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ISOLATED SUBGROUPS OF FINITE ABELIAN GROUPS

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Abstract. We say that a subgroup H is isolated in a group G if for every $x \in G$ we have either $x \in H$ or $\langle x \rangle \cap H = 1$. We describe the set of isolated subgroups of a finite abelian group. The technique used is based on an interesting connection between isolated subgroups and the function sum of element orders of a finite group.

Keywords: finite abelian group; isolated subgroup; sum of element orders

MSC 2020: 20K01, 20K27

1. Introduction

Let G be a finite group. We say that a subgroup H of G is isolated in G if for every $x \in G$ we have either $x \in H$ or $\langle x \rangle \cap H = 1$. Groups with isolated subgroups were studied in [2], [3]. However, this concept appears much earlier (see for instance Section 66 of [7] and the entry "isolated subgroup" in Encyclopedia of Mathematics, cf. [5]). The starting point for our discussion is given by Janko's paper (see [6]) that investigates isolated subgroups for certain classes of nonabelian p-groups.

In the following, we determine these subgroups for finite abelian groups. The problem is reduced to finite abelian p-groups. Our main result can be summarized as follows.

Theorem 1.1. *Let p be a prime number and $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p-group, where $1 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_k$.*

(a) *If $\alpha_1 > 1$, then the unique isolated subgroups of G are 1 and G.*

(b) *If $1 = \alpha_1 = \alpha_2 = \ldots = \alpha_r < \alpha_{r+1} \leqslant \ldots \leqslant \alpha_k$ and $A = \mathbb{Z}_{p^{\alpha_{r+1}}} \times \ldots \times \mathbb{Z}_{p^{\alpha_k}}$, then the isolated subgroups of G are G and all subgroups $H \leqslant G$ with $H \cap A = 1$.*

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The main tool used is the function sum of element orders of G,

$$\psi(G)=\sum_{x\in G}o(x),$$

defined by Amiri, Jafarian Amiri and Isaacs in [1]. Given a subgroup H of G, this has been generalized in [9] to the function

$$\psi_H(G)=\sum_{x\in G}o_H(x),$$

where $o_H(x)$ denotes the order of x relative to H, i.e., the smallest positive integer m such that $x^m\in H$. Clearly, for $H=1$ we have $\psi_H(G)=\psi(G)$.

We remark that

$$\psi_H(G)=\sum_{x\in H}o_H(x)+\sum_{x\in G\setminus H}o_H(x)=|H|+\sum_{x\in G\setminus H}\frac{o(x)}{|\langle x\rangle\cap H|}$$

and therefore H is isolated in G if and only if

$$\psi_H(G)=|H|+\sum_{x\in G\setminus H}o(x)=|H|+\psi(G)-\psi(H).$$

Since for $H\lhd G$ we have $\psi_H(G)=|H|\psi(G/H)$, we infer that a normal subgroup H is isolated in G if and only if

$$\psi(G)-\psi(H)=|H|(\psi(G/H)-1). \tag{1.1}$$

In particular, this equivalence holds for all subgroups H of a finite abelian group G. It will be used in what follows, together with Theorem 1 of [10]:

Theorem 1.2. *Let $G=\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{p^{\alpha_2}}\times\ldots\times\mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p-group, where $1\leqslant\alpha_1\leqslant\alpha_2\leqslant\ldots\leqslant\alpha_k$. Then*

$$\psi(G)=1+\sum_{\alpha=1}^{\alpha_k}(p^{2\alpha}f_{(\alpha_1,\alpha_2,\ldots,\alpha_k)}(\alpha)-p^{2\alpha-1}f_{(\alpha_1,\alpha_2,\ldots,\alpha_k)}(\alpha-1)), \tag{1.2}$$

where

$$f_{(\alpha_1,\alpha_2,\ldots,\alpha_k)}(\alpha)=\begin{cases}p^{(k-1)\alpha} & \text{if } 0\leqslant\alpha\leqslant\alpha_1,\\ p^{(k-2)\alpha+\alpha_1} & \text{if } \alpha_1\leqslant\alpha\leqslant\alpha_2,\\ \vdots & \\ p^{\alpha_1+\alpha_2+\ldots+\alpha_{k-1}} & \text{if } \alpha_{k-1}\leqslant\alpha.\end{cases}$$

We note that (1.2) gives a formula for the sum of element orders of an arbitrary finite abelian group because the function ψ is multiplicative. Also, we note that $\psi(G)$ in Theorem 1.2 is a polynomial in p of degree $2\alpha_k + \alpha_{k-1} + \ldots + \alpha_1$. An alternative way of writing it is

$$\begin{aligned}(1.3)\qquad \psi(G) &= p^{2\alpha_k+\alpha_{k-1}+\ldots+\alpha_1} - (p-1)\sum_{\alpha=0}^{\alpha_k-1} p^{2\alpha} f_{(\alpha_1,\alpha_2,\ldots,\alpha_k)}(\alpha)\\ &= p^{2\alpha_k+\alpha_{k-1}+\ldots+\alpha_1} + \ldots + p^{k+1} - p + 1.\end{aligned}$$

Most of our notation is standard and is usually not introduced here. Elementary notions and results on groups can be found in [4], [7], [8].

2. Proofs of the main results

We start with the following lemma whose proof is elementary and thus omitted.

Lemma 2.1. *Let G be a finite abelian group and H be a subgroup of G. Write G and H as the direct products of their Sylow subgroups*

$$G = G_1 \times G_2 \times \ldots \times G_m \quad \text{and} \quad H = H_1 \times H_2 \times \ldots \times H_m,$$

respectively. Then H is isolated in G if and only if there are $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, m\}$ such that H_{i_j} is isolated in G_{i_j} for all $j = 1, 2, \ldots, k$ and $H_i = 1$ for all $i \neq i_1, i_2, \ldots, i_k$.

Lemma 2.1 shows that our study can be reduced to finite abelian p-groups via the description of the structure of finite abelian groups.

Lemma 2.2. *Let $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p-group, where $1 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_k$, and H be a maximal subgroup of G. If H is isolated in G, then G is elementary abelian and H is a direct factor of G.*

P r o o f. Let $n = \alpha_1 + \alpha_2 + \ldots + \alpha_k$. Since H is maximal and isolated in G, by the equality (1.1) it follows that

$$\psi(G) - \psi(H) = p^{n-1}(\psi(\mathbb{Z}_p) - 1) = p^{n+1} - p^n$$

and so $\psi(G)$ is a polynomial in p of degree $n+1$. On the other hand, by (1.3) we know that $\psi(G)$ is a polynomial in p of degree $2\alpha_k + \alpha_{k-1} + \ldots + \alpha_1 = n + \alpha_k$. Thus $n + \alpha_k = n + 1$, that is $\alpha_k = 1$, implying that G is elementary abelian. The second conclusion is obvious. □

From Lemma 2.2 we infer that if H is a proper isolated subgroup of a finite abelian p-group G, then

$$H \subset \Omega_1(G) = \{x \in G\colon\ x^p = 1\} \tag{2.1}$$

and, in particular, H is p-elementary abelian.

Indeed, take a subgroup K of G such that H is maximal in K. Then H is isolated in K and Lemma 2.2 shows that K must be elementary abelian, i.e., $K \subseteq \Omega_1(G)$. Hence, H is strictly contained in $\Omega_1(G)$.

Lemma 2.3. *Let $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p-group with $1 < \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_k$. Then G has no isolated proper subgroup.*

Proof. Assume that H is an isolated proper subgroup of G and let $|H| = p^m$. Then

$$\psi(G) - \psi(H) = p^m(\psi(G/H) - 1). \tag{2.2}$$

By (2.1) we know that H is elementary abelian and $m < k$. Then $\psi(H) = p^{m+1} - p + 1$ and therefore the left side of (2.2) becomes

$$\psi(G) - \psi(H) = p^{2\alpha_k + \alpha_{k-1} + \ldots + \alpha_1} + \ldots + p^{k+1} - p^{m+1}.$$

On the other hand, (2.1) shows that G/H has also k direct factors

$$G/H = \mathbb{Z}_{p^{\beta_1}} \times \mathbb{Z}_{p^{\beta_2}} \times \ldots \times \mathbb{Z}_{p^{\beta_k}},$$

where either $\beta_i = \alpha_i$ or $\beta_i = \alpha_i - 1$ for all $i = 1, 2, \ldots, k$. Thus, the right side of (2.2) becomes

$$\begin{aligned} p^m(\psi(G/H) - 1) &= p^m(p^{2\beta_k + \beta_{k-1} + \ldots + \beta_1} + \ldots + p^{k+1} - p) \\ &= p^{m + 2\beta_k + \beta_{k-1} + \ldots + \beta_1} + \ldots + p^{m+k+1} - p^{m+1}. \end{aligned}$$

Hence, (2.2) leads to $p^{k+1} = p^{m+k+1}$, i.e., $m = 0$, a contradiction. □

In the following, let $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p-group, where $1 = \alpha_1 = \alpha_2 = \ldots = \alpha_r < \alpha_{r+1} \leqslant \ldots \leqslant \alpha_k$, and H be a subgroup of order p of G. Then H is isolated in G if and only if $\langle x \rangle \cap H = 1$ for all $x \in G \setminus H$, that is if and only if H is contained in no cyclic subgroup of order p^s with $s \geqslant 2$ of G. This is equivalent with the fact that H is not contained in $A = \mathbb{Z}_{p^{\alpha_{r+1}}} \times \ldots \times \mathbb{Z}_{p^{\alpha_k}}$.

The above remark can be easily extended to arbitrary proper subgroups H of G, namely H is isolated in G if and only if H contains no subgroup of order p of A.

Indeed, if H is not isolated in G, then there is $x \in G \setminus H$ such that $\langle x \rangle \cap H \neq 1$. Take a subgroup K of order p of $\langle x \rangle \cap H$. Then K is also not isolated in G and thus $K \subseteq A$. Consequently, H contains a subgroup of order p of A, a contradiction. The converse is obvious.

Hence, we proved the next lemma.

Lemma 2.4. *Let $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \ldots \times \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p-group, where $1 = \alpha_1 = \alpha_2 = \ldots = \alpha_r < \alpha_{r+1} \leqslant \ldots \leqslant \alpha_k$, and let $A = \mathbb{Z}_{p^{\alpha_{r+1}}} \times \ldots \times \mathbb{Z}_{p^{\alpha_k}}$. Then a proper subgroup H of G is isolated in G if and only if $H \cap A = 1$.*

In particular, Lemma 2.4 shows that all subgroups of an elementary abelian p-group are isolated.

We are now able to prove our main result.

P r o o f of Theorem 1.1. It follows from Lemmas 2.3 and 2.4. □

Finally, we mention that the computation of isolated subgroups of finite abelian p-groups can be done by using well-known Goursat's lemma (see, e.g., the result (4.19) of [8]). We exemplify it in three particular cases:

Example 2.1.

(1) The group $\mathbb{Z}_p \times \mathbb{Z}_{p^m}$ with $m \geqslant 2$ has $p+2$ isolated subgroups, namely 1, G and p subgroups of order p.

(2) The group $\mathbb{Z}_p \times \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ with $m, n \geqslant 2$ has p^2+2 isolated subgroups, namely 1, G and p^2 subgroups of order p.

(3) The group $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^m}$ with $m \geqslant 2$ has $2p^2 + p + 2$ isolated subgroups, namely 1, G, $p^2 + p$ subgroups of order p and p^2 subgroups of order p^2.

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