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NEW EINSTEIN METRICS ON Sp(n) WHICH ARE NON-NATURALLY REDUCTIVE

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Abstract. We prove that there are at least two new non-naturally reductive $\mathrm{Ad}(\mathrm{Sp}(l)\times \mathrm{Sp}(k)\times\mathrm{Sp}(k)\times\mathrm{Sp}(k))$ invariant Einstein metrics on $\mathrm{Sp}(l+3k)$ $(k<l)$. It implies that every compact simple Lie group $\mathrm{Sp}(n)$ for $n=l+3k>4$ admits at least $2[\frac{1}{4}(n-1)]$ non-naturally reductive $\mathrm{Ad}(\mathrm{Sp}(l)\times\mathrm{Sp}(k)\times\mathrm{Sp}(k)\times\mathrm{Sp}(k))$ invariant Einstein metrics.

Keywords: Einstein metric; non-naturally reductive metric; compact Lie group; symplectic group

MSC 2020: 53C25, 53C30, 65H10

1. Introduction

A Riemannian manifold (M,g) is called *Einstein* if there is a constant ϱ such that Ricci tensor satisfies $\mathrm{Ric}(g)=\varrho g$. Einstein manifolds play an important role in Riemannian geometry and general relativity; see the survey book [4] and the articles [13], [14]. In particular, it has been an interesting and important problem to study left-invariant Einstein metrics on compact Lie groups.

As is well known, any bi-invariant Riemannian metric on a compact simple Lie group must be Einstein, which are globally symmetric. As a generalization of symmetric metrics, naturally reductive metrics play important roles in many topics. In 1979, D'Atri and Ziller in [10] established many important results on naturally reductive Einstein metrics on the classical compact groups $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$; for example, they pointed that every compact simple Lie group except $\mathrm{SO}(3)$ admits at least two left-invariant naturally reductive Einstein metrics. They also found that

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it is difficult to obtain left-invariant Einstein metrics on compact simple Lie groups, which are non-naturally reductive. From then on, many results have been established on left-invariant non-naturally reductive Einstein metrics on compact simple Lie groups. Mori initiated the study of this problem by showing that there exist non-naturally reductive left-invariant Einstein metrics on $\mathrm{SU}(n)$ for $n \geqslant 6$ in [11]. Later, the authors in [2] got the existence of such metrics on the compact Lie groups $\mathrm{SO}(n)$ for $n \geqslant 11$, $\mathrm{Sp}(n)$ for $n \geqslant 3$, E_6, E_7 and E_8. Chen and Liang in [8], Chrysikos and Sakane in [9], and H. Chen, Z. Chen and Deng in [5] also obtained some new non-naturally reductive Einstein metrics on exceptionally simple Lie groups.

Recently, many researchers are studying the problem of lower bounds of the number for non-naturally reductive Einstein metrics on compact simple Lie groups. Yan and Deng in [16] found a new method to construct left-invariant non-naturally reductive Einstein metrics on compact simple Lie groups through the study of standard triples. They got lower bounds of the number for compact simple Lie groups $\mathrm{SO}(2n)$ and $\mathrm{Sp}(2n)$, i.e., $\mathrm{SO}(2n)$ admits at least $(l_1+1)(l_2+1)\ldots(l_s+1)-3$ non-naturally reductive Einstein metrics and $\mathrm{Sp}(2n)$ admits at least $(l_1+1)(l_2+1)\ldots(l_s+1)-1$ non-naturally reductive Einstein metrics for any $n = {p_1}^{l_1}{p_2}^{l_2}\ldots{p_s}^{l_s}$ with p_i prime. Later on, H. Chen, Z. Chen and Deng proved that $\mathrm{SO}(n)$ admits at least $2([\frac{1}{3}(n-1)]-2)$ left-invariant non-naturally reductive Einstein metrics in [6]. Furthermore, the authors in [7] gave lower bounds on the number of $\mathrm{Sp}(n)$ $(n \geqslant 4)$. In this paper, we generalize the results in [17] and prove the following theorem.

Theorem 1.1. *For $n = 3k + l > 4$ $(k < l)$, the compact simple Lie group $\mathrm{Sp}(n)$ admits at least two new $\mathrm{Ad}(\mathrm{Sp}(l)\times\mathrm{Sp}(k)\times\mathrm{Sp}(k)\times\mathrm{Sp}(k))$-invariant Einstein metrics, which are non-naturally reductive. Moreover, every compact simple Lie group $\mathrm{Sp}(n)$ admits at least $2[\frac{1}{4}(n-1)]$ non-naturally reductive $\mathrm{Ad}(\mathrm{Sp}(l)\times\mathrm{Sp}(k)\times\mathrm{Sp}(k)\times\mathrm{Sp}(k))$ invariant Einstein metrics.*

The paper is organized as follows. In Section 2, we recall some fundamental results about the Ricci tensors and naturally reductive metrics on compact Lie groups. In Section 3, we calculate the Ricci tensors and naturally reductive metrics for Lie group $\mathrm{Sp}(n)$ in detail. Finally in Section 4, we finish the proof of Theorem 1.1.

2. The Ricci tensor and naturally reductive metrics on reductive homogeneous spaces

Let G be a compact simple Lie group, K a connected closed subgroup of G with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, respectively. The Killing form B of $\mathfrak{g}$ is negative definite, so we can define an $\mathrm{Ad}(G)$-invariant inner product $-B$ on $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a reductive decomposition with respect to $-B$. Then $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $\mathfrak{m} \cong T_o(G/K)$. We assume that $\mathfrak{m}$ can be decomposed into mutually non-equivalent irreducible $\mathrm{Ad}(K)$-modules as follows:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_q. \tag{2.1}$$

Then any G-invariant Riemannian metric on G/K can be expressed as

$$\langle \cdot, \cdot \rangle = x_1(-B)\big|_{\mathfrak{m}_1} + \ldots + x_q(-B)\big|_{\mathfrak{m}_q}, \tag{2.2}$$

where $x_1, \ldots, x_q$ are positive real numbers.

The Ricci tensor r of a G-invariant Riemannian metric on G/K is of the same form as in (2.2), that is, there exist real numbers $y_1, \ldots, y_q$ such that

$$r = y_1(-B)\big|_{\mathfrak{m}_1} + \ldots + y_q(-B)\big|_{\mathfrak{m}_q}. \tag{2.3}$$

Let $\{e_\alpha\}$ be a $(-B)$-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, i.e., $e_\alpha \in m_i$ for some i, and $\alpha < \beta$ if $i < j$. We put $A^\gamma_{\alpha\beta} = -B([e_\alpha, e_\beta], e_\gamma)$, that is $[e_\alpha, e_\beta] = \sum\limits_\gamma A^\gamma_{\alpha\beta} e_\gamma$. Set $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i$, $e_\beta \in \mathfrak{m}_j$, $e_\gamma \in \mathfrak{m}_k$, see [15].

Let $d_i = \dim \mathfrak{m}_i$ with respect to the metric given as above. We define $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum\limits_{ijk} \langle [e^i_\alpha, e^j_\beta], e^k_\gamma \rangle^2$, where i, j, k vary from 1 to d_α, d_β, $d\gamma$, respectively. Then it is easily seen that the positive numbers $\begin{bmatrix} k \\ ij \end{bmatrix}$ are independent of the B-orthonormal bases chosen for $\mathfrak{m}_i$, $\mathfrak{m}_j$, $\mathfrak{m}_k$, and we have $\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}$.

Then we have the following:

Lemma 2.1 ([12])**.** *The components $r_1, \ldots, r_q$ of the Ricci tensor r of the metric of the form (2.2) on G/K are given by*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \begin{bmatrix} k \\ ij \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 1, \ldots, q), \tag{2.4}$$

where the sum is taken over $i, j = 1, \ldots, q$, $r_k = r(X^{(k)}_l, X^{(k)}_l)$, and $\{X^{(k)}_j\}^{d_k}_{j=1}$ is an $\langle \cdot, \cdot \rangle$-orthonormal basis of $\mathfrak{m}_k$.

As is well known, there is a one-to-one correspondence between G-invariant metrics on G/H and $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{m}$.

Definition 2.1. A Riemannian homogeneous space $(M = G/K, \langle\cdot,\cdot\rangle)$ with a reductive complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$ is called *naturally reductive* if

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle = 0 \quad \forall\, X,Y,Z \in \mathfrak{m}.$$

Under the notions above, let $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_p$ be a decomposition of $\mathfrak{k}$ into ideals, where $\mathfrak{k}_0$ is the center of $\mathfrak{k}$ and $\mathfrak{k}_i$ $(i = 1,\ldots,p)$ are simple ideals of $\mathfrak{k}$. Let $A_0\big|_{\mathfrak{k}_0}$ be an arbitrary metric on $\mathfrak{k}_0$. In [10], D'Atri and Ziller investigated naturally reductive metrics among left-invariant metrics on compact Lie groups. And they gave the following useful result.

Theorem 2.1 ([10])**.** *A left-invariant metric on G of the form*

$$\langle\cdot,\cdot\rangle = x(-B)\big|_{\mathfrak{m}} + A_0\big|_{\mathfrak{k}_0} + u_1(-B)\big|_{\mathfrak{k}_1} + \ldots + u_p(-B)\big|_{\mathfrak{k}_p}, \quad x, u_1, \ldots, u_p > 0 \tag{2.5}$$

is naturally reductive with respect to $G \times K$, where $G \times K$ acts on G by

$$(g,k)y = gyk^{-1}.$$

Moreover, if a left-invariant metric $\langle\cdot,\cdot\rangle$ on a compact simple Lie group G is naturally reductive, then there is a closed subgroup K of G and the metric $\langle\cdot,\cdot\rangle$ is given by the form (2.5)*.*

3. The compact simple Lie groups $\mathrm{Sp}(n)$

In this section, we describe a decomposition of the Lie algebra of $\mathrm{Sp}(n)$. Then we obtain the Ricci tensors and the naturally reductive metrics.

Let $G = \mathrm{Sp}(n) = \mathrm{Sp}(k_1+k_2+k_3+k_4)$ and $K = \mathrm{Sp}(k_1)\times\mathrm{Sp}(k_2)\times\mathrm{Sp}(k_3)\times\mathrm{Sp}(k_4)$, where $k_1,k_2,k_3,k_4 \geqslant 1$. Obviously, K is the closed subgroup of G. Then the tangent space $\mathfrak{sp}(k_1+k_2+k_3+k_4)$ of the symplectic group $G = \mathrm{Sp}(k_1+k_2+k_3+k_4)$ can be written as a direct sum of $\mathrm{Ad}(K)$-invariant modules as

$$\mathfrak{sp}(k_1+k_2+k_3+k_4) = \mathfrak{sp}(k_1)\oplus\mathfrak{sp}(k_2)\oplus\mathfrak{sp}(k_3)\oplus\mathfrak{sp}(k_4)\oplus\mathfrak{m}, \tag{3.1}$$

where $\mathfrak{m}$ corresponds to the tangent space of G/K.

For $i = 1,2,3,4$, we embed the Lie subalgebra

$$\mathfrak{sp}(k_i) = \left\{ \begin{pmatrix} X_i & -\overline{Y}_i \\ Y_i & \overline{X}_i \end{pmatrix} : \ X_i \in \mathfrak{u}(k_i),\ Y_i \in \mathbb{C}^{k_i\times k_i},\ Y_i' = Y_i \right\}$$

into the Lie algebra $\mathfrak{sp}(k_1+k_2+k_3+k_4)$ as

$$\mathfrak{m}_1 \triangleq \left\{ \left(\begin{array}{cccc|cccc} X_1 & 0 & 0 & 0 & -\overline{Y}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline Y_1 & 0 & 0 & 0 & \overline{X}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\}, \ \mathfrak{m}_2 \triangleq \left\{ \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_2 & 0 & 0 & 0 & -\overline{Y}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_2 & 0 & 0 & 0 & \overline{X}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\},$$

$$\mathfrak{m}_3 \triangleq \left\{ \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_3 & 0 & 0 & 0 & -\overline{Y}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Y_3 & 0 & 0 & 0 & \overline{X}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\}, \ \mathfrak{m}_4 \triangleq \left\{ \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_4 & 0 & 0 & 0 & -\overline{Y}_4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_4 & 0 & 0 & 0 & \overline{X}_4 \end{array} \right) \right\}.$$

The tangent space $\mathfrak{m}$ of G/K at the origin is given by $\mathfrak{k}^\perp$ in $\mathfrak{g} = \mathfrak{sp}(k_1+k_2+k_3+k_4)$ with respect to the $\mathrm{Ad}(G)$-invariant inner product $-B$. We denote by $M(p,q)$ the set of all $p \times q$ matrices. Then we set

$$\mathfrak{m}_{12} = \left\{ \left(\begin{array}{cccc|cccc} 0 & A_{12} & 0 & 0 & 0 & -\overline{B}_{12} & 0 & 0 \\ -\bar{A}'_{12} & 0 & 0 & 0 & -\overline{B}'_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & B_{12} & 0 & 0 & 0 & \bar{A}_{12} & 0 & 0 \\ B'_{12} & 0 & 0 & 0 & -A'_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\},$$

$$\mathfrak{m}_{13} = \left\{ \left(\begin{array}{cccc|cccc} 0 & 0 & A_{13} & 0 & 0 & 0 & -\overline{B}_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\bar{A}'_{13} & 0 & 0 & 0 & -\overline{B}'_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & B_{13} & 0 & 0 & 0 & \bar{A}_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B'_{13} & 0 & 0 & 0 & -A'_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\},$$

$$\mathfrak{m}_{14} = \left\{ \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & A_{14} & 0 & 0 & 0 & -\overline{B}_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\bar{A}'_{14} & 0 & 0 & 0 & -\overline{B}'_{14} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & B_{14} & 0 & 0 & 0 & \bar{A}_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B'_{14} & 0 & 0 & 0 & -A'_{14} & 0 & 0 & 0 \end{array} \right) \right\},$$

$$\mathfrak{m}_{23} = \left\{ \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{23} & 0 & 0 & 0 & -\overline{B}_{23} & 0 \\ 0 & -\bar{A}'_{23} & 0 & 0 & 0 & -\overline{B}'_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{23} & 0 & 0 & 0 & \bar{A}_{23} & 0 \\ 0 & B'_{23} & 0 & 0 & 0 & -A'_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \right\},$$

$$\mathfrak{m}_{24} = \left\{ \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{24} & 0 & 0 & 0 & -\overline{B}_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\bar{A}'_{24} & 0 & 0 & 0 & -\overline{B}'_{24} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{24} & 0 & 0 & 0 & \bar{A}_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B'_{24} & 0 & 0 & 0 & -A'_{24} & 0 & 0 \end{array} \right) \right\},$$

$$\mathfrak{m}_{34} = \left\{ \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{34} & 0 & 0 & 0 & -\overline{B}_{34} \\ 0 & 0 & -\bar{A}'_{34} & 0 & 0 & 0 & -\overline{B}'_{34} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{34} & 0 & 0 & 0 & \bar{A}_{34} \\ 0 & 0 & B'_{34} & 0 & 0 & 0 & -A'_{34} & 0 \end{array} \right) \right\},$$

where

$$A_{ij}, B_{ij} \in M(k_i, k_j), \quad 1 \leqslant i < j \leqslant 4.$$

The subspaces $\mathfrak{m}_{ij}$ are given as the corresponding orthogonal complements with respect to the negative of the Killing form. Note that the subspaces $\mathfrak{m}_{ij}$ are the irreducible $\mathrm{Ad}(K)$-submodules and that the irreducible submodules $\mathfrak{m}_{ij}$ are mutually non-equivalent.

Therefore, decomposition (3.1) of the Lie algebra of the symplectic group $G = \mathrm{Sp}(k_1 + k_2 + k_3 + k_4)$ is

$$\mathfrak{g} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 + \mathfrak{m}_4 + \mathfrak{m}_{12} + \mathfrak{m}_{13} + \mathfrak{m}_{14} + \mathfrak{m}_{23} + \mathfrak{m}_{24} + \mathfrak{m}_{34}, \tag{3.2}$$

where we denote by $\mathfrak{m}_j$ the subalgebras $\mathfrak{sp}(k_j)$, $1 \leqslant j \leqslant 4$.

Now we consider left-invariant metrics on G which are determined by the $\mathrm{Ad}(\mathrm{Sp}(k_1)\times\mathrm{Sp}(k_2)\times\mathrm{Sp}(k_3)\times\mathrm{Sp}(k_4))$-invariant scalar products on $\mathfrak{sp}(k_1+k_2+k_3+k_4)$ given by

$$\langle\cdot,\cdot\rangle = \sum_{i=1}^{4} x_i(-B)\big|_{\mathfrak{sp}(k_i)} + \sum_{i=2}^{4} + x_{1i}(-B)\big|_{\mathfrak{m}_{1i}} + \sum_{i=3}^{4} x_{2i}(-B)\big|_{\mathfrak{m}_{2i}} + x_{34}(-B)\big|_{\mathfrak{m}_{34}}, \tag{3.3}$$

where x_i $(i = 1, 2, 3, 4)$ and x_{ij} $(1 \neq i < j \neq 4)$ are all positive real numbers. Then we have the following lemma.

Lemma 3.1. *The submodules in decomposition* (3.2) *satisfy the following bracket relations:*

$$\begin{array}{llll} [\mathfrak{m}_i, \mathfrak{m}_i] = \mathfrak{m}_i, & [\mathfrak{m}_i, \mathfrak{m}_{ij}] = \mathfrak{m}_{ij}, & [\mathfrak{m}_j, \mathfrak{m}_{ij}] = \mathfrak{m}_{ij}, & [\mathfrak{m}_{ij}, \mathfrak{m}_{jk}] = \mathfrak{m}_{ik}, \\ [\mathfrak{m}_{ij}, \mathfrak{m}_{ik}] = \mathfrak{m}_{jk}, & [\mathfrak{m}_{ik}, \mathfrak{m}_{jk}] = \mathfrak{m}_{ij}, & [\mathfrak{m}_{ij}, \mathfrak{m}_{ij}] = \mathfrak{m}_i + \mathfrak{m}_j, & \end{array}$$

where $1 \leqslant i < j < k \leqslant 4$ *and all the other pairs of subspaces not appearing in the above list are all multiplication commutative.*

The proof is direct and we omit it.

It follows that the only non zero symbols (up to permutations of indices) are

$$\begin{bmatrix} i \\ ii \end{bmatrix}, \quad \begin{bmatrix} (ij) \\ i(ij) \end{bmatrix}, \quad \begin{bmatrix} (ij) \\ j(ij) \end{bmatrix}, \quad \begin{bmatrix} (ik) \\ (ij)(jk) \end{bmatrix}, \quad \begin{bmatrix} (jk) \\ (ij)(ik) \end{bmatrix}, \quad \begin{bmatrix} (ij) \\ (ik)(jk) \end{bmatrix}. \tag{3.4}$$

Denote by d_i and d_{ij} the dimensions of the modules $\mathfrak{m}_i$ and $\mathfrak{m}_{ij}$, respectively. Since $\dim \mathfrak{m}_{ij} = 4k_ik_j$ $(1 \leqslant i < j \leqslant 4)$, it is easily seen that $d_i = 2k_i^2 + k_i$, $d_{ij} = 4k_ik_j$.

Then we consider $\mathrm{Ad}(\mathrm{Sp}(k_1) \times \mathrm{Sp}(k_2) \times \mathrm{Sp}(k_3) \times \mathrm{Sp}(k_4))$-invariant metrics of the form (3.3). We first prove the following proposition.

Proposition 3.1. *If a left invariant metric* $\langle\cdot,\cdot\rangle$ *of the form* (3.3) *on* $\mathrm{Sp}(n)$ *is naturally reductive with respect to* $\mathrm{Sp}(n) \times L$ *for some closed subgroup* L *of* $\mathrm{Sp}(n)$*, then one of the following holds:*

1) $x_1 = x_2 = x_{12}$, $x_{13} = x_{14} = x_{23} = x_{24} = x_{34}$;
2) $x_1 = x_2 = x_{12}$, $x_3 = x_4 = x_{34}$, $x_{13} = x_{14} = x_{23} = x_{24}$;
3) $x_1 = x_2 = x_3 = x_{12} = x_{13} = x_{23}$, $x_{14} = x_{24} = x_{34}$;
4) $x_1 = x_2 = x_4 = x_{12} = x_{14} = x_{24}$, $x_{13} = x_{23} = x_{34}$;
5) $x_1 = x_3 = x_{13}$, $x_{12} = x_{14} = x_{23} = x_{24} = x_{34}$;
6) $x_1 = x_3 = x_{13}$, $x_2 = x_4 = x_{24}$, $x_{12} = x_{14} = x_{23} = x_{34}$;
7) $x_1 = x_3 = x_4 = x_{13} = x_{14} = x_{34}$, $x_{12} = x_{23} = x_{24}$;
8) $x_1 = x_4 = x_{14}$, $x_{12} = x_{13} = x_{23} = x_{24} = x_{34}$;
9) $x_1 = x_4 = x_{14}$, $x_2 = x_3 = x_{23}$, $x_{12} = x_{13} = x_{24} = x_{34}$;
10) $x_2 = x_3 = x_{23}$, $x_{12} = x_{13} = x_{14} = x_{24} = x_{34}$;
11) $x_2 = x_3 = x_{23}$, $x_1 = x_4 = x_{14}$, $x_{12} = x_{24} = x_{13} = x_{34}$;
12) $x_2 = x_3 = x_4 = x_{23} = x_{24} = x_{34}$, $x_{12} = x_{13} = x_{14}$;
13) $x_2 = x_4 = x_{24}$, $x_{12} = x_{14} = x_{23} = x_{13} = x_{34}$;
14) $x_3 = x_4 = x_{34}$, $x_{12} = x_{13} = x_{14} = x_{23} = x_{24}$;
15) $x_{12} = x_{13} = x_{14} = x_{23} = x_{24} = x_{34}$.

Conversely, if one of the above conditions is satisfied, then the metric $\langle \cdot, \cdot \rangle$ *of the form* (3.3) *is naturally reductive with respect to* $\mathrm{Sp}(n) \times L$ *for some closed subgroup* L *of* $\mathrm{Sp}(n)$.

P r o o f. Let $\mathfrak{l}$ be the Lie algebra of L. Then we have either $\mathfrak{l} \subset \mathfrak{k}$ or $\mathfrak{l} \not\subseteq \mathfrak{k}$. First we consider the case $\mathfrak{l} \not\subseteq \mathfrak{k}$. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{l}$ and $\mathfrak{k}$. Since $\mathfrak{g}$ can be decomposed into the sum of mutually non-equivalent irreducible $\mathrm{Ad}(K)$-modules as (3.2), it is easily seen that $\mathfrak{h}$ contains at least one of $\mathfrak{m}_{12}$, $\mathfrak{m}_{13}$, $\mathfrak{m}_{14}$, $\mathfrak{m}_{23}$, $\mathfrak{m}_{24}$, or $\mathfrak{m}_{34}$.

We first consider the case when $\mathfrak{h}$ contains $\mathfrak{m}_{12}$. Note that $[\mathfrak{m}_{12}, \mathfrak{m}_{12}] = \mathfrak{m}_1 + \mathfrak{m}_2$. Thus, $\mathfrak{h}$ contains $\mathfrak{sp}(k_1+k_2) \oplus \mathfrak{sp}(k_3) \oplus \mathfrak{sp}(k_4)$, $\mathfrak{sp}(k_1+k_2) \oplus \mathfrak{sp}(k_3+k_4)$, $\mathfrak{sp}(k_1+k_2+k_3) \oplus \mathfrak{sp}(k_4)$ or $\mathfrak{sp}(k_1+k_2+k_4) \oplus sp(k_3)$. If $\mathfrak{h} = \mathfrak{sp}(k_1+k_2) \oplus \mathfrak{sp}(k_3) \oplus \mathfrak{sp}(k_4)$, then we have an irreducible decomposition $\mathfrak{sp}(k_1+k_2+k_3+k_4) = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n} = \mathfrak{m}_{13} \oplus \mathfrak{m}_{14} \oplus \mathfrak{m}_{23} \oplus \mathfrak{m}_{24} \oplus \mathfrak{m}_{34}$. Notice that the terms of composition (2.5) are $\mathrm{Ad}(L)$-modules. If the metric of the form (3.3) is naturally reductive, then it must satisfy $x_1 = x_2 = x_{12}$ and $x_{13} = x_{14} = x_{23} = x_{24} = x_{34}$. The other conclusions can be proved similarly.

Next we consider the case when $\mathfrak{l} \subseteq \mathfrak{k}$. In this case we have $\mathfrak{l}^{\perp} \supset \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{14} \oplus \mathfrak{m}_{23} \oplus \mathfrak{m}_{24} \oplus \mathfrak{m}_{34}$. It follows that $x_{12} = x_{13} = x_{14} = x_{23} = x_{24} = x_{34}$. The converse statement is a direct consequence of Theorem 2.1. □

Now we calculate the Ricci tensor of the left-invariant metrics on $\mathrm{Sp}(n) = \mathrm{Sp}(k_1+k_2+k_3+k_4)$, determined by the $\mathrm{Ad}(K) = \mathrm{Ad}(\mathrm{Sp}(k_1) \times \mathrm{Sp}(k_2) \times \mathrm{Sp}(k_3) \times \mathrm{Sp}(k_4))$-invariant scalar products of the form (3.3). Note that the Ricci tensor r of the metric (3.3) is also $\mathrm{Ad}(K)$-invariant. Write

$$\mathfrak{g} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 + \mathfrak{m}_4 + \mathfrak{m}_5 + \mathfrak{m}_6 + \mathfrak{m}_7 + \mathfrak{m}_8 + \mathfrak{m}_9 + \mathfrak{m}_{10},$$

where $\mathfrak{m}_5 = \mathfrak{m}_{12}$, $\mathfrak{m}_6 = \mathfrak{m}_{13}$, $\mathfrak{m}_7 = \mathfrak{m}_{14}$, $\mathfrak{m}_8 = \mathfrak{m}_{23}$, $\mathfrak{m}_9 = \mathfrak{m}_{24}$ and $\mathfrak{m}_{10} = \mathfrak{m}_{34}$. Then by Lemma 3.1, it is easy to see the following lemma.

Lemma 3.2. *For an* $\mathrm{Ad}(K)$*-invariant symmetric* 2*-tensor* ϱ *on* $\mathfrak{sp}(k_1+k_2+k_3+k_4)$ *we have* $\varrho(\mathfrak{m}_i, \mathfrak{m}_j) = (0)$ *for* $i \neq j$. *In particular, for the Ricci tensor* r *of the metric* (3.3) *we have* $r(\mathfrak{m}_i, \mathfrak{m}_j) = (0)$ *for* $i \neq j$.

Using Lemma 2.1 and (3.4), we get the following:

Proposition 3.2. *The components of the Ricci tensor* r *for the left-invariant metric* $\langle , \rangle$ *on* $\mathrm{Sp}(n)$ *defined by* (3.3) *are given as follows:*

$$
\begin{aligned}
r_i &= \frac{1}{2x_i} + \frac{1}{4d_i}\left(\begin{bmatrix} i \\ ii \end{bmatrix}\frac{1}{x_i} + \sum_{j=1, j\neq i}^{4}\begin{bmatrix} i \\ (ij)(ij) \end{bmatrix}\frac{x_i}{x_{ij}^2}\right) \\
&\quad - \frac{1}{2d_i}\left(\begin{bmatrix} i \\ ii \end{bmatrix}\frac{1}{x_i} + \sum_{j=1, j\neq i}^{4}\begin{bmatrix} (ij) \\ i(ij) \end{bmatrix}\frac{1}{x_i}\right), \\
r_{ij} &= \frac{1}{2x_{ij}} + \frac{1}{2d_{ij}}\left(\begin{bmatrix} (ij) \\ i(ij) \end{bmatrix}\frac{1}{x_i} + \begin{bmatrix} (ij) \\ j(ij) \end{bmatrix}\frac{1}{x_j} + \sum_{k=1, k\neq i,j}^{4}\begin{bmatrix} (ij) \\ (ik)(jk) \end{bmatrix}\frac{x_{ij}}{x_{ik}x_{jk}}\right. \\
&\quad - \begin{bmatrix} (ij) \\ (ij)i \end{bmatrix}\frac{1}{x_i} - \begin{bmatrix} (ij) \\ (ij)j \end{bmatrix}\frac{1}{x_j} - \sum_{k=1, k\neq i,j}^{4}\begin{bmatrix} (ik) \\ (ij)(jk) \end{bmatrix}\frac{x_{ik}}{x_{ij}x_{jk}} \\
&\quad \left. - \sum_{k=1, k\neq i,j}^{4}\begin{bmatrix} (jk) \\ (ij)(ik) \end{bmatrix}\frac{x_{jk}}{x_{ij}x_{ik}} - \begin{bmatrix} i \\ (ij)(ij) \end{bmatrix}\frac{x_i}{x_{ij}^2} - \begin{bmatrix} j \\ (ij)(ij) \end{bmatrix}\frac{x_j}{x_{ij}^2}\right),
\end{aligned}
$$

where $i, j = 1, 2, 3, 4$ *and* $i \neq j$.

We recall the following result by Arvanitoyeorgos, Dzhepko and Nikonorov:

Lemma 3.3 ([1], [3])**.** *The following relations hold:*

$$
\begin{bmatrix} a \\ aa \end{bmatrix} = \frac{k_a(k_a+1)(2k_a+1)}{n+1}, \quad \begin{bmatrix} a \\ (ab)(ab) \end{bmatrix} = \frac{k_a k_b(2k_a+1)}{n+1},
$$
$$
\begin{bmatrix} b \\ (ab)(ab) \end{bmatrix} = \frac{k_a k_b(2k_b+1)}{n+1}, \quad \begin{bmatrix} (ab) \\ (bc)(ac) \end{bmatrix} = \frac{2k_a k_b k_c}{n+1},
$$

where $a, b, c = 1, 2, 3$ *or* 4 *and* $(a-b)(b-c)(c-a) \neq 0$.

Combining the above results we get the components of the Ricci tensor for metrics (3.3).

Proposition 3.3. *The components of the Ricci tensor r for the left-invariant metric $\langle\cdot,\cdot\rangle$ on G defined by* (3.3) *are given as follows:*

$$r_i = \frac{k_i+1}{4(n+1)x_i} + \frac{1}{4(n+1)}\Big(\sum_{j=1,j\neq i}^{4} k_j \frac{x_i}{{x_{ij}}^2}\Big),$$

$$r_{ij} = \frac{1}{2x_{ij}} + \sum_{l=1,l\neq i,j}^{4} \frac{k_l}{4(n+1)}\Big(\frac{x_{ij}}{x_{il}x_{jl}} - \frac{x_{il}}{x_{ij}x_{jl}} - \frac{x_{jl}}{x_{ij}x_{il}}\Big)$$
$$- \frac{2k_i+1}{8(n+1)}\frac{x_i}{x_{ij}^2} - \frac{2k_j+1}{8(n+1)}\frac{x_j}{x_{ij}^2},$$

where $i, j = 1, 2, 3, 4$ and $i < j$.

4. Proof of Theorem 1.1

Consider the case of $k_1 = l$, $k_2 = k_3 = k_4 = k$ in the formula given in Proposition 3.3. By the definition of r_k, we only need to deal with the equations

$$r_1 = r_2 = r_3 = r_4 = r_{12} = r_{13} = r_{14} = r_{23} = r_{24} = r_{34}. \tag{4.1}$$

Setting the metric with

$$x_{12} = x_{13} = x_{14} = 1, \quad x_{24} = x_{34} = x_{23}, \quad x_2 = x_3 = x_4,$$

we can reduce equations (4.1) to

$$r_1 = r_2, \quad r_2 = r_{12}, \quad r_2 = r_{23}. \tag{4.2}$$

Then finding Einstein metrics of the form (3.3) reduces to finding positive solutions of the system of equations (4.2). It is easy to deduce the system of algebraic equations:

$$\begin{cases} f_1 = 3kx_1^2x_2x_{23}^2 - lx_1x_2^2x_{23}^2 - 2kx_1x_2^2 - kx_1x_{23}^2 + lx_2x_{23}^2 - x_1x_{23}^2 + x_2x_{23}^2 = 0, \\ f_2 = 2lx_1x_2x_{23}^2 + 2kx_2^2x_{23}^2 + 2lx_2^2x_{23}^2 + 4kx_2x_{23}^3 - 12kx_2x_{23}^2 - 4lx_2x_{23}^2 + 4kx_2^2 \\ \qquad + 2kx_{23}^2 + x_1x_2x_{23}^2 + x_2^2x_{23}^2 - 4x_2x_{23}^2 + 2x_{23}^2 = 0, \\ f_3 = lx_2^2x_{23}^2 - lx_2x_{23}^3 + 4kx_2^2 - 5kx_2x_{23} + kx_{23}^2 + x_2^2 - 2x_2x_{23} + x_{23}^2 = 0, \end{cases}$$

where $f_1 \triangleq r_1 - r_2$, $f_2 \triangleq r_2 - r_{12}$, $f_3 \triangleq r_2 - r_{23}$.

In particular,

$$f_3 = (x_2 - x_{23})(lx_2x_{23}^2 + 4kx_2 - kx_{23} + x_2 - x_{23}) = 0.$$

If $x_2 = x_{23}$, by Proposition 3.1 we have that it contradicts the non-naturally reductive Einstein metrics. So we assume $x_2 \neq x_{23}$. Together with $x_2, x_{23} \neq 0$, we get

$$x_2 = \frac{(k+1)x_{23}}{lx_{23}^2 + 4k + 1}. \tag{4.3}$$

Substituting (4.3) into f_1 and f_2, we have

$$\begin{aligned}
g_1 &= x_{23}^2(k+1)(3klx_1^2x_{23}^3 + 12k^2x_1^2x_{23} + 3kx_1^2x_{23} - l^2x_1x_{23}^4 - 9klx_1x_{23}^2 - 3lx_1x_{23}^2 \\
&\quad - 18k^2x_1 - 10kx_1 - x_1 + l^2x_{23}^3 + lx_{23}^3 + 4klx_{23} + 4kx_{23} + lx_{23} + x_{23}) \\
&= 3klx_1^2x_{23}^3 + 12k^2x_1^2x_{23} + 3kx_1^2x_{23} - l^2x_1x_{23}^4 - 9klx_1x_{23}^2 - 3lx_1x_{23}^2 - 18k^2x_1 \\
&\quad - 10kx_1 - x_1 + l^2x_{23}^3 + lx_{23}^3 + 4klx_{23} + 4kx_{23} + lx_{23} + x_{23} = 0, \\
g_2 &= x_{23}^2(k+1)(2l^2x_1x_{23}^3 + lx_1x_{23}^3 + 8klx_1x_{23} + 4kx_1x_{23} + 2lx_1x_{23} + x_1x_{23} + klx_{23}^4 \\
&\quad + 2l^2x_{23}^4 - 12klx_{23}^3 - 4l^2x_{23}^3 - 4lx_{23}^3 + 18k^2x_{23}^2 + 18klx_{23}^2 + 7kx_{23}^2 + 6lx_{23}^2 \\
&\quad + x_{23}^2 - 48k^2x_{23} - 16klx_{23} - 28kx_{23} - 4lx_{23} - 4x_{23} + 36k^2 + 20k + 2) \\
&= 2l^2x_1x_{23}^3 + lx_1x_{23}^3 + 8klx_1x_{23} + 4kx_1x_{23} + 2lx_1x_{23} + x_1x_{23} + klx_{23}^4 + 2l^2x_{23}^4 \\
&\quad - 12klx_{23}^3 - 4l^2x_{23}^3 - 4lx_{23}^3 + 18k^2x_{23}^2 + 18klx_{23}^2 + 7kx_{23}^2 + 6lx_{23}^2 + x_{23}^2 \\
&\quad - 48k^2x_{23} - 16klx_{23} - 28kx_{23} - 4lx_{23} - 4x_{23} + 36k^2 + 20k + 2 = 0.
\end{aligned}$$

Consider the polynomial ring $R = \mathbb{Q}[z, x_1, x_{23}]$ and the ideal I generated by $\{g_1, g_2, zx_1x_{23} - 1\}$, and take a lexicographic order '$>$' with $z > x_1 > x_{23}$ for a monomial ordering on R. We need to find positive solutions of the system. With the help of a computer, we obtain a Gröbner basis containing the polynomials $\{h(x_{23}), h_1(x_{23}, x_1)\}$, where $h(x_{23})$ is the polynomial of x_{23} given by

$$\begin{aligned}
h(x_{23}) = {} & 2l^2(2k+1)(12k^2 + 6kl + 2l^2 + l)x_{23}^8 \\
& - 4l^2(3k+l+1)(24k^2 + 12kl + 2l^2 + l)x_{23}^7 \\
& + l(432k^4 + 1080k^3l + 168k^3 + 612k^2l^2 + 570k^2l + 24k^2 \\
& + 120kl^3 + 242kl^2 + 79kl + 4l^4 + 32l^3 + 19l^2 + 2l)x_{23}^6 \\
& - 4l(3k+l+1) \\
& \times (204k^3 + 156k^2l + 66k^2 + 26kl^2 + 61kl + 6k + 8l^2 + 4l)x_{23}^5 \\
& + (972k^5 + 6264k^4l + 756k^4 + 4176k^3l^2 + 5286k^3l + 255k^3 \\
& + 852k^2l^3 + 2780k^2l^2 + 1525k^2l + 42k^2 + 32kl^4 + 456kl^3 \\
& + 580kl^2 + 174kl + 3k + 8l^4 + 60l^3 + 38l^2 + 5l)x_{23}^4 \\
& - 4(3k+l+1)(432k^4 + 648k^3l + 276k^3 + 108k^2l^2 + 426k^2l \\
& + 66k^2 + 62kl^2 + 79kl + 6k + 8l^2 + 4l)x_{23}^3
\end{aligned}$$

$$
\begin{aligned}
&+ (10800k^5 + 9144k^4l + 12060k^4 + 2064k^3l^2 + 8556k^3l \\
&+ 5082k^3 + 64k^2l^3 + 1664k^2l^2 + 2860k^2l + 998k^2 + 32kl^3 \\
&+ 424kl^2 + 398kl + 85k + 4l^3 + 32l^2 + 19l + 2)x_{23}^2 \\
&- 4(4k+1)(18k^2+10k+1)(3k+l+1)(12k+2l+1)x_{23} \\
&+ 2(18k^2+10k+1)^2(6k+2l+1).
\end{aligned}
$$

For every $k, l \in \mathbb{Z}^+$, if $l > k$, we have

$$
\begin{aligned}
&h(0) = 2(18k^2+10k+1)^2(6k+2l+1) > 0, \\
&h(1) = (k-l)(2k+1)(3k+l)^2(6k+2l+1) < 0, \\
&h(\infty) \to \infty.
\end{aligned}
$$

Thus, the equation $h(x_{23}) = 0$ has at least two positive roots α and β, where $0<\alpha<1$, $\beta > 1$. On the other hand, $h_2(x_{23}, x_1)$ is the polynomial of x_{23} and x_1 given by

$$
\begin{aligned}
h_2(x_{23}, x_1) = &- 2l^2(2k+l)(12k^2+6kl+2l^2+l) \\
&\times (6k^3 + 30k^2l + 9k^2 + 10kl^2 + 17kl + 3k + 4l^2 + 2l)x_{23}^7 \\
&+ 4l^2(3k+l+1)(24k^2+12kl+2l^2+l) \\
&\times (6k^3 + 30k^2l + 9k^2 + 10kl^2 + 17kl + 3k + 4l^2 + 2l)x_{23}^6 \\
&- l(2592k^7 + 14256k^6l + 4896k^6 + 33480k^5l^2 + 21780k^5l \\
&+ 2952k^5 + 25992k^4l^3 + 40692k^4l^2 + 12516k^4l + 720k^4 + 8592k^3l^4 \\
&+ 26076k^3l^3 + 18084k^3l^2 + 3129k^3l + 72k^3 + 1176k^2l^5 + 7120k^2l^4 \\
&+ 8974k^2l^3 + 3424k^2l^2 + 303k^2l + 40kl^6 + 788kl^5 + 1810kl^4 \\
&+ 1179kl^3 + 245kl^2 + 6kl + 16l^6 + 128l^5 + 132l^4 + 44l^3 + 4l^2)x_{23}^5 \\
&+ 4l(3k+l+1) \\
&\times (1224k^6 + 4464k^5l + 2232k^5 + 4716k^4l^2 + 5346k^4l \\
&+ 1242k^4 + 1692k^3l^3 + 4716k^3l^2 + 2367k^3l + 252k^3 \\
&+ 188k^2l^4 + 1484k^2l^3 + 1625k^2l^2 + 429k^2l + 18k^2 \\
&+ 144kl^4 + 396kl^3 + 210kl^2 + 24kl + 28l^4 + 28l^3 + 7l^2)x_{23}^4 \\
&- (5832k^8 + 40824k^7l + 13284k^7 + 118152k^6l^2 + 95616k^6l \\
&+ 11250k^6 + 103608k^5l^3 + 197748k^5l^2 + 82170k^5l + 4815k^5 \\
&+ 37128k^4l^4 + 142128k^4l^3 + 129222k^4l^2 + 33702k^4l + 1161k^4 \\
&+ 5304k^3l^5 + 43524k^3l^4 + 74790k^3l^3 + 41175k^3l^2 + 6981k^3l \\
&+ 153k^3 + 176k^2l^6 + 5352k^2l^5 + 18540k^2l^4 + 18442k^2l^3 \\
&+ 6528k^2l^2 + 678k^2l + 9k^2 + 128kl^6 + 1792kl^5 + 3336kl^4
\end{aligned}
$$

$$
\begin{aligned}
&+ 2056kl^3 + 452kl^2 + 21kl + 24l^6 + 196l^5 + 210l^4 + 75l^3 + 8l^2)x_{23}^3 \\
&+ 4(3k + l + 1)(2592k^7 + 5184k^6 l + 5544k^6 + 8856k^5 l^2 \\
&+ 11052k^5 l + 4176k^5 + 3888k^4 l^3 + 12672k^4 l^2 + 8136k^4 l \\
&+ 1458k^4 + 432k^3 l^4 + 4716k^3 l^3 + 6768k^3 l^2 + 2619k^3 l + 252k^3 \\
&+ 476k^2 l^4 + 2024k^2 l^3 + 1631k^2 l^2 + 369k^2 l + 18k^2 + 172kl^4 \\
&+ 352kl^3 + 169kl^2 + 18kl + 20l^4 + 20l^3 + 5l^2)x_{23}^2 \\
&- (53136k^8 + 71712k^7 l + 139104k^7 + 68112k^6 l^2 + 176400k^6 l \\
&+ 140508k^6 + 30624k^5 l^3 + 136848k^5 l^2 + 162984k^5 l + 71700k^5 \\
&+ 4848k^4 l^4 + 51888k^4 l^3 + 105276k^4 l^2 + 73812k^4 l + 19902k^4 \\
&+ 64k^3 l^5 + 7312k^3 l^4 + 32856k^3 l^3 + 39188k^3 l^2 + 17510k^3 l \\
&+ 2934k^3 + 112k^2 l^5 + 3896k^2 l^4 + 9544k^2 l^3 + 7302k^2 l^2 \\
&+ 2099k^2 l + 195k^2 + 56kl^5 + 844kl^4 + 1250kl^3 + 621kl^2 \\
&+ 106kl + 3k + 8l^5 + 60l^4 + 58l^3 + 17l^2 + l)x_{23} \\
&+ (k + 1)(2k + 1)(18k^2 + 10k + 1)(2l + 1)(6k + 2l + 1) \\
&\times (12k^2 + 6kl + 3k + 2l^2 + l)x_1 \\
&+ 24k(k + 1)(2k + 1)(18k^2 + 10k + 1)(3k + l + 1) \\
&\times (12k^2 + 6kl + 3k + 2l^2 + l).
\end{aligned}
$$

By the equation $h_2(x_{23}, x_1) = 0$, x_1 can be expressed by a polynomial of x_{23}. Since there exist at least two real roots α and β of $f(x_{23}) = 0$, there are at least two real solutions for $h_2(x_{23}, x_1) = 0$. Then we show that the real solutions x_1 of $h_2(x_{23}, x_1) = 0$ are positive. We take a lexicographic order '$>$' with $z > x_{23} > x_1$ for a monomial ordering on R. With the help of a computer, we obtain that a Gröbnerbasis contains the polynomial $y_1(x_1)$ of x_1 given by

$$
\begin{aligned}
y(x_1) = &(k + 1)(2k + 1)(6k + 2l + 1)(12k^2 + 6kl + 2l^2 + l) \\
&\times (12k^2 + 6kl + 3k + 2l^2 + l)^2 x_1^8 \\
&- 8(k + 1)(2k + 1)(3k + l + 1)(12k^2 + 6kl + 3k + 2l^2 + l) \\
&\times (72k^4 + 180k^3 l + 18k^3 + 132k^2 l^2 + 48k^2 l \\
&+ 48kl^3 + 30kl^2 + 3kl + 8l^4 + 8l^3 + 2l^2)x_1^7 \\
&+ 2(k + 1)(2k + 1) \\
&\times (5832k^7 + 46764k^6 l + 12420k^6 + 85320k^5 l^2 + 54324k^5 l + 6228k^5 \\
&+ 77976k^4 l^3 + 81810k^4 l^2 + 20763k^4 l + 1170k^4 + 41940k^3 l^4 + 62028k^3 l^3 \\
&+ 27045k^3 l^2 + 3330k^3 l + 72k^3 + 13824k^2 l^5 + 26472k^2 l^4 + 16680k^2 l^3
\end{aligned}
$$

$$
\begin{aligned}
&+ 3858k^2l^2 + 222k^2l + 2640kl^6 + 6312kl^5 + 5248kl^4 + 1798kl^3 + 223kl^2 \\
&+ 6kl + 224l^7 + 664l^6 + 716l^5 + 338l^4 + 63l^3 + 2l^2)x_1^6 \\
&- 8(k+1)(2k+1)(3k+l+1) \\
&\times (2322k^5l + 864k^5 + 6156k^4l^2 + 3915k^4l + 432k^4 + 6300k^3l^3 + 6426k^3l^2 \\
&+ 1386k^3l + 54k^3 + 3312k^2l^4 + 4692k^2l^3 + 1794k^2l^2 + 180k^2l + 936kl^5 \\
&+ 1728kl^4 + 958kl^3 + 176kl^2 + 9kl + 112l^6 + 268l^5 + 212l^4 + 63l^3 + 5l^2)x_1^5 \\
&+ (k+1)(2k+1) \\
&\times (13608k^6l + 7776k^6 + 83376k^5l^2 + 78192k^5l + 14256k^5 + 143496k^4l^3 \\
&+ 198252k^4l^2 + 71754k^4l + 6576k^4 + 117360k^3l^4 + 220284k^3l^3 + 127278k^3l^2 \\
&+ 23980k^3l + 1128k^3 + 52032k^2l^5 + 126384k^2l^4 + 103200k^2l^3 + 32556k^2l^2 \\
&+ 3443k^2l + 60k^2 + 12000kl^6 + 36784kl^5 + 40584kl^4 + 19212kl^3 + 3604kl^2 \\
&+ 200kl + 1120l^7 + 4240l^6 + 6124l^5 + 4168l^4 + 1321l^3 + 160l^2 + 4l)x_1^4 \\
&- 8(k+1)(2k+1)(l+1)(3k+l+1) \\
&\times (774k^4l + 288k^4 + 2052k^3l^2 + 1305k^3l + 144k^3 + 1896k^2l^3 \\
&+ 1944k^2l^2 + 414k^2l + 18k^2 + 768kl^4 + 1172kl^3 + 454kl^2 \\
&+ 44kl + 112l^5 + 240l^4 + 158l^3 + 35l^2 + 2l)x_1^3 \\
&+ 4(k+1)(2k+1)(l+1)^2 \\
&\times (324k^5 + 2598k^4l + 690k^4 + 4514k^3l^2 + 2841k^3l + 314k^3 \\
&+ 3168k^2l^3 + 3640k^2l^2 + 960k^2l + 49k^2 + 984kl^4 + 1740kl^3 \\
&+ 882kl^2 + 129kl + 2k + 112l^5 + 276l^4 + 224l^3 + 67l^2 + 6l)x_1^2 \\
&- 16(k+1)(2k+1)(l+1)^3(3k+l+1)(4k+2l+1) \\
&\times (4k^2 + 10kl + k + 4l^2 + l)x_1 \\
&+ 4(k+1)(2k+1)(l+1)^4(2k+l)(4k+2l+1)^2.
\end{aligned}
$$

Since the coefficients of the polynomial $y(x_1)$ are positive for even degree terms and negative for odd degree terms, all the real solutions of the polynomial $y(x_1) = 0$ are positive. Therefore, we obtain at least two solutions of the homogeneous Einstein equations of the form

$$
\begin{aligned}
&x_1 = p(x_{23}), && x_2 = x_3 = x_4 = \frac{(k+1)x_{23}}{lx_{23}^2 + 4k + 1}, \\
&x_{12} = x_{13} = x_{14} = 1, && x_{23} = x_{24} = x_{34} \neq 1.
\end{aligned}
$$

We conclude that these metrics are not naturally reductive by Proposition 3.1. This completes the proof of Theorem 1.1. □

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