemes. Let $p, r, r_1, \ldots, r_p$ be positive integers. The $(p+2)$-tuple $\mathcal{E} = (p, r; r_1, \ldots, r_p)$ is called a *decomposition ensemble* if

$$r_j \leqslant r, \quad j = 1, \ldots, p, \quad r \leqslant r_1 + \ldots + r_p.$$

Clearly, this definition is motivated by Theorem 3.1.

Lemma 4.1. *Let $\mathcal{E} = (p, r; r_1, \ldots, r_p)$ be a decomposition ensemble. Then there exists a matrix $\Lambda = [\lambda_{l,j}]_{l=1,j=1}^{p,r} \in \mathbb{C}^{p \times r}$ such that in each column of Λ the entries add up to* 1*, while for $l = 1, \ldots, p$ the number of nonvanishing entries in the lth row of Λ is equal to r_l.*

Such a matrix is called a *distribution scheme* associated with $\mathcal{E}$.

From the proof as given below it will appear that there is always a distribution scheme associated with $\mathcal{E}$ having rational entries, even being positive whenever nonzero. It will also become clear that generally, one can associate many different distribution schemes with a given decomposition ensemble. Anticipating on what we will see in Section 5, we mention here already that there is an exception to this standard state of affairs: when $r = r_1 + \ldots + r_p$, there is just one scheme modulo the trivial changes brought about by column permutations.

P r o o f. As $r_1 \leqslant r$, there exist integers t among $1, \ldots, p$ such that $r_1 + \ldots + r_t \leqslant r$. Write s for the largest of these and introduce

$$\theta_{l,j} = \begin{cases} 0, & j = 1, \ldots, r_1 + \ldots + r_{l-1}, \\ 1, & j = r_1 + \ldots + r_{l-1} + 1, \ldots, r_1 + \ldots + r_l, \\ 0, & j = r_1 + \ldots + r_l + 1, \ldots, r, \end{cases} \qquad l = 1, \ldots, s,$$

$$\theta_{l,j} = \begin{cases} 0, & j = 1, \ldots, r - r_l, \\ 1, & j = r - r_l + 1, \ldots, r, \end{cases} \qquad l = s+1, \ldots, p.$$

Then $\Theta = [\theta_{l,j}]_{l=1,j=1}^{p,r}$ is a real (in fact a zero/one) matrix, and for $l = 1, \ldots, p$, the lth row of Θ contains precisely r_l nonzero entries (all equal to 1). Also, in each column the entries add up to a positive integer.

Now let $\Omega = [\omega_{l,j}]_{l=1,j=1}^{p,r}$ be a complex matrix with nonzero entries and satisfying

$$\omega_{1,j}\theta_{1,j} + \ldots + \omega_{p,j}\theta_{p,j} = 1, \quad j = 1, \ldots, r.$$

Further introduce

$$\lambda_{l,j} = \omega_{l,j}\theta_{l,j}, \quad l = 1, \ldots, p, \ j = 1, \ldots, r.$$

Then $\Lambda = [\lambda_{l,j}]_{l=1,j=1}^{p,r}$ has the desired features.

Perhaps needless to point out, but there does exist a matrix Ω with the properties required above. Indeed, one can take

$$\omega_{l,j} = \frac{1}{\theta_{1,j} + \ldots + \theta_{p,j}}, \quad l = 1, \ldots, p, \ j = 1, \ldots r.$$

This choice leads to a distribution scheme associated with $\mathcal{E}$ having rational entries, even being positive whenever nonzero. □

We illustrate the foregoing with an example.

Example 4.1. Let $\mathcal{E} = (6, 7; 4, 2, 2, 5, 3, 1)$. Then $\mathcal{E}$ is a decomposition ensemble. Note that $r_1 + r_2 = 6 < 7 = r$ and $r_1 + r_2 + r_3 = 8 > 7 = r$. So $s = 2$, using the notation of the proof of Lemma 4.1. In line with what we did there, we now produce the 6×7 matrix

$$\Theta = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The corresponding column sums are 1, 1, 2, 2, 3, 4, 4. Using their inverses as multiplication factors for the respective columns, we obtain

$$\Lambda = \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix},$$

which is, indeed, a distribution scheme associated with $\mathcal{E}$.

Other schemes can, of course, be obtained by permuting columns. This, by far, does not exhaust all possibilities. Just by way of example, we exhibit

$$\begin{bmatrix} 1 & 1 & -4+\mathrm{i} & -1+31 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3-4\mathrm{i} & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -3+\mathrm{i} \\ 0 & 0 & 5+\mathrm{i} & 2-3\mathrm{i} & 5 & -3 & 5+4\mathrm{i} \\ 0 & 0 & 0 & 0 & -1+4\mathrm{i} & 6 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5\mathrm{i} \end{bmatrix},$$

involving complex numbers, some of them nonreal.

4.2. Lean forms. Let M be an $m \times n$ matrix. Here m and n are positive integers. We say that M is in *lean form* or alternatively, has *lean column structure* if the nonzero columns of M are linearly independent. This, of course, implies that the number of nonzero columns of M is equal to the rank of M. If M happens to be the $m \times n$ zero matrix, then M is in lean form (trivially).

Let A be an arbitrary $m \times n$ matrix. Then A can be brought in lean form, without changing its rank, via multiplication on the right with a suitable upper triangular $n \times n$ matrix. A simple Gaussian elimination type procedure for doing this is as follows. Write $A = [a_1 \dots a_n]$ with $a_1, \dots, a_n \in \mathbb{C}^n$ being the columns of A. For $l = 1, \dots, n$, we leave the lth column of A unchanged whenever a_l is not a linear combination of the columns $a_1, \dots, a_{l-1}$ preceding a_l; otherwise we replace a_l by a zero column. (Here, of course, a vector is a linear combination of an empty collection of columns if and only if it is the zero vector. In this way, the case $l = 1$ is covered without ambiguity.) The resulting matrix $\widehat{A}$ certainly has lean column structure, and its rank is equal to that of A. Also, $\widehat{A}$ can be written in the form $A\widehat{U}$ with $\widehat{U}$ a *monic* $n \times n$ matrix, i.e., an upper triangular matrix with ones on the diagonal.

Generally this is not the only way to bring A in lean form. For proving Theorem 3.1, it needs to be done in a special manner. This is the motivation for the next lemma, which is crucial for what follows. A square matrix is said to be *monic* if it is upper triangular and all its diagonal entries are equal to one.

Lemma 4.2. *Let $\mathcal{Z}$ be an L-free directed bipartite graph from $\mathbf{M} = \{1, \ldots, m\}$ to $\mathbf{N} = \{1, \ldots, n\}$. Then given $A \in \mathbb{C}^{m\times n}[\mathcal{Z}]$, there exists a monic $n \times n$ matrix U such that AU is in lean form, $AU \in \mathbb{C}^{m\times n}[\mathcal{Z}]$, and $AUDU^{-1} \in \mathbb{C}^{m\times n}[\mathcal{Z}]$ for every diagonal $n \times n$ matrix D.*

A direct proof of Lemma 4.2 is possible but very cumbersome, not in the least notationally. Therefore, we will take advantage of Theorem 4.2 in [4] in the proof of which these complications have already been dealt with, following up on the material in [5], highly intricate in its own right. Theorem 4.2 in [4] has an upper triangularity aspect to it. Thus, falling back on it involves an embedding trick of a type also employed in [3] and [4].

All in all, the argument is of a constructive nature and provides an algorithm for obtaining the matrix U. This is relevant for constructing decompositions of the type considered in Theorem 3.1 in concrete examples.

P r o o f. Write $A = [a_{i,j}]_{i=1,j=1}^{m,n}$, put $k = m + n$ and introduce the square matrix $\widehat{A} = [\widehat{a}_{i,j}]_{i,j=1}^{k}$ via

$$\widehat{a}_{i,j} = \begin{cases} a_{i,j-m}, & i = 1, \ldots, m \text{ and } j = m+1, \ldots, k, \\ 0, & i = m+1, \ldots, k \text{ or } j = 1, \ldots, m. \end{cases}$$

In other words,

$$\widehat{A} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}. \tag{4.1}$$

With the directed bipartite graph $\mathcal{Z}$ we now associate a directed graph $\widehat{\mathcal{Z}}$ with ground set $\mathbf{K} = \{1, \ldots, k\}$. In suggestive shorthand

$$\widehat{\mathcal{Z}} = \begin{pmatrix} 0 & \mathcal{Z} \\ 0 & 0 \end{pmatrix}, \tag{4.2}$$

analogously to (4.1). More precisely, using the notation $\to_{\widehat{\mathcal{Z}}}$ for the arrows determining $\widehat{\mathcal{Z}}$,

$$i, j \in \mathbf{K},\ i \to_{\widehat{\mathcal{Z}}} j \quad \text{if and only if} \quad i \in \mathbf{M},\ j \in \mathbf{K} \setminus \mathbf{M},\ i \to^{\mathcal{Z}} (j - m).$$

By assumption $\mathcal{Z}$ is an L-free directed bipartite graph. For the directed graph $\widehat{\mathcal{Z}}$ this translates into

$$\left.\begin{array}{l} p \to_{\widehat{\mathcal{Z}}} r \leftarrow_{\widehat{\mathcal{Z}}} q \to_{\widehat{\mathcal{Z}}} s \\ p < q, \ r < s \end{array}\right\} \Rightarrow p \to_{\widehat{\mathcal{Z}}} s.$$

This means that $\widehat{\mathcal{Z}}$ is L-free as this was defined for directed graphs in [4]. Again borrowing terminology from [4], we also observe that $\widehat{\mathcal{Z}}$ is of upper triangular type, meaning that $i \not\to_{\widehat{\mathcal{Z}}} j$ for all $i, j \in \mathbf{K}$, $i > j$.

Theorem 4.2 in [4] – specialized to the situation, where the blocks are scalar, i.e., have size 1×1 – now guarantees the existence of a monic $k \times k$ matrix $\widehat{U}$ such that $\widehat{A}\widehat{U}$ is in lean form, $\widehat{A}\widehat{U} \in \mathbb{C}^{k\times k}[\widehat{\mathcal{Z}}]$ and $\widehat{A}\widehat{U}\widehat{D}\widehat{U}^{-1} \in \mathbb{C}^{k\times k}[\widehat{\mathcal{Z}}]$ for every diagonal $k \times k$ matrix $\widehat{D}$. The remainder of the argument consists of translating this back to the originally given matrix A and directed bipartite graph $\mathcal{Z}$.

In line with (4.1) and (4.2), and taking into account that $\widehat{U}$ is upper triangular, we decompose $\widehat{U}$ as

$$\widehat{U} = \begin{bmatrix} U_- & U_+ \\ 0 & U \end{bmatrix}.$$

Identifying U with an $n \times n$ matrix, which is then clearly monic, one gets

$$\widehat{A}\widehat{U} = \begin{bmatrix} 0 & AU \\ 0 & 0 \end{bmatrix},$$

implying that AU has lean form and belongs to $\mathbb{C}^{m\times n}[\mathcal{Z}]$.

Finally, let $D \in \mathbb{C}^{n\times n}$ be a diagonal matrix, and introduce

$$\widehat{D} = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}.$$

Then $\widehat{D}$ is (more precisely, can be identified with) a diagonal $k \times k$ matrix, and so $\widehat{A}\widehat{U}\widehat{D}\widehat{U}^{-1} \in \mathbb{C}^{k\times k}[\widehat{\mathcal{Z}}]$. Straightforward computation gives

$$\widehat{A}\widehat{U}\widehat{D}\widehat{U}^{-1} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_- & U_+ \\ 0 & U \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} U_-^{-1} & -U_-^{-1}U_+U^{-1} \\ 0 & U^{-1} \end{bmatrix} = \begin{bmatrix} 0 & AUDU^{-1} \\ 0 & 0 \end{bmatrix},$$

and it follows that $AUDU^{-1} \in \mathbb{C}^{m\times n}[\mathcal{Z}]$, as was claimed. □

The L-freeness requirement in Lemma 4.2 is essential. This appears from the following example.

Example 4.2. Let the directed bipartite graph $\mathcal{Z}$ and $A \in \mathbb{C}^{3\times 3}$ be given by

$$\mathcal{Z} = \left(\begin{array}{cccc} & \mathit{1} & \mathit{2} & \mathit{3} \\ \mathit{1} & * & 0 & * \\ \mathit{2} & * & * & 0 \\ \mathit{3} & 0 & * & * \end{array}\right), \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \tag{4.3}$$

Then $A \in \mathbb{C}^{3\times 3}[\mathcal{Z}]$ and $\mathcal{Z}$ is not L-free. The latter can, of course, be checked directly, but can also be read off from Example 3.1 (case $n = 3$). We shall make clear that with this choice for $\mathcal{Z}$ and A there is no matrix U with the properties mentioned in Lemma 4.2.

Suppose there is one, say

$$U = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

By assumption $AU \in \mathbb{C}^{3\times 3}[\mathcal{Z}]$, i.e.,

$$\begin{bmatrix} 1 & x & y+1 \\ 1 & x-1 & y-z \\ 0 & 1 & z+1 \end{bmatrix} \in \mathbb{C}^{3\times 3}[\mathcal{Z}],$$

and it ensues that $x = 0$ and $y = z$. From this we get

$$U = \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -y \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}^{-1}, \quad AU = \begin{bmatrix} 1 & 0 & y+1 \\ 1 & -1 & 0 \\ 0 & 1 & y+1 \end{bmatrix}.$$

Now, again by hypothesis, the matrix AU has lean column structure, yielding $y = -1$ and

$$U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}, \quad AU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Write D for the 3×3 diagonal matrix having one at the first diagonal position and zero at the two others. Then

$$AUDU^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{4.4}$$

This matrix has a nonzero entry in the position $(2, 1)$, hence it does not belong to $\mathbb{C}^{3\times 3}[\mathcal{Z}]$.

We close this subsection with another comment on Lemma 4.2. Considering the situation of the lemma, involving an L-free graph, the following question arises. Can it happen that for some monic $n \times n$ matrix U, the matrix AU is lean and belongs to $\mathbb{C}^{m\times n}[\mathcal{Z}]$ while, nevertheless, there does exist an $n \times n$ diagonal matrix D such that $AUDU^{-1}$ fails to be in $\mathbb{C}^{m\times n}[\mathcal{Z}]$? Clearly, such a matrix U has to be different from the one whose existence is guaranteed by Lemma 4.2. As will appear from Example 4.3 in Subsection 4.4, the answer is affirmative.

4.3. Proof of main result. In the situation of Theorem 3.1, writing r for the (positive) rank of the given matrix A, the $(p+2)$-tuple $\mathcal{E} = (p, r; r_1, \ldots, r_p)$ is a decomposition ensemble. By Lemma 4.1, there exists a distribution scheme associated with $\mathcal{E}$. Choose one, $\Lambda = [\lambda_{l,j}]_{l=1,j=1}^{p,r}$, say. Further, referring to Lemma 4.2, we take U to be a monic $n \times n$ matrix such that AU is in lean form, $AU \in \mathbb{C}^{m\times n}[\mathcal{Z}]$, and $AUDU^{-1} \in \mathbb{C}^{m\times n}[\mathcal{Z}]$ for every diagonal $n \times n$ matrix D. Using these ingredients we shall construct an additive decomposition of A of the type indicated in Theorem 3.1.

Put $B = AU$. The nonvanishing columns in B are linearly independent, hence the number of them is $\operatorname{rank} B = \operatorname{rank} A = r$. Let us denote their positions by $l_1, \ldots, l_r$, taken in standard order, so $l_1 < \ldots < l_r$. Also write $u_1, \ldots, u_n$ for the standard unit vectors in $\mathbb{C}^n$. Clearly

$$B = \sum_{j=1}^{r} B u_{l_j} u_{l_l}^{\top} = B \sum_{j=1}^{r} u_{l_j} u_{l_l}^{\top}. \tag{4.5}$$

Note that $u_{l_j} u_{l_j}^{\top}$ is the diagonal $n \times n$ matrix having zeros on the diagonal except for the l_jth position, where the entry is 1. Thus, $B u_{l_j} u_{l_l}^{\top}$ is the matrix obtained from B by leaving the l_jth column intact and replacing all the others by zero columns. Hence, along with B, the matrix $B u_{l_j} u_{l_l}^{\top}$ belongs to $\mathbb{C}^{m\times n}[\mathcal{Z}]$.

For $k = 1, \ldots, p$, introduce

$$B_k = \sum_{j=1}^{r} \lambda_{k,j} B u_{l_j} u_{l_j}^{\top}. \tag{4.6}$$

The matrices $B_1, \ldots, B_p$ have (positive) rank $r_1, \ldots, r_p$, respectively, and they belong to $\mathbb{C}^{m\times n}[\mathcal{Z}]$. Also $B = B_1 + \ldots + B_p$. For this we argue as follows.

Let t be one of the integers $1, \ldots, n$. Then, employing the Kronecker delta notation,

$$\left(\sum_{k=1}^{p} B_k\right) u_t = \sum_{k=1}^{p}\sum_{j=1}^{r} \lambda_{k,j} B u_{l_j} u_{l_j}^{\top} u_t = \sum_{k=1}^{p}\sum_{j=1}^{r} \lambda_{k,j} \delta_{l_j,t} B u_{l_j},$$

and hence

$$\begin{aligned}\left(\sum_{k=1}^{p} B_k\right) u_t &= \begin{cases} 0, & t \neq l_1, \ldots, l_r, \\ \displaystyle\sum_{k=1}^{p} \lambda_{k,s} B u_{l_s}, & t = l_s,\ s = 1, \ldots, r, \end{cases} \\ &= \begin{cases} B u_t, & t \neq l_1, \ldots, l_r, \\ \displaystyle\left(\sum_{k=1}^{p} \lambda_{k,s}\right) B u_t, & t = l_s,\ s = 1, \ldots, r. \end{cases}\end{aligned}$$

The sum in the latter expression is equal to 1, and we conclude that

$$\left(\sum_{k=1}^{p} B_k\right) u_t = Bu_t.$$

As t was taken arbitrarily among $1, \ldots, p$, it ensues that $B = B_1 + \ldots + B_p$, indeed.

We have now obtained a decomposition of $B = AU$ in which the summands $B_1, \ldots, B_p$ have rank $r_1, \ldots, r_p$, respectively, and are in $\mathbb{C}^{m\times n}[\mathcal{Z}]$. For $k = 1, \ldots, p$, define $A_k = B_k U^{-1}$. Then $A = (AU)U^{-1} = A_1 + \ldots + A_p$, and this is a decomposition of A involving terms $A_1, \ldots, A_p$ having rank $r_1, \ldots, r_p$, respectively. Later we shall see that these terms feature the desired zero pattern. But first we consider their images and null spaces.

For $k = 1, \ldots, p$, we have $\operatorname{Im} B_k \subset \operatorname{Im} B$. This is clear from the definition of B_k via (4.6). Evidently, $\operatorname{Im} A_k = \operatorname{Im} B_k$ and $\operatorname{Im} A = \operatorname{Im} B$. Thus,

$$\operatorname{Im} A_1 + \ldots + \operatorname{Im} A_p = \operatorname{Im} B_1 + \ldots + \operatorname{Im} B_p \subset \operatorname{Im} B = \operatorname{Im} A.$$

Also $\operatorname{Im} A \subset \operatorname{Im} A_1 + \ldots + \operatorname{Im} A_p$ because $A = A_1 + \ldots + A_p$, and it follows that $\operatorname{Im} A_1 + \ldots + \operatorname{Im} A_p = \operatorname{Im} A$.

Next we look at the null spaces of A and $A_1, \ldots, A_p$. Again taking into account the identity $A = A_1 + \ldots + A_p$, it ensues that

$$\operatorname{Ker} A_1 \cap \ldots \cap \operatorname{Ker} A_p \subset \operatorname{Ker} A.$$

Let $x \in \operatorname{Ker} A$, and put $y = U^{-1}x$, so that $By = 0$. Write $B = [Bu_1 \ldots Bu_n]$ and $y = [y_1 \ldots y_n]^\top$. Then $y_1 Bu_1 + \ldots y_n Bu_n = 0$. The columns of B not in the positions $l_1, \ldots, l_r$ vanish, hence $y_{l_1} Bu_{l_1} + \ldots + y_{l_r} Bu_{l_r} = 0$. Also $Bu_{l_1} + \ldots + Bu_{l_r}$ are linearly independent, it entails that $y_{l_j}, \ldots, y_{l_r}$ vanish. But then

$$B_k y = \sum_{j=1}^{r} \lambda_{k,j} Bu_{l_j} u_{l_j}^\top y = \sum_{j=1}^{r} \lambda_{k,j} y_{l_j} Bu_{l_j} = 0, \quad k = 1, \ldots, p.$$

Recalling that $A_k = B_k U^{-1}$ and $y = U^{-1}x$, we arrive at $A_k x = B_k y = 0$, i.e., $x \in \operatorname{Ker} A_k$, which is what we wanted to establish.

It remains to prove that $A_1, \ldots, A_p \in \mathbb{C}^{m\times n}[\mathcal{Z}]$. Take k among $1, \ldots, p$, and define the diagonal $n \times n$ matrix by $D_k = \lambda_{k,1} u_{l_1} u_{l_1}^\top + \ldots + \lambda_{k,r} u_{l_r} u_{l_r}^\top$. Then $AUD_l U^{-1}$ is in $\mathbb{C}^{m\times n}[\mathcal{Z}]$. On the other hand,

$$A_k = B_k U^{-1} = \left(\sum_{j=1}^{r} \lambda_{k,j} Bu_{l_j} u_{l_j}^\top\right) U^{-1} = B\left(\sum_{j=1}^{r} \lambda_{k,j} u_{l_j} u_{l_j}^\top\right) U^{-1} = AUD_k U^{-1},$$

and the upshot is that $A_k \in \mathbb{C}^{m\times n}[\mathcal{Z}]$, as desired. □

4.4. Example. We illustrate what we did in the preceding subsections with an elucidating example. It will underscore that a special monic matrix U is needed to bring the given matrix A in lean form. The crux lies here in the necessity to have $AUDU^{-1}$ in $\mathbb{C}^{m\times n}[\mathcal{Z}]$ whenever D is a diagonal matrix, as stated in Lemma 4.2. The example will also exhibit a concrete decomposition of A obtained along the lines suggested by the proof of Theorem 3.1 the way it is given in the preceding subsection.

Example 4.3. Let the directed bipartite graph $\mathcal{Z}$ and the matrix A be given by

$$\mathcal{Z} = \begin{pmatrix} & \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} & \mathit{5} & \mathit{6} & \mathit{7} \\ \mathit{1} & * & * & 0 & * & * & * & * \\ \mathit{2} & 0 & 0 & * & 0 & * & * & * \\ \mathit{3} & 0 & 0 & * & 0 & * & * & * \\ \mathit{4} & 0 & * & 0 & * & * & 0 & * \\ \mathit{5} & * & * & 0 & 0 & 0 & 0 & * \end{pmatrix}, \quad A = \begin{bmatrix} 1 & 3 & \mathbf{0} & 1 & 3 & 0 & 5 \\ \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & 2 & 3 & 6 \\ \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} & 4 & 6 & 6 \\ \mathbf{0} & 1 & \mathbf{0} & 1 & 3 & \mathbf{0} & 2 \\ 1 & 2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}$$

with the zero entries forced upon A by the required zero pattern emphasized. Then $\mathcal{Z}$ is L-free and $A \in \mathbb{C}^{5\times 7}[\mathcal{Z}]$. We first bring A in lean form via the procedure described in the second paragraph of Subsection 4.2. Note that the columns of A at the positions 1, 2, 3 and 7 are linearly independent. So the rank four matrix

$$\widehat{A} = \begin{bmatrix} 1 & 3 & \mathbf{0} & 0 & 0 & 0 & 5 \\ \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & 0 & 0 & 6 \\ \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} & 0 & 0 & 6 \\ \mathbf{0} & 1 & \mathbf{0} & 0 & 0 & \mathbf{0} & 2 \\ 1 & 2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \end{bmatrix}$$

has lean column structure. Introducing the monic matrix

$$\widehat{U} = \begin{bmatrix} 1 & 0 & 0 & 2 & 6 & 0 & 0 \\ 0 & 1 & 0 & -1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -2 & -6 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

we have $\widehat{A} = A\widehat{U} \in \mathbb{C}^{5\times 7}[\mathcal{Z}]$.

Now let $\widehat{D}$ be the diagonal 7×7 matrix having all diagonal entries equal to zero, except for the second, which is one. Straightforward calculation shows that the last row in $A\widehat{U}\widehat{D}\widehat{U}^{-1} = \widehat{A}\widehat{D}\widehat{U}^{-1}$ is $[0 \quad 2 \quad 0 \quad 2 \quad 6 \quad 0 \quad 0]$. But then $A\widehat{U}\widehat{D}\widehat{U}^{-1}$ fails to be in $\mathbb{C}^{5\times 7}[\mathcal{Z}]$. Note that we have a validation here of the comment on Lemma 4.2 made at the end of Subsection 4.2.

As we see, the straightforward approach taken above, fails to bring in a crucial element of Lemma 4.2. Indeed, the lemma guarantees the existence of a monic 7×7 matrix U such that not only AU is in lean form and $AU \in \mathbb{C}^{5\times 7}[\mathcal{Z}]$, but also $AUDU^{-1} \in \mathbb{C}^{5\times 7}[\mathcal{Z}]$ for every diagonal $n \times n$ matrix D. In the present situation, it is not completely trivial to identify such a matrix. It can be done by using the material developed in [4].

Leaving the details for what they are, we give the end result, namely the matrix

$$U = \begin{bmatrix} 1 & -2 & 0 & 2 & 6 & 0 & 3 \\ 0 & 1 & 0 & -1 & -3 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

yielding for $B = AU \in \mathbb{C}^{5\times 7}[\mathcal{Z}]$ and U^{-1}

$$B = \begin{bmatrix} 1 & 1 & \mathbf{0} & 0 & 0 & 0 & 2 \\ \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & 0 & 0 & 3 \\ \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} & 0 & 0 & 0 \\ \mathbf{0} & 1 & \mathbf{0} & 0 & 0 & \mathbf{0} & 0 \\ 1 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that, indeed, B is in $\mathbb{C}^{5\times 7}[\mathcal{Z}]$ and has lean column structure.

Suppose we are looking for an additive decomposition of $A = A_1 + A_2 + A_3 + A_4$, involving summands A_1, A_2, A_3, A_4 belonging to $A \in \mathbb{C}^{5\times 7}[\mathcal{Z}]$, and having rank 3, 3, 2, 1, respectively. Theorem 3.1 assures that such decompositions exist. Here is how we can get one in concrete form.

Let $\mathcal{E}$ be the decomposition ensemble $(4, 4; 3, 3, 2, 1)$, and let

$$\Lambda = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} \\ \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} \\ \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} \\ \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} \end{bmatrix} \tag{4.7}$$

be a distribution schemes associated with $\mathcal{E}$. For $k = 1, 2, 3, 4$, in line with the expression (4.6), introduce

$$B_k = \lambda_{k,1} B u_1 u_1^\top + \lambda_{k,2} B u_2 u_2^\top + \lambda_{k,3} B u_3 u_3^\top + \lambda_{k,4} B u_7 u_7^\top$$

and put $A_k = B_k U^{-1}$, i.e.,

$$(4.8)\quad A_k = \begin{bmatrix} \lambda_{k,1} & 2\lambda_{k,1} + \lambda_{k,2} & \mathbf{0} & \lambda_{k,2} & 3\lambda_{k,2} & 0 & \lambda_{k,1} + 2\lambda_{k,2} + 2\lambda_{k,4} \\ \mathbf{0} & \mathbf{0} & \lambda_{k,3} & \mathbf{0} & 2\lambda_{k,3} & 3\lambda_{k,3} & 3\lambda_{k,3} + 3\lambda_{k,4} \\ \mathbf{0} & \mathbf{0} & 2\lambda_{k,3} & \mathbf{0} & 4\lambda_{k,3} & 6\lambda_{k,3} & 6\lambda_{k,3} \\ \mathbf{0} & \lambda_{k,2} & \mathbf{0} & \lambda_{k,2} & 3\lambda_{k,2} & \mathbf{0} & 2\lambda_{k,2} \\ \lambda_{k,1} & 2\lambda_{k,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \lambda_{k,1} \end{bmatrix}.$$

Then $A = A_1 + A_2 + A_3 + A_4$ is a decomposition of A respecting the zero pattern and having the desired rank characteristics.

Specializing the right-hand side of (4.7) to

$$(4.9)\quad \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \end{bmatrix},$$

we obtain concrete numerical instances for A_1, A_2, A_3, A_4, namely

$$A_1 = \frac{1}{6}\begin{bmatrix} 2 & 6 & 0 & 2 & 6 & 0 & 6 \\ 0 & 0 & 3 & 0 & 6 & 9 & 9 \\ 0 & 0 & 6 & 0 & 12 & 18 & 18 \\ 0 & 2 & 0 & 2 & 6 & 0 & 4 \\ 2 & 4 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad A_2 = \frac{1}{6}\begin{bmatrix} 0 & 2 & 0 & 2 & 6 & 0 & 16 \\ 0 & 0 & 3 & 0 & 6 & 9 & 27 \\ 0 & 0 & 6 & 0 & 12 & 18 & 18 \\ 0 & 2 & 0 & 2 & 6 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \frac{1}{3}\begin{bmatrix} 1 & 3 & 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 3 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_4 = \frac{1}{3}\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Inspection confirms that $A = A_1 + A_2 + A_3 + A_4$ with A_1, A_2, A_3, A_4 belonging to $\mathbb{C}^{5\times 7}[\mathcal{Z}]$ and having rank 3, 3, 2, 1, respectively. Also it can be checked by hand that

$$\operatorname{Im} A = \operatorname{Im} A_1 + \operatorname{Im} A_2 + \operatorname{Im} A_3 + \operatorname{Im} A_4$$

$$\operatorname{Ker} A = \operatorname{Ker} A_1 \cap \operatorname{Ker} A_2 \cap \operatorname{Ker} A_3 \cap \operatorname{Ker} A_4,$$

which was to be expected in view of Theorem 3.1.

5. Minimal decompositions

In this section we consider decompositions that are minimal in the sense that the ranks of the summands add up exactly to the rank of the given matrix.

5.1. Decomposition under minimality requirements. Let m, n be positive integers, and suppose $A \in \mathbb{C}^{m\times n}$. Consider a decomposition $A = A_1 + \ldots + A_p$ with p a positive integer and $A_1, \ldots, A_p$ nonzero matrices in $\mathbb{C}^{m\times n}$. Then, clearly, $\operatorname{rank} A \leqslant \operatorname{rank} A_1 + \ldots + \operatorname{rank} A_p$, and we have equality here as the extreme possibility. In that case, when

$$\operatorname{rank} A = \operatorname{rank} A_1 + \ldots + \operatorname{rank} A_p,$$

we call the decomposition *minimal*. This obviously implies that p cannot exceed $\operatorname{rank} A$ and that $\operatorname{rank} A_j \leqslant \operatorname{rank} A$, $j = 1, \ldots, p$.

Theorem 5.1. *Let m, n be positive integers, let $\mathcal{Z}$ be a directed bipartite graph from $\mathbf{M} = \{1, \ldots, m\}$ to $\mathbf{N} = \{1, \ldots, n\}$, and suppose $\mathcal{Z}$ is* L*-free. Then, given A in $\mathbb{C}^{m\times n}[\mathcal{Z}]$, a positive integer p (not exceeding* $\operatorname{rank} A$*) and positive integers $r_1, \ldots, r_p$ satisfying* $\operatorname{rank} A = r_1 + \ldots + r_p$*, there exists a minimal decomposition $A = A_1 + \ldots + A_p$ such that*

(1) *$A_1, \ldots, A_p$ belong to $\mathbb{C}^{m\times n}[\mathcal{Z}]$, are linearly independent, and have rank $r_1, \ldots, r_p$, respectively,*

(2) $\operatorname{Im} A$ *is the direct sum of* $\operatorname{Im} A_1, \ldots, \operatorname{Im} A_p$*, and* $\operatorname{Ker} A$ *is the interlaced intersection of* $\operatorname{Ker} A_1, \ldots, \operatorname{Ker} A_p$.

By saying that a subspace N of a linear space X is the *interlaced intersection* of the subspaces $N_1, \ldots, N_p$ of X we mean that $N = N_1 \cap \ldots \cap N_p$ and

$$X = N_s + (N_1 \cap \ldots \cap N_{s-1} \cap N_{s+1} + \ldots \cap N_p), \quad s = 1, \ldots, p.$$

Recall that a subspace R of X is the direct sum of the subspaces $R_1, \ldots, R_p$ of X if $R = R_1 + \ldots + R_p$ and

$$\{0\} = R_s \cap (R_1 + \ldots + R_{s-1} + R_{s+1} + \ldots + R_p), \quad s = 1, \ldots, p.$$

Thus, the notion of an interlaced intersection is, so to speak, the intersection counterpart of the familiar concept of a direct sum.

P r o o f. The proof is a continuation of the argument given in Subsection 4.3. So we elaborate on this argument under the additional assumption that the positive integers $r_1, \ldots, r_p$ add up to r, where $r = \operatorname{rank} A$.

Recall that $\Lambda = [\lambda_{l,j}]_{l=1,j=1}^{p,r}$ is a distribution scheme associated with the decomposition ensemble $\mathcal{E} = (p, r, r_1, \ldots, r_p)$. For $l = 1, \ldots, p$, let Λ_l be the set of all j among $1, \ldots, r$ such that $\lambda_{l,j} \neq 0$. Then the cardinality $\sharp\Lambda_l$ of Λ_l is equal to r_l. Now take k in $\{1, \ldots, r\}$. By assumption, $\lambda_{1,k} + \ldots + \lambda_{p,k} = 1$, and so the terms in the sum cannot all vanish. Thus, $k \in \Lambda_l$ for some l among $1, \ldots, p$, and we conclude that $\{1, \ldots, r\}$ is the union of the (nonempty) sets $\Lambda_1, \ldots, \Lambda_p$. But then

$$\{1, \ldots, r\} = \bigcup_{l=1}^{p} (\Lambda_l \setminus (\Lambda_1 \cup \ldots \cup \Lambda_{l-1}))$$

as well. This, however, is a disjoint union, so

$$r = \sum_{l=1}^{p} \sharp(\Lambda_l \setminus (\Lambda_1 \cup \ldots \cup \Lambda_{l-1})) \leqslant \sum_{l=1}^{p} \sharp\Lambda_l = \sum_{l=1}^{p} r_l = r.$$

From this we get

$$\Lambda_j \setminus (\Lambda_1 \cup \ldots \cup \Lambda_{j-1}) = \Lambda_j, \quad j = 1, \ldots, r,$$

and it follows that $\{1, \ldots, r\}$ is not only the union, but in fact the disjoint union of the (nonempty) sets $\Lambda_1, \ldots, \Lambda_p$.

We can prove even more. Take j in Λ_l, where l is one of the integers $1, \ldots, p$. Then $\lambda_{l,j} \neq 0$. For $t = 1, \ldots, p$, $t \neq l$, we have that $\Lambda_l \cap \Lambda_t = \emptyset$, therefore $\lambda_{t,j} = 0$. But $\lambda_{1,j} + \ldots + \lambda_{p,j} = 1$, and it ensues that $\lambda_{l,j} = 1$. Thus, the nonzero entries of Λ are all equal to 1. Also each column of Λ contains precisely one nonzero entry, which is equal to 1.

For B as in Subsection 4.3, we have $B = AU$ with U as in Lemma 4.2. Also, the matrices $B_1, \ldots, B_p$ are given by (4.6), so in the present situation

$$B_k = \sum_{j=1}^{r} \lambda_{k,j} B u_{l_j} u_{l_j}^\top = \sum_{j \in \Lambda_k} B u_{l_j} u_{l_j}^\top$$

with $l_1, \ldots, l_p$ being the consecutive positions of the nonzero columns in the matrix B. As $\{1, \ldots, r\}$ is the disjoint union of the sets $\Lambda_1, \ldots, \Lambda_p$, we have, using (4.5),

$$\sum_{k=1}^{p} B_k = \sum_{k=1}^{p} \sum_{j \in \Lambda_k} B u_{l_j} u_{l_j}^\top = B \sum_{j=1}^{r} u_{l_j} u_{l_j}^\top = B.$$

Further, $B_1, \ldots, B_p$ belong to $\mathbb{C}^{m \times n}[\mathcal{Z}]$ and have rank $r_1, \ldots, r_p$, respectively. As B is in lean form, its nonzero columns $Bu_{l_1}, \ldots, Bu_{l_r}$ are linearly independent vectors in $\mathbb{C}^m$. It immediately follows that $B_1, \ldots, B_p$ are linearly independent and that $\operatorname{Im} B$

is the direct sum of $\operatorname{Im} B_1, \ldots, \operatorname{Im} B_p$. The conclusion that $\operatorname{Ker} A$ is the interlaced intersection of $\operatorname{Ker} A_1, \ldots, \operatorname{Ker} A_p$ can be reached by a straightforward expansion of the reasoning contained in the all but last paragraph of Subsection 4.3.

For $k = 1, \ldots, p$, define $A_k = B_k U^{-1}$. Then $A = BU^{-1} = A_1 + \ldots + A_p$, and this is a decomposition of A involving linearly independent terms $A_1, \ldots, A_p$ from $\mathbb{C}^{m \times n}[\mathcal{Z}]$ having rank $r_1, \ldots, r_p$, respectively. Thus, statement (1) in the theorem that we are proving is correct. Claim (2) holds as well. This is immediate from the validity of the corresponding assertions for the decomposition $B = B_1 + \ldots + B_p$. The minimality of the decomposition $A = A_1 + \ldots + A_p$ stems directly from the assumption $\operatorname{rank} A = r_1 + \ldots + r_p$. $\square$

Just as this was the case for Theorem 3.1, the L-freeness hypothesis cannot be missed in Theorem 5.1. In fact, the example which was used to make this clear (see Example 3.1 from Section 3) is concerned with a minimal decomposition in the sense considered here.

A simple instance of a minimal decomposition can be obtained from Example 4.3, replacing the decomposition ensemble $(4, 4; 3, 3, 2, 1)$ there by $(3, 4; 1, 2, 1)$, and the distribution scheme (4.9) by, for instance,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We refrain from working out the details. Instead, we move on to the next subsection, where examples of minimal decompositions will come up too.

5.2. Counting minimal decompositions and examples. Let us circle back to Theorem 5.1 and its proof. Fix a monic matrix U with the properties indicated in Lemma 4.2. Then, corresponding to every distribution scheme $\Lambda = [\lambda_{l,j}]_{l=1,j=1}^{p,r}$, the given matrix $A \in \mathbb{C}^{m \times n}[\mathcal{Z}]$ admits a minimal decomposition involving summands having the properties mentioned in the theorem. Different schemes may give rise to the same decompositions, that is, if a simple change in the order of the terms is not counted as a difference. Such a change corresponds to a permutation of the rows of the scheme. Now take into account the special character of the distribution schemes here at hand, as exhibited in the second paragraph of the proof of Theorem 5.1. This gives that there is a one-to-one correspondence between partitions of the set $\{1, \ldots, r\}$ into nonempty subsets on the one hand and the minimal decompositions we are interested in on the other. Thus, counting the minimal decompositions of the type obtained in the proof of Theorem 5.1 comes down to counting the number of partitions of the set $\{1, \ldots, r\}$ into nonempty subsets. As is well-known, the outcome is the rth Bell number (in the literature usually denoted by B_r).

Looking at the counting issue more closely, let p be a positive integer among $1,\dots,r$. Then the number of minimal decompositions involving p terms, again of the type obtained in the proof of Theorem 5.1, is equal to the number of partitions of the set $\{1,\dots,r\}$ involving p nonempty subsets. This is the so called *Stirling partition number* (written $\{{n \atop p}\}$). The sum of these numbers for p ranging from 1 to r is (of course) equal to the rth Bell number mentioned above. For details about these special numbers, see [7], [9] and [10]. They play a role too in counting issues dealt with in [2] and [3].

The matrix U featuring in the preceding paragraph is not uniquely determined. So letting it range through different possibilities might add to the number of minimal decompositions one obtains. We will now present two examples featuring totally different outcomes, in fact from uniqueness of the 'right multiplier' U to a situation, where the number of minimal decompositions is infinite.

Example 5.1. Let the directed bipartite graph $\mathcal{Z}$ and $A \in \mathbb{C}^{3\times 3}$ be given by

$$\mathcal{Z} = \left(\begin{array}{cccc} & \mathit{1} & \mathit{2} & \mathit{3} \\ \mathit{1} & 0 & * & * \\ \mathit{2} & * & * & * \\ \mathit{3} & * & * & 0 \end{array}\right), \quad A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \tag{5.1}$$

Then $\mathcal{Z}$ is L-free, A belongs to $\mathbb{C}^{3\times 3}[\mathcal{Z}]$ and the rank of A is 2. Take $p = 2$ and $r_1 = r_2 = 1$. Then the conditions imposed on p, r_1, r_2 in Theorem 5.1 (for $m = n = 3$ and $\operatorname{rank} A = 2$) are met. In accordance with the theorem, there exist matrices $A_1, A_2 \in \mathbb{C}^{3\times 3}[\mathcal{Z}]$, both of rank one and such that $A = A_1 + A_2$. Indeed,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5.2}$$

and this is a minimal decomposition. The linear independence of the summands and statement (2) of Theorem 5.1 narrowed down to the situation here at hand, can be easily checked directly.

The minimal decomposition (5.2) can be obtained via the route indicated in the proofs of Theorems 3.1 and 5.1. The distribution scheme in question is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and for the monic right multiplier U, one can (in fact must — as we will see later) take

$$U = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1},$$

resulting in the matrix

$$AU = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

which, indeed, has lean column structure, belongs to $\mathbb{C}^{3\times 3}[\mathcal{Z}]$, and has the additional property that $AUDU^{-1} \in \mathbb{C}^{3\times 3}[\mathcal{Z}]$ for every diagonal 3×3 matrix D. Indeed, with u_1, u_2 and u_3 standing, respectively, for the first, second and third unit vector in $\mathbb{C}^{3\times 3}$,

$$AUu_1u_1^\top U^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \quad AUu_2u_2^\top U^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

while $AUu_3u_3^\top U^{-1} = 0$.

Anticipating on what we will do in a moment, we mention that (5.2) is the only (minimal) decomposition of A involving two rank one terms from $\mathbb{C}^{3\times 3}[\mathcal{Z}]$. Here, as already indicated, decompositions are identified when they can be obtained from each other by reordering of the summands.

Taking into account that, trivially, $A = A$ is a minimal decomposition of A, we conclude that there are precisely two minimal decompositions of A involving summands from $\mathbb{C}^{3\times 3}[\mathcal{Z}]$. This fits with the fact that the second Bell number is (obviously) equal to two.

Let us now address the uniqueness claim put forward above. Suppose $A^{(1)}$ and $A^{(2)}$ are rank one matrices in $\mathbb{C}^{3\times 3}[\mathcal{Z}]$ adding up to A, and write

$$A^{(1)} = \begin{bmatrix} 0 & a_{1,2}^{(1)} & a_{1,3}^{(1)} \\ a_{2,1}^{(1)} & a_{2,2}^{(1)} & a_{2,3}^{(1)} \\ a_{3,1}^{(1)} & a_{3,2}^{(1)} & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & a_{1,2}^{(2)} & a_{1,3}^{(2)} \\ a_{2,1}^{(2)} & a_{2,2}^{(2)} & a_{2,3}^{(2)} \\ a_{3,1}^{(2)} & a_{3,2}^{(2)} & 0 \end{bmatrix}.$$

As $a_{3,1}^{(1)} + a_{3,1}^{(2)} = 1$, we may assume without loss of generality that $a_{3,1}^{(1)} \neq 0$. Consider the submatrix of A_1 obtained by deleting the second row and the second column, and observe that its rank is (at most) one. Hence, $a_{1,3}^{(1)} = 0$. A similar reasoning gives $a_{1,2}^{(1)} = a_{2,3}^{(1)} = 0$. But then

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ a_{2,1}^{(1)} & a_{2,2}^{(1)} & 0 \\ a_{3,1}^{(1)} & a_{3,2}^{(1)} & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & 1 & 1 \\ a_{2,1}^{(2)} & a_{2,2}^{(2)} & 1 \\ a_{3,1}^{(2)} & a_{3,2}^{(2)} & 0 \end{bmatrix}.$$

Bringing into play that $A^{(2)}$ has rank one, we get

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = A_1, \quad A^{(2)} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = A_2,$$

as desired.

We now come back to the uniqueness of the right multiplier U already claimed in passing above. Suppose the monic matrix

$$V = \begin{bmatrix} 1 & v_{1,2} & v_{1,3} \\ 0 & 1 & v_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$$

is such that AV is in lean form and the matrices

$$AV, \quad AVu_1u_1^\top V^{-1}, \quad AVu_2u_2^\top V^{-1}, \quad AVu_3u_3^\top V^{-1}$$

all belong to $\mathbb{C}^{3\times3}[\mathcal{Z}]$. We shall prove that $V = U$. Here is the argument.

Multiplying A and V gives

$$AV = \begin{bmatrix} 0 & 1 & v_{2,3}+1 \\ 1 & v_{1,2} & v_{1,3}+1 \\ 1 & v_{1,2}-1 & v_{1,3}-v_{2,3} \end{bmatrix}.$$

The first two columns of this matrix are linearly independent. So, as AV has lean column structure, the third column of AV has to vanish. Thus, $v_{1,3} = v_{2,3} = -1$ and

$$V = \begin{bmatrix} 1 & v_{1,2} & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad AV = \begin{bmatrix} 0 & 1 & 0 \\ 1 & v_{1,2} & 0 \\ 1 & v_{1,2}-1 & 0 \end{bmatrix}.$$

The latter matrix already belongs to $\mathbb{C}^{3\times3}[\mathcal{Z}]$, so this requirement does not impose a restriction on $v_{1,2}$. As is easily checked, the inverse of V is given by

$$V^{-1} = \begin{bmatrix} 1 & -v_{1,2} & 1-v_{1,2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and it follows that

$$AVu_1u_1^\top V^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -v_{1,2} & 1-v_{1,2} \\ 1 & -v_{1,2} & 1-v_{1,2} \end{bmatrix}.$$

This matrix should be in $\mathbb{C}^{3\times 3}[\mathcal{Z}]$. This yields $v_{1,2} = 1$ with in its wake

$$V = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

in other words $V = U$, as claimed.

In the preceding example, the number of different minimal decompositions is two, so a fortiori it is finite. As we will now see, this need not be the case.

Example 5.2. Let the directed bipartite graph $\mathcal{Z}$ and $A \in \mathbb{C}^{3\times 3}$ be given by

$$\mathcal{Z} = \left(\begin{array}{c|ccc} & \mathit{1} & \mathit{2} & \mathit{3} \\ \mathit{1} & 0 & * & * \\ \mathit{2} & * & * & * \\ \mathit{3} & * & * & 0 \end{array}\right), \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that $\mathcal{Z}$ is the same graph as the one featuring in Example 5.1. As already observed there, it is L-free. Obviously, the matrix A belongs to $\mathbb{C}^{3\times 3}[\mathcal{Z}]$.

Again take $p = 2$ and $r_1 = r_2 = 1$, so that the conditions imposed on p, r_1, r_2 in Theorem 5.1 (for $m = n = 3$ and $r = 2$) are met. In sharp contrast to what we encountered in Example 5.1, there are now infinitely many minimal decompositions of A involving two terms from $\mathbb{C}^{3\times 3}[\mathcal{Z}]$. Indeed, the decompositions of this type can be parameterized as

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1-\alpha & -\beta & 0 \\ -\gamma & \alpha & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \alpha & \beta & 0 \\ \gamma & 1-\alpha & 0 \end{bmatrix} \tag{5.3}$$

with α, β and γ scalars such that $\alpha^2 - \alpha + \beta\gamma = 0$. Here is how this can be seen.

Suppose α, β and γ are scalars as indicated in the previous paragraph. Then (5.3) is a minimal decomposition of A involving two rank one summands from $\mathbb{C}^{3\times 3}[\mathcal{Z}]$. The linear independence of the summands and statement (2) of Theorem 5.1, specialized to the situation here at hand, can be easily verified by inspection.

Conversely, suppose $A = A^{(1)} + A^{(2)}$ is a decomposition of the type under consideration, and write

$$A^{(1)} = \begin{bmatrix} 0 & a_{1,2}^{(1)} & a_{1,3}^{(1)} \\ a_{2,1}^{(1)} & a_{2,2}^{(1)} & a_{2,3}^{(1)} \\ a_{3,1}^{(1)} & a_{3,2}^{(1)} & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} 0 & a_{1,2}^{(2)} & a_{1,3}^{(2)} \\ a_{2,1}^{(2)} & a_{2,2}^{(2)} & a_{2,3}^{(2)} \\ a_{3,1}^{(2)} & a_{3,2}^{(2)} & 0 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_{1,2}^{(1)} & a_{1,3}^{(1)} \\ a_{2,1}^{(1)} & a_{2,2}^{(1)} & a_{2,3}^{(1)} \\ a_{3,1}^{(1)} & a_{3,2}^{(1)} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{1,2}^{(2)} & a_{1,3}^{(2)} \\ a_{2,1}^{(2)} & a_{2,2}^{(2)} & a_{2,3}^{(2)} \\ a_{3,1}^{(2)} & a_{3,2}^{(2)} & 0 \end{bmatrix}.$$

Clearly $a_{2,1}^{(2)}a_{3,2}^{(2)} = a_{2,2}^{(2)}a_{3,1}^{(2)}$. Also, as $a_{2,1}^{(1)} + a_{2,1}^{(2)} = 1$, we may assume without loss of generality that $a_{2,1}^{(1)} \neq 0$. Inspection of appropriately chosen submatrices yields $a_{1,2}^{(1)} = a_{1,3}^{(1)} = 0$, and we get

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 - a_{2,1}^{(2)} & -a_{2,2}^{(2)} & -a_{2,3}^{(2)} \\ -a_{3,1}^{(2)} & 1 - a_{3,2}^{(2)} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ a_{2,1}^{(2)} & a_{2,2}^{(2)} & a_{2,3}^{(2)} \\ a_{3,1}^{(2)} & a_{3,2}^{(2)} & 0 \end{bmatrix}.$$

Considering the submatrices in the summands $A^{(1)}$ and $A^{(2)}$ obtained from deleting the first row and the first column, we conclude that $-a_{2,3}^{(2)}(1 - a_{3,2}^{(2)}) = 0 = a_{2,3}^{(2)}a_{3,2}^{(2)}$, and thus $a_{2,3}^{(2)} = 0$, leading to

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 - a_{2,1}^{(2)} & -a_{2,2}^{(2)} & 0 \\ -a_{3,1}^{(2)} & 1 - a_{3,2}^{(2)} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ a_{2,1}^{(2)} & a_{2,2}^{(2)} & 0 \\ a_{3,1}^{(2)} & a_{3,2}^{(2)} & 0 \end{bmatrix}.$$

Now introduce $\alpha = a_{2,1}^{(2)}$, $\beta = a_{2,2}^{(2)}$, $\gamma = a_{3,1}^{(2)}$ and $\delta = a_{3,2}^{(2)}$. As the summands in the above expression supposedly are of rank one, we have

$$(1 - \alpha)(1 - \delta) = \beta\gamma, \quad \alpha\delta = \beta\gamma,$$

and these imply that $\delta = 1 - \alpha$, which can be rewritten as $a_{3,2}^{(2)} = 1 - \alpha$. With this, we have arrived at the desired form (5.3).

How does all of this relate to the way minimal decompositions were obtained in the proof of Theorem 5.1? Here is the pertinent analysis.

Let

$$U = \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -x & xz - y \\ 0 & 1 & -z \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

be a monic matrix. Then

$$AU = \begin{bmatrix} 0 & 0 & 0 \\ 1 & x & y \\ 0 & 1 & z \end{bmatrix}.$$

This matrix belongs to $\mathbb{C}^{3\times 3}[\mathcal{Z}]$ and has lean form if and only if $y = z = 0$.

Suppose this is the case, so

$$U = \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} 1 & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad AU = \begin{bmatrix} 0 & 0 & 0 \\ 1 & x & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Write u_1 and u_2 for the first and second standard unit vector in $\mathbb{C}^{3\times 3}$. What we want to have is that $AUu_1u_1^\top U^{-1}$ and $AUu_2u_2^\top U^{-1}$ belong to $\mathbb{C}^{3\times 3}[\mathcal{Z}]$. Straightforward computation shows that

$$AUu_1u_1^\top U^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -x & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad AUu_2u_2^\top U^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and these matrices do indeed belong to $\mathbb{C}^{3\times 3}[\mathcal{Z}]$. The resulting minimal decomposition of A, involving two rank one matrices in $\mathbb{C}^{3\times 3}[\mathcal{Z}]$, is

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -\beta & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

with $\beta = x$. This fits with (5.3): just take $\alpha = 0$, $\gamma = 0$ and $\delta = 1$.

So we see that following the path of the proof of Theorem 5.1 gives only a subset of the collection of all minimal decompositions. In a sense, this is not surprising. Indeed, the approach in the present paper, following the set up in [4] and [5], is column based, and it is also possible to work on a row basis. This would give minimal decompositions of the type (5.3) with $\alpha = 0$, $\beta = 0$, $\delta = 1$, i.e.,

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -\gamma & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma & 1 & 0 \end{bmatrix}.$$

These can also be obtained by taking transposes.

Thus, with the approach via lean forms, we find the minimal decompositions of A, involving two rank one terms, of the form

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -\beta & 0 \\ -\gamma & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ \gamma & 1 & 0 \end{bmatrix}$$

with β or γ equal to zero. Evidently this does not exhaust all possibilities. For instance, the decomposition

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -3 & -6 & 0 \\ 2 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 6 & 0 \\ -2 & -3 & 0 \end{bmatrix}$$

is not covered. It is obtained from (5.3) by taking $\alpha = 4$, $\beta = 6$ and $\gamma = -2$, scalars that are easily seen to satisfy the identity $\alpha^2 - \alpha + \beta\gamma = 0$.

The above example warrants the conclusion that a set up giving all possible minimal decompositions meeting the zero pattern requirement should go beyond the approach utilizing lean forms. As the paragraph directly below Lemma 4.2 suggests, this is most probably a highly challenging matter.

6. Concluding remarks and open problems

In this section, we finish the paper with some remarks leading to issues open for further research.

6.1. Norm optimization of distribution schemes. The distribution scheme given by (4.9) in Example 4.3 has nonnegative rational entries not exceeding one. We could have used one having integer entries, such as

$$\Lambda = \begin{bmatrix} 4 & 4 & 2 & 0 \\ 0 & -2 & -1 & 1 \\ -2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{6.1}$$

This would have resulted in a decomposition which is not essentially different from the one we obtained via (4.9). The drawback, however, is that with this 'integer approach' the summands in the decomposition will generally have larger norms, assuming of course that we think of the underlying matrix algebra as being equipped with one of the usual norms. One can drive this to extremes by employing distribution schemes featuring entries with comparatively large absolute values, such as

$$\Lambda = \begin{bmatrix} 4.10^{100}+4 & 4.10^{100}+4 & 10^{1000}+2 & 0 \\ 0 & -2.10^{100}-2 & -10^{1000}-1 & 1 \\ -2.10^{100}-2 & -2.10^{100}-1 & 0 & 0 \\ -2.10^{100}-1 & 0 & 0 & 0 \end{bmatrix} \tag{6.2}$$

for instance. From the point of view of working with norms there is an optimization issue here.

6.2. Permutation L-freeness. As before, let m, n be positive integers, and let $\mathcal{Z}$ be a binary relation between the sets $\mathbf{M} = \{1, \ldots, m\}$ and $\mathbf{N} = \{1, \ldots, n\}$. The notion of being L-free depends on the natural order of $\mathbf{M}$ and $\mathbf{N}$. In the last paragraph of Section 2, it has been indicated that this forms an obstacle for an adequate characterization. As has been mentioned there, this is different for the less general concept of N-freeness, which is purely graph theoretical.

This state of affairs leads to the following definition. Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two directed bipartite graphs from $\mathbf{M}$ to $\mathbf{N}$. We say that $\mathcal{Z}$ is *permutation equivalent* to $\mathcal{Z}'$ if the matrix diagram for $\mathcal{Z}$ can be obtained from the matrix diagram for $\mathcal{Z}'$ by reordering of the rows and by reordering of the columns. Thus, $\mathcal{Z}$ is permutation equivalent to $\mathcal{Z}'$ if, using a suggestive notation, $\mathcal{Z} = P\mathcal{Z}'Q$ for certain permutation matrices P and Q of order m and n, respectively.

Evidently $\mathcal{Z}$ is permutation equivalent to $\mathcal{Z}'$ if and only if $\mathcal{Z}'$ is permutation equivalent to $\mathcal{Z}$. Also, in that case, if $A, A' \in \mathbb{C}^{m\times n}$, then $A \in \mathbb{C}^{m\times n}[\mathcal{Z}]$ if and only if $P^{-1}AQ^{-1} \in \mathbb{C}^{m\times n}[\mathcal{Z}']$.

We call the directed bipartite graph $\mathcal{Z}$ *permutation* L*-free* if it is permutation equivalent to a directed bipartite graph which is L-free. Note that this property does not depend on the natural orders of $\mathbf{M}$ and $\mathbf{N}$. So permutation L-freeness is a genuine graph theoretical concept. Also observe that (trivially) each L-free bipartite graph is permutation L-free. The following simple example shows that the converse does not hold.

Example 6.1. Let the directed bipartite graphs $\mathcal{Z}$ and $\mathcal{Z}'$ be given by the matrix diagrams

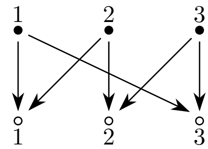
$$\mathcal{Z} = \begin{pmatrix} & \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} \\ \mathit{1} & * & 0 & 0 & 0 \\ \mathit{2} & * & * & 0 & * \\ \mathit{3} & * & 0 & * & * \end{pmatrix}, \quad \mathcal{Z}' = \begin{pmatrix} & \mathit{1} & \mathit{2} & \mathit{3} & \mathit{4} \\ \mathit{1} & 0 & * & * & * \\ \mathit{2} & * & * & * & 0 \\ \mathit{3} & 0 & * & 0 & 0 \end{pmatrix}. \tag{6.3}$$

Then $\mathcal{Z}$ is obtained from $\mathcal{Z}'$ by interchanging the first and the third row, and also interchanging the first and the second column as well as the third and the fourth. Hence, $\mathcal{Z}$ is permutation equivalent to $\mathcal{Z}'$. Inspection shows that $\mathcal{Z}'$ is L-free but $\mathcal{Z}$ is not. So $\mathcal{Z}$ is a permutation L-free bipartite graph which fails to be L-free.

The relevance of the new notion lies in the fact that the main results in this paper, Theorems 3.1 and 5.1, remain true when the requirement of L-freeness for the given directed bipartite graph $\mathcal{Z}$ is replaced by the condition that it is permutation L-free. The argument showing this is straightforward and left to the reader.

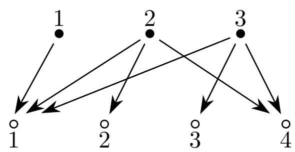
The directed graph in Example 3.1 is not permutation L-free. This is clear from the fact that the graph in question does not allow for the conclusion of Theorem 3.1. It can also be verified directly, especially for the situation where $n = 3$ and (3.3) is the graph featuring in Example 4.2, namely

$$\begin{array}{cc} & \begin{array}{ccc} \mathit{1} & \mathit{2} & \mathit{3} \end{array} \\ \begin{array}{c} \mathit{1} \\ \mathit{2} \\ \mathit{3} \end{array} & \begin{pmatrix} * & 0 & * \\ * & * & 0 \\ 0 & * & * \end{pmatrix} \end{array},$$



where the solid dots and the circles correspond, respectively, to the rows and the columns in the matrix diagram. This direct verification, which can be done in several ways, e.g., by checking 36 cases, is left to the reader.

We close this subsection (and the paper) by putting forward another open issue. As has been indicated, permutation L-freeness is a genuine graph theoretical notion in the sense that it does not rely on the order of the underlying sets of nodes. Is it possible to characterize the concept without direct reference to L-freeness? Concretely, the arrow diagram for the graph $\mathscr{Z}$ in Example 6.1 is



where, again, the solid dots and the circles correspond, respectively, to the rows and the columns in the first matrix diagram in (6.3). How to read off from this arrow diagram that the graph in question is permutation L-free?

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