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BAYESIAN REFERENCE ANALYSIS FOR PROPORTIONAL HAZARDS MODEL OF RANDOM CENSORSHIP WITH WEIBULL DISTRIBUTION

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This article deals with the objective Bayesian analysis of random censorship model with informative censoring using Weibull distribution. The objective Bayesian analysis has a long history from Bayes and Laplace through Jeffreys and is reaching the level of sophistication gradually. The reference prior method of Bernardo is a nice attempt in this direction. The reference prior method is based on the Kullback-Leibler divergence between the prior and the corresponding posterior distribution and easy to implement when the information matrix exists in closed-form. We apply this method to Weibull random censorship model and compare it with Jeffreys and maximum likelihood methods. It is observed that the closed-form expressions for the Bayes estimators are not possible; we use importance sampling technique to obtain the approximate Bayes estimates. The behaviour of maximum likelihood and Bayes estimators is observed via extensive numerical simulation. The proposed methodology is used for the analysis of a real-life data for illustration and appropriateness of the model is tested by Henze goodness-of-fit test.

Keywords: Jeffreys prior method, reference prior method, random censorship model, Kaplan–Meier survival estimate, Henze goodness-of-fit test

Classification: 62N01, 62N05, 62F10, 62F15

1. INTRODUCTION

The nice development of reference prior method enables the Bayesian researchers to fit more and more realistic and sophisticated models which were not previously possible. The reference prior method, initiated in [5] and further developed by Berger and Bernardo [3, 4], is the best known of available objective Bayesian methods in the sense of invariance, consistent marginalization, consistent sampling properties, generality, and admissibility. The essence of any Bayesian analysis is the posterior distribution which combines the experimental data with the available prior information. However, the formal Bayesian analysis has a reputation of producing vested results using the priors which fit into the model at hand. Although there are well known non-informative prior methods in the form of Laplace and Jeffreys, these prior methods have inconsistency and dimensional issues in some cases. The need for model-based prior that has minimal

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effect on the posterior inference relative to the data at hand was being felt and the reference prior is a good attempt in this direction.

The Weibull distribution is one the most widely used distributions for modeling failure time data from reliability and life testing experiments. It is mainly due to the variety of shapes of its density function and the behaviors of its failure rate function. Literally thousands of references to the Weibull distribution can be found in the statistical literature. The distributional properties, estimation of parameters and applications in different fields can be seen in [15]. Readers are referred to [21] for a detailed comprehensive overview of the Weibull family of distributions, its modifications, and its relation to other distributions. Hossain and Zimmer [13] compared several methods for estimating the parameters of two-parameter Weibull distribution with complete, multiply time censored and type II censored samples based on extensive simulation study. Kundu [20] dealt with the progressively censored Weibull distribution for the Bayesian estimation of unknown parameters. Abu-Taleb, Smadi and Alawneh [1] conducted the maximum likelihood estimation of randomly censored exponential distribution. Joarder, Krishna and Kundu [14] considered the statistical inferences of the unknown parameters of Weibull distribution when the data are type-I censored and proposed a simple algorithm to compute the maximum likelihood estimators and the approximate maximum likelihood estimators based on Taylor series expansions. Danish and Aslam [7] considered the Bayesian inference of the unknown parameters of the randomly censored Weibull distribution assuming survival time and censoring time variables have the same shape parameter but different scale parameters. More recent treatment on the topic may be found in Krishna et al. [19]; Garg et al. [10] and Danish et al. [8].

Although much work has been done on the statistical inferences for the parameter of Weibull distribution both in classical and Bayesian contexts, however, we provide an alternative methodology for the analysis of failure time data based on reference prior method and compare it with the Jeffreys and maximum likelihood methods both in term of simulation study and actual data application.

The rest of the article is organized as follows. In the next section we define the Weibull random censorship model and associated assumptions. Section 3 provides maximum likelihood (ML) estimators and corresponding Fisher information matrix. In Section 4 we derive Jeffreys and reference priors and obtain the Bayes estimates using importance sampling procedure. This section also describes the procedure to obtain the highest posterior density (HPD) intervals. A simulation study is carried out in Section 5 and finally a real data analysis is performed in Section 6.

2. PROBLEM FORMULATION

Suppose a sample of n identical patients enter a life testing experiment after some medical treatment and their survival times during the experiment are recorded. Each patient in the sample will have either failure time or censoring time. Let the random variables $X_1,\ X_2,\dots,X_n$ with distribution function $F_X(x)$ and density function $f_X(x)$ represent their failure times and the random variables $T_1,\ T_2,\dots,T_n$ with distribution function $G_T(t)$ and density function $g_T(t)$ denote their censoring times. Since we do not know for each patient whether the failure will happen first or the censoring, we define $Y_i = \min(X_i, T_i)$ and $D_i = I(X_i \le T_i)$ for $i = 1, 2, \dots, n$. The fundamental assumption in the theory of survival analysis is the independence of censoring and failure times. This is due to the fact that in medical investigations the patients often have random entry into the study with fixed censoring time at the end of the study. If it is reasonable to assume that the variables X and T are independent, then joint density of Y and D can be expressed as

$$f_{(Y,D)}(y,d) = [f_X(y)\{1 - G_T(y)\}]^d [g_T(y)\{1 - F_X(y)\}]^{(1-d)}. \tag{1}$$

There are two cases of this random censorship model in medical studies: informative and non-informative censoring. In informative censoring the censoring time variable is related to survival time variable in terms of distribution function. In this case Koziol and Green [18] introduced a special model assuming that the survival time variable and the censoring time variable are independent, and they are connected by the relation

$$G_T(y) = 1 - \{1 - F_X(y)\}^{\beta}; \quad \forall \ y \geq 0 \tag{2}$$

for some positive constant β. [17] obtained the same Koziol–Green model under the assumption of independence of the observable survival time variable Y and the censoring indicator variable D. The Koziol-.Green model allowed the censored observations to the estimation of survival function and started the era of informative censoring in the life testing application.

The parameter β in (2) is defined as

$$p = P(D = 1) = P[(X \leq T) = (1 + \beta)^{-1}. \tag{3}$$

Under the assumption (2), the random censorship model in (1) takes the form

$$f_{Y,D}(y,d) = f_X(y)[\{1 - F_X(y)\}]^{\beta} \beta^{1-d}. \tag{4}$$

The density function of Weibull distribution is

$$f_X(x;\theta,\lambda) = \theta\lambda x^{\theta-1} e^{-\lambda x^{\theta}}; \ x > 0, \theta > 0, \lambda > 0, \tag{5}$$

and corresponding distribution function is

$$F_X(x;\lambda) = 1 - e^{-\lambda x^{\theta}}. \tag{6}$$

Using (5) and (6) in (4), we obtain the joint distribution Y and D with density function

$$f_{Y,D}(y,d;\theta,\lambda,\beta) = \theta\lambda y^{\theta-1} e^{-(1+\beta)\lambda y^{\theta}} \beta^{1-d}; \quad y > \theta. \tag{7}$$

3. MAXIMUM LIKELIHOOD ESTIMATION

For an observed sample (y_1, d_1),...,$(y_n, d_n) = (y, d)$ from (7), the likelihood function is

$$l(\theta,\lambda,\beta) = \theta^n \lambda^n \prod_{i=1}^{n} y_i^{\theta-1} e^{-(1+\beta)\lambda \sum_{i=1}^{n} y_i^{\theta}} \beta^{n-\sum_{i=1}^{n} d_i}. \tag{8}$$

The corresponding log-likelihood function is

$$l(\theta,\lambda,\beta) = nln\theta + nln\lambda + (\theta-1)\sum_{i=1}^{n} lny_i - \lambda(1+\beta)\sum_{i=1}^{n} y_i^{\theta} + (n-\sum_{i=1}^{n} d_i)ln\beta. \quad (9)$$

The ML estimates are obtained from (9) as

$$\beta = \frac{(n-\sum_{i=1}^{n} d_i)}{\sum_{i=1}^{n} d_i}, \qquad \lambda(\theta) = \frac{\sum_{i=1}^{n} d_i}{\sum_{i=1}^{n} y_i^{\theta}} \quad \text{and} \quad \theta = h(\theta),$$

where

$$\frac{1}{h(\theta)} = \frac{\sum_{i=1}^{n} y_i^{\theta} lny_i}{\sum_{i=1}^{n} y_i^{\theta}} - \frac{\sum_{i=1}^{n} lny_i}{n}.$$

Once the ML estimate of θ is obtained from the nonlinear equation $\theta = h(\theta)$ using any of the available iterative procedure, the ML estimate of λ can be obtained from $\lambda(\hat{\theta})$. The Fisher information matrix is derived as

$$\mathbf{I}(\theta,\lambda,\beta) = \begin{bmatrix} \frac{n}{\theta^2}(\frac{\pi^2}{6}+k^2) & \frac{nk}{\theta\lambda} & \frac{nk}{\theta(1+\beta)} \\ \frac{nk}{\theta\lambda} & \frac{n}{\lambda^2} & \frac{n}{\lambda(1+\beta)} \\ \frac{nk}{\theta(1+\beta)} & \frac{n}{\lambda(1+\beta)} & \frac{n}{\beta(1+\beta)} \end{bmatrix},$$

where $k = \Psi(2) - ln(\lambda(1+\beta))$ and $\Psi(.)$ is digamma function.

4. OBJECTIVE BAYESIAN ANALYSIS

This section provides the objective Bayesian estimation of unknown parameters with respect to squared error loss function. For this purpose, we consider two different priors: (a) Jeffreys prior and (b) Reference prior.

4.1. Jeffreys prior

In some realistic sense the non-informative priors make the Bayesian analysis a different entity and the Jeffreys prior is the most commonly used non-informative prior. It is defined as $\pi_J(\theta,\lambda,\beta) \propto |\mathbf{I}(\theta,\lambda,\beta)|^{1/2}$. The determinant $|\mathbf{I}(\theta,\lambda,\beta)|$ of the Fisher information matrix is

$$|\mathbf{I}(\theta,\lambda,\beta)| = \frac{n^3\pi^2}{6\theta^2\lambda^2\beta(1+\beta)^2}$$

so

$$\pi_J(\theta,\lambda,\beta) = \frac{1}{\theta\lambda(1+\beta)\beta^{1/2}}. \quad (10)$$

4.2. Reference prior

Suppose $f(x|\theta)$ is a parametric model indexed with parameter vector $\theta \in \Theta$ and $I(\theta)$ is the corresponding Fisher information matrix with full rank, where $\theta = (\theta_1, \theta_2, \dots, \theta_m)$. Define $S(\theta) = I^{-1}(\theta)$, $\theta^{[j]} = (\theta_1, \dots, \theta_j)$ with $\theta^{[0]} = \{\}$, $\theta_{[j]} = (\theta_{j+1}, \dots, \theta_m)$ with $\theta_{[0]} = \theta = (\theta_1, \theta_2, \dots, \theta_m)$, $S_j(\theta)$= upper left $j \times j$ submatrix of $S(\theta)$ and $h_j(\theta)$= lower right element of $S_j^{-1}(\theta)$. Further assume that for $\Theta = \Theta_1 \times \ldots \times \Theta_m$ with $\theta_i \in \Theta_i$, $\{\Theta_i^l\}$, $i = 1, 2, \ldots, m$, $l = 1, 2, \ldots$, is an increasing sequence of compact subsets of Θ_i and $\Theta_{[j]}^l = \Theta_{j+1}^l \times \ldots \times \Theta_m^l$. We borrow the following algorithm from Theorem 22 in [6] for further development.

Algorithm 1
The reference prior $\pi(\theta)$ relative to the ordered parameterisation $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$ is

$$\pi(\theta) = \lim_{l \to \infty} \frac{\pi^l(\theta)}{\pi^l(\theta^*)},$$

for some $\theta^* \in \Theta$, where $\pi^l(\theta)$ is defined by the following three steps:

(i) For $j = m$ and $\theta_m \in \Theta_m^l$

$$\pi_m^l\left(\theta_{[m-1]} | \theta^{[m-1]}\right) = \pi_m^l(\theta_m | \theta_1, \ldots, \theta_{[m-1]}) = \frac{\{h_m(\theta)\}^{1/2}}{\int_{\Theta_m^l} \{h_m(\theta)\}^{1/2} \, \mathrm{d}\theta_m}.$$

(ii) For $j = m-1, m-2, \ldots, 2$, and $\theta_j \in \Theta_j^l$,

$$\pi_j^l\left(\theta_{[j-1]} | \theta^{[j-1]}\right) = \pi_{j+1}^l(\theta_{[j]} | \theta^{[j]}) \frac{exp\{E_j^l[ln\{h_j(\theta)\}^{1/2}]\}}{\int_{\Theta_j^l} exp\{E_j^l[ln\{h_j(\theta)\}^{1/2}]\} \, \mathrm{d}\theta_j},$$

where

$$E_j^l[ln\{h_j(\theta)\}^{1/2}] = \int_{\Theta_{[j]}^l} [ln\{h_j(\theta)\}^{1/2}] \pi_{j+1}^l(\theta_{[j]} | \theta^{[j]}) \, \mathrm{d}\theta_{[j]}.$$

(iii) For $j = 1$, $\theta_{[0]} = \theta = (\theta_1, \theta_2, \ldots, \theta_m)$ and

$$\pi^l(\theta) = \pi_1^l(\theta_{[0]} | \theta^{[0]}) = \pi_1^l(\theta_1, \theta_2, \ldots, \theta_m).$$

The interested readers are referred to [6] for further detail on the development of reference prior Algorithm 1 and for the following heuristic result.

Let us consider what can be said about the form of the posterior distribution of θ, for an observed data x of large size n, given by

$$\pi(\theta|x) \propto l(\theta)\pi(\theta) \propto exp\{ln\pi(\theta) + lnl(\theta)\}. \tag{11}$$

Expansion of the two logarithmic terms about their respective maxima, m_θ and $\hat{\theta}_n$, gives

$$ln\pi(\theta) = ln\pi(m_0) - \frac{1}{2}(\theta - m_0)' H_0 (\theta - m_0) + R_0$$

$$lnl(\theta) = lnl(\hat{\theta}_n) - \frac{1}{2}(\theta - \hat{\theta}_n)' H(\hat{\theta}_n)(\theta - \hat{\theta}_n) + R_n,$$

where R_0, R_n denote remainder terms and H_0, $H(\hat{\theta}_n)$ are minus the hessians of $ln\pi(\theta)$ and $lnl(\theta)$, respectively evaluated at their respective maxima.

For small R_0 and R_n, expression (11) can be written as

$$\pi(\theta|x) \propto exp\{ln\pi(m_0) - \frac{1}{2}(\theta - m_0)' H_0(\theta - m_0) + lnl(\hat{\theta}_n) - \frac{1}{2}(\theta - \hat{\theta}_n)' H(\hat{\theta}_n)(\theta - \hat{\theta}_n)\}$$

$$\propto exp\{-\frac{1}{2}(\theta - m_n)' H_n(\theta - m_n)\}, \tag{12}$$

where

$$H_n = H_0 + H(\hat{\theta}_n)$$

and

$$m_0 = H_n^{-1}\big(H_0 m_0 + H(\hat{\theta}_n)\hat{\theta}\big).$$

We note that the expression in (12) resemble the multivariate normal distribution mean vector m_n and covariance matrix H_n.

Theorem 4.1. Consider the probability model in (7) with parameter space $\Theta = \{\Theta_1 \times \Theta_2 \times \Theta_3\} = \{(\theta, \lambda, \beta) : 0 < \theta < \infty, 0 < \lambda < \infty, 0 < \beta < \infty\}$. Suppose that the asymptotic posterior distribution of (θ, λ, β) is multivariate normal with covariance matrix $S(\hat{\theta}, \hat{\lambda}, \hat{\beta}) = \mathbf{I}^{-1}(\hat{\theta}, \hat{\lambda}, \hat{\beta})$, where $\hat{\theta}$, $\hat{\lambda}$ and $\hat{\beta}$ are the consistent estimates of θ, λ and β.

(a) If θ is parameter of interest and ordered parameterisation is (θ, λ, β), then the reference prior is same as the Jeffreys prior, that is

$$\pi_R(\theta, \lambda, \beta) = \pi_J(\theta, \lambda, \beta) = \frac{1}{\theta\lambda(1+\beta)\beta^{1/2}}$$

(b) If θ is parameter of interest and the natural parameterisation is (θ, λ, β), then the reference prior is

$$\pi_R(\theta, \lambda, \beta) = \frac{1}{\theta\lambda(1+\beta)^{1/2}\beta^{1/2}}. \tag{13}$$

Proof. (a) Using Algorithm 1 with $\theta = (\theta_1, \theta_2, \theta_3) = (\theta, \beta, \lambda)$, $H((\theta, \beta, \lambda)) = \mathbf{I}(\theta, \beta, \lambda)$,

$$H^{-1}(\theta, \beta, \lambda) = S(\theta, \beta, \lambda) \begin{bmatrix} \frac{6\theta^2}{n\pi^2} & 0 & -\frac{6\theta\lambda k}{n\pi^2} \\ 0 & \frac{\beta(1+\beta)^2}{n} & -\frac{\lambda\beta(1+\beta)}{n} \\ -\frac{6\theta\lambda k}{n\pi^2} & -\frac{\lambda\beta(1+\beta)}{n} & \frac{\lambda^2\{\pi^2(1+\beta)+6k^2\}}{n\pi^2} \end{bmatrix},$$

presuming the assumptions of the theorem, we have
$h_1(\theta, \beta, \lambda) = \frac{n\pi^2}{6\theta^2}$, $h_2(\theta, \beta, \lambda) = \frac{n}{\beta(1+\beta)^2}$ and $h_3(\theta, \beta, \lambda) = \frac{n}{\lambda^2}$.

(i) For $j = 3$ and $\lambda \in \Theta_3 = \{\lambda : 0 < \lambda < \infty\}$, the conditional reference prior of λ given θ and β defined by

$$\pi_3(\lambda|\theta, \beta) = \frac{\{h_3(\theta, \beta, \lambda)\}^{1/2}}{\int_0^\infty \{h_3(\theta, \beta, \lambda)\}^{1/2} d\lambda} = \frac{1}{\lambda \int_0^\infty \lambda^{-1}\, \mathrm{d}\lambda}$$

is not proper, consider a nested sequence $\Theta_3^l = (\lambda : l^{-1} < \lambda < l), l = 1, 2, \ldots$ of compact sets of Θ_3 such that

$$\pi_3^l(\lambda|\theta,\beta) = \frac{1}{\lambda \int_{l-1}^{\infty} \lambda^{-1}\,\mathrm{d}\lambda} = \frac{1}{2\lambda ln(l)}, \quad \lambda \in \Theta_3^l,$$

is proper density.

(ii) For $j = 2$ and $\beta \in \Theta_2 = \{\beta : \beta > 0\}$, the conditional reference prior of λ, β given θ is

$$\pi_2^l(\beta,\lambda|\theta) = \pi_3^l(\lambda|\theta,\beta)\frac{exp\{E_2^l[\{h_2(\theta,\beta,\lambda)\}^{1/2}]\}}{\int_0^{\infty} exp\{E_2^l[\{h_2(\theta,\beta,\lambda)\}^{1/2}]\}\,\mathrm{d}\beta};\ \beta \in \Theta_2, \lambda \in \Theta_3^l,$$

where
$E_2^l[\{h_2(\theta,\beta,\lambda)\}^{1/2}] = \int_{1/l}^{l} ln[\frac{n}{\beta(1+\beta)^2}]^{1/2}\pi_3^l(\lambda|\theta,\beta)\,\mathrm{d}\lambda = ln[\frac{n}{\beta(1+\beta)^2}]^{1/2}.$
Substituting this in $\pi_2^l(\beta,\lambda|\theta)$ above, we have

$$\pi_2^l(\beta,\lambda|\theta) = \frac{1}{2\lambda ln(l)}\frac{\beta^{-1/2}(1+\beta)^{-1}}{\int_0^{\infty}\beta^{-1/2}(1+\beta)^{-1}\,\mathrm{d}\beta} = \frac{\pi}{2\lambda ln(l)(1+\beta)\beta^{1/2}},\ \beta \in \Theta_2, \lambda \in \Theta_3^l.$$

(iii) For $j = 1$ and $\theta \in \Theta_3^l = (\theta : \theta^{-1} < \theta < l)$, the joint reference prior of (θ,β,λ), is

$$\pi_1^l(\theta,\beta,\lambda) = \pi_2^l(\beta,\lambda|\theta)\frac{exp\{E_1^l[\{h_1(\theta,\beta,\lambda)\}^{1/2}]\}}{\int_{1/l}^{l} exp\{E_1^l[\{h_1(\theta,\beta,\lambda)\}^{1/2}]\}\,\mathrm{d}\theta},$$

where
$E_1^l[\{h_1(\theta,\beta,\lambda)\}^{1/2}] = \int_0^{\infty}\int_{1/l}^{l} ln[\frac{n\pi^2}{6\theta^2}]^{1/2}\pi_2^l(\beta,\lambda|\theta)\,\mathrm{d}\beta d\lambda = [\frac{n\pi^2}{6\theta^2}]^{1/2}.$
Substituting this in $\pi_1^l(\theta,\beta,\lambda)$ above, we have

$$\pi_1^l(\theta,\beta,\lambda) = \frac{\pi}{2\lambda ln(l)(1+\beta)\beta^{1/2}}\frac{\theta^{-1}}{\int_{1/l}^{l}\theta^{-1}\,\mathrm{d}\theta} = \frac{\pi}{4\{ln(l)\}^2\theta\lambda(1+\beta)\beta^{1/2}}.$$

Thus reference prior for (θ,β,λ) is

$$\pi(\theta,\beta,\lambda) = \lim_{l\to\infty}\frac{\frac{\pi}{4\{ln(l)\}^2\theta\lambda(1+\beta)\beta^{1/2}}}{\frac{\pi}{4\{ln(l)\}^2\theta_0\lambda_0(1+\beta_0)\beta_0^{1/2}}} = \frac{1}{\theta\lambda(1+\beta)\beta^{1/2}},$$

where $(\theta_0,\beta_0,\lambda_0)$ is some fixed point in Θ.

(b) To prove this part take $(\theta_1,\theta_2,\theta_3) = (\theta,\lambda,\beta)$, $H(\theta,\beta,\lambda) = \mathbf{I}(\theta,\beta,\lambda)$ so that

$$H^{-1}(\theta,\lambda,\beta) = S(\theta,\lambda\beta)\begin{bmatrix} \frac{6\theta^2}{n\pi^2} & -\frac{6\theta\lambda k}{n\pi^2} & 0 \\ -\frac{6\theta\lambda k}{n\pi^2} & \frac{\lambda^2\{\pi^2(1+\beta)+6k^2\}}{n\pi^2} & -\frac{\lambda\beta(1+\beta)}{n} \\ 0 & -\frac{\lambda\beta(1+\beta)}{n} & \frac{\beta(1+\beta)^2}{n} \end{bmatrix},$$

$h_1(\theta,\lambda,\beta) = \frac{n\pi^2}{6\theta^2}$, $h_2(\theta,\lambda,\beta) = \frac{n}{\lambda^2(1+\beta)}$ and $h_3(\theta,\lambda,\beta) = \frac{n}{\beta(1+\beta)}$.

(i) For $j = 3$ and $\beta \in \Theta_3 = \{\beta : 0 < \beta < \infty\}$, the conditional reference prior of β given θ and λ defined by

$$\pi_3(\beta|\theta,\lambda) = \frac{\{h_3(\theta,\lambda,\beta)\}^{1/2}}{\int_0^\infty \{h_3(\theta,\lambda,\beta)\}^{1/2}\,\mathrm{d}\beta} = \frac{1}{\sqrt{\beta(1+\beta)}\int_0^\infty \sqrt{\beta(1+\beta)}\,\mathrm{d}\beta}$$

is not proper, consider a nested sequence $\Theta_3^l = (\beta : l^{-1} < \beta < l), l = 1, 2, \ldots$ of compact sets of $\Theta_3 = \{\beta : 0 < \beta < \infty\}$ such that

$$\pi_3^l(\beta|\theta,\lambda) = \frac{\left((\beta+\beta^2)\right)^{-1/2}}{\int_{l^{-1}}^\infty \sqrt{\beta(1+\beta)}\,\mathrm{d}\beta} = \frac{\left((\beta+\beta^2)\right)^{-1/2}}{A_1(l)}, \quad \beta \in \Theta_3^l,$$

is proper, where

$$A_1(l) = ln[\frac{l\{2l+1+2\sqrt{l(l+1)}\}}{2+l+2\sqrt{1+l}}].$$

(ii) For $j = 2$ and $\lambda \in \Theta_2^l = \{\lambda : l^{-1} < \lambda < l\}$, the conditional reference prior of λ, β given θ is

$$\pi_2^l(\lambda,\beta|\theta) = \frac{\left((\beta+\beta^2)\right)^{-1/2}}{2\lambda A_1(l)ln(l)}, \quad \beta \in \Theta_3^l, \lambda \in \Theta_2^l.$$

(iii) Finally, for $j = 1$ and $\theta \in \Theta_3^l = (\theta : l^{-1} < \theta < l)$, the marginal reference prior of θ can be obtained as

$$\pi_1^l(\theta) = \frac{exp\{E_1^l[\{h_1(\theta,\lambda,\beta)\}^{1/2}]\}}{\int_{1/l}^l exp\{E_1^l[\{h_1(\theta,\lambda,\beta)\}^{1/2}]\}\,\mathrm{d}\theta},$$

where
$E_1^l[\{h_1(\theta,\lambda,\beta)\}^{1/2}] = \int_{1/l}^l \int_{1/l}^l ln[\frac{\pi n^{1/2}}{\theta(6)^{1/2}}]\frac{(\beta+\beta^2)^{-1/2}}{2\lambda A_1(lln(l))}\,\mathrm{d}\lambda\mathrm{d}\beta = ln(\frac{\pi n^{1/2}}{\theta 6^{1/2}})$.
Therefore, the marginal reference prior of θ is

$$\pi_1^l(\theta) = \frac{exp\{ln(\frac{\pi n^{1/2}}{\theta 6^{1/2}})\}}{\int_{1/l}^l \{ln(\frac{\pi n^{1/2}}{\theta 6^{1/2}})\}\,\mathrm{d}\theta} = \frac{1}{2\theta ln(l)}.$$

The joint reference prior with respect to the compact space Θ_3^l is

$$\pi_1^l(\theta,\lambda,\beta) = \pi_2^l(\lambda,\beta|\theta)\pi_1^l(\theta) = \frac{(\beta+\beta^2)^{-1/2}}{2\lambda A_1(l)ln(l)}\frac{1}{2\theta ln(l)}.$$

Thus, the reference prior for (θ,λ,β) is

$$\pi(\theta,\lambda,\beta) = \lim_{l\to\infty} \frac{\frac{1}{4A_1(l)ln(l)\theta\lambda(1+\beta)\beta^{1/2}}}{\frac{1}{4A_1(l)ln(l)\theta_0\lambda_0(1+\beta_0)\beta_0^{1/2}}} = \frac{1}{\theta\lambda(1+\beta)^{1/2}\beta^{1/2}},$$

This completes the proof. □

Remark 1. The reference prior $\pi(\theta, \lambda, \beta)$ is invariant under scale transformation of the observed data.

Proof. First note from the joint density in (7) that Y is independently distributed from D as Weibull with density

$$f_Y(y; \theta, \lambda, \beta) = \theta\lambda(1+\beta)y^{\theta-1}e^{-\lambda(1+\beta)y^\theta}$$

and D as Bernoulli with density

$$f_D(d; \beta) = (1+\beta)^{-1}\beta^{1-d}.$$

Suppose the original data follow a Weibull distribution with density

$$f_{Y'}(y'; \theta, \lambda', \beta) = \theta\lambda'(1+\beta)y'^{\theta-1}e^{-\lambda'(1+\beta)y^\theta}.$$

Let $y = y'/a$, so that $y' = ay$ and $dy'/dy = a$. The distribution of the transformed variable is

$$f_Y(y; \theta, \lambda', \beta) = \theta(1+\beta)\lambda'(ay)^{\theta-1}e^{-\lambda'(1+\beta)(ay)^\theta} = \lambda'(1+\beta)a^\theta y^{\theta-1}e^{-\lambda'(1+\beta)a^\theta y^\theta}$$

or

$$f_Y(y; \theta, \lambda, \beta) = \lambda(1+\beta)y^{\theta-1}e^{-\lambda(1+\beta)y^\theta},$$

where $\lambda = \lambda' a^\theta$. Now suppose that we have constructed the reference prior under the transformed model $f_Y(y; \theta, \lambda, \beta)$ with density

$$\pi(\theta, \lambda, \beta) \propto \frac{1}{\theta\lambda(1+\beta)^{1/2}\beta^{1/2}}$$

and want to find the reference prior under the original model $f_{Y'}(y'; \theta, \lambda', \beta)$. Set $\theta' = \theta$, $\lambda' = \lambda/a^\theta$ and $\beta' = \beta$, so that the inverse transformation is $\theta = \theta$, $\lambda = \lambda'/a^{\theta'}$ and $\beta = \beta'$. The Jacobian of transformation is $a^{\theta'}$ and the reference prior in terms of original parametrization is

$$\pi(\theta, \lambda', \beta) = \frac{1}{\theta\lambda' a^\theta(1+\beta)^{1/2}\beta^{1/2}a^\theta} = \frac{1}{\theta\lambda'(1+\beta)^{1/2}\beta^{1/2}}.$$

□

4.3. Posterior analysis

The Jeffreys prior in (10) and the reference prior in (13) can be expressed in more general form as

$$\pi_b(\theta, \lambda, \beta) = \frac{1}{\theta\lambda(1+\beta)^b\beta^{1/2}}, \qquad b \geq 0, \tag{14}$$

where $b = 1$ represents the Jeffreys prior and $b = 0.5$ the reference prior. Combining the likelihood function in (8) and the joint prior in (14), the joint posterior distribution of (θ, λ, β) given data (y, d) can be written as

$$\pi(\theta, \lambda, \beta | y, d) \propto \theta^n\lambda^n \prod_{i=1}^{n} y_i^{\theta-1} e^{-(1+\beta)\lambda\sum_{i=1}^n y_i^\theta} \beta^{n-\sum_{i=1}^n d_i} \times \frac{1}{\theta\lambda(1+\beta)^b\beta^{1/2}}$$

$$\propto \theta^{n-1}\lambda^{n-1}e^{\theta\sum_{i=1}^{n} lny_i}e^{-(1+\beta)\lambda\sum_{i=1}^{n} y_i^\theta}\beta^{n-\sum_{i=1}^{n} d_i-0.5}\frac{1}{(1+\beta)^b},$$

$$\propto \theta^{n-1}e^{\theta\sum_{i=1}^{n} lny_i}\frac{\left((1+\beta)\sum_{i=1}^{n} y_i^\theta\right)^n}{\left((1+\beta)\sum_{i=1}^{n} y_i^\theta\right)^n}\lambda^{n-1}e^{-(1+\beta)\lambda\sum_{i=1}^{n} y_i^\theta}\frac{\beta^{n-\sum_{i=1}^{n} d_i-0.5}}{(1+\beta)^b},$$

$$\propto \frac{\theta^{n-1}e^{\theta\sum_{i=1}^{n} lny_i}}{(\sum_{i=1}^{n} y_i^\theta)^n}\left((1+\beta)\sum_{i=1}^{n} y_i^\theta\right)^n\lambda^{n-1}e^{-(1+\beta)\lambda\sum_{i=1}^{n} y_i^\theta}\frac{\beta^{n-\sum_{i=1}^{n} d_i-0.5}}{(1+\beta)^{b+n}},$$

$$\propto g_1\left(n,-\sum_{i=1}^{n} lny_i\right)h(\theta)g_2\left(n,(1+\beta)\sum_{i=1}^{n} y_i^\theta\right)b_{11}\left(\sum_{i=1}^{n} d_i+b-\frac{1}{2},n-\sum_{i=1}^{n} d_i+\frac{1}{2}\right), \quad (15)$$

where

$$g_1\left(n,-\sum_{i=1}^{n} lny_i\right)=\frac{(-\sum_{i=1}^{n} lny_i)^n}{\Gamma(n)}\theta^{n-1}e^{\theta\sum_{i=1}^{n} lny_i},$$

$$g_2\left(n,(1+\beta)\sum_{i=1}^{n} y_i^\theta\right)=\frac{\{(1+\beta)\sum_{i=1}^{n} y_i^2\}^n}{\Gamma(n)}\lambda^{n-1}e^{-(1+\beta)\lambda\sum_{i=1}^{n} y_i^\theta},$$

$$b_{11}\left(\sum_{i=1}^{n} d_i+b-\frac{1}{2},n-\sum_{i=1}^{n} d_i+\frac{1}{2}\right)=\frac{\Gamma(n+b)\beta^{n-\sum_{i=1}^{n} d_i+0.5}(1+\beta)^{-n-b}}{\Gamma(\sum_{i=1}^{n} d_i+b-0.5)\Gamma(n-\sum_{i=1}^{n} d_i+0.5)}$$

and

$$h(\theta)=\left(\sum_{i=1}^{n} y_i^\theta\right)^{-n}.$$

We see from (15) that it is possible to use the importance sampling procedure to generate the posterior samples and to obtain the Bayes estimates and then in turn to obtain the HPD credible intervals. We propose the following algorithm for this purpose assuming there exists at least one observation non-censored and $\sum_{i=1}^{n} lny_i < 0$. Dividing data values by their maximum will put all the observations in interval $(0,1]$ which guaranteed that $\sum_{i=1}^{n} lny_i < 0$.

Algorithm 2

(i) Generate $\beta^{(1)} \sim b_{11}\left(\sum_{i=1}^{n} d_i+b-0.5,\ n-\sum_{i=1}^{n} d_i=0.5\right)$.

(ii) Generate $\theta^{(1)} \sim g_1\big(n,\ -\sum_{i=1}^{n} lny_i\big)$.

(iii) Generate $\lambda^{(1)}|\beta^{(1)},\theta^{(1)} \sim g_2\left(n,\ (1+\beta^{(1)})\sum_{i=1}^{n} y_i^{\theta(1)}\right)$.

(iv) Repeat steps (i-iii) M times to obtain $\big(\theta^{(1)},\lambda^{(1)},\beta^{(1)}\big),\cdots,\big(\theta^{(M)},\lambda^{(M)},\beta^{(M)}\big)$.

(v) The Bayes estimate of any inference function, say $u(\theta,\lambda,\beta)$, with respect to squared error loss function is

$$\hat{u}(\theta,\lambda,\beta)=\frac{\sum_{j=1}^{M}u(\theta^{(j)},\lambda^{(j)},\beta^{(j)})h(\theta^{(j)})}{\sum_{j=1}^{M}h(\theta^{(j)})}.$$

The Bayes estimate of $\hat{u}(\theta,\lambda',\beta)$ with reference to Remark 1 can be obtained as

$$\hat{u'}(\theta,\lambda',\beta)=\frac{\sum_{j=1}^{M}u(\theta^{(j)},\lambda'^{(j)},\beta^{(j)})h(\theta^{(j)})}{\sum_{j=1}^{M}h(\theta^{(j)})},$$

where

$$\lambda'^{(j)}=\frac{\lambda^{(j)}}{a^{\theta(j)}};\quad j=1,2,3,\ldots,M.$$

The HPD credible interval for $u(\theta,\lambda,\beta)$ can be obtained using the above generated importance sampling procedure as follows: Let $w_j=\frac{h(\theta^{(j)})}{h(\sum_{j=1}^{M}\theta^{(j)})}$ and $u_j=u(\theta^{(j)},\lambda^{(j)},\beta^{(j)})$. Arrange (u_1,w_1),$\cdots$, (u_M,w_M) as $(u_{(1)},w_{(1)})$,$\ldots$, $(u_{(M)},w_{(M)})$, where $u_{(1)}\le\cdots\le u_{(M)}$ and $w_{(j)}$'s are not ordered but are associated with $u_{(j)}$'s. Construct all the $100(1-\alpha)\%$ credible intervals for $u(\theta,\lambda,\beta)$ as $\left(u_{[M_k]},u_{[M_{k+1-\alpha}]}\right)$ for $k=w(1),w(1)+w(2),\cdots,\sum_{j=1}^{M_{1-\alpha}}w_{(j)}$, where $[M_p]$ is the integer satisfying $\sum_{j=1}^{M_p}w_{(j)}\le p<\sum_{j=1}^{M_p+1}w_{(j)}$. Now the HPD credible interval for $u(\theta,\lambda,\beta)$ is the interval which has the shortest length.

5. SIMULATION

In this section we perform a simulation study to observe the behaviour of Bayes estimators and to compare with the ML estimators for different sample sizes, different parameters values and for different censoring rates. To control the non-censoring rate to some extent, we used the following mechanism. The generated sample is accepted only if $r-2\le\sum_{i=1}^{n}d_i\le r+2$ for $n=20$, $r-4\le\sum_{i=1}^{n}d_i\le r+4$ for $n=40$ and if $r-6\le\sum_{i=1}^{n}d_i\le r+6$ for $n=60$, where $r=n*p$ and $p=(1+\beta)^{(-1)}$, otherwise rejected. We compute the average values of Bayes estimates and corresponding mean square errors, ML estimates and corresponding mean square errors, average lengths of 95% confidence/credible intervals and corresponding coverage percentages based on 1000 importance samples. The results are reported in Tables 1 - 6. It is observed that as the sample size increases the biases, MSEs and lengths of confidence intervals of the estimators decrease. The Bayes estimators perform slightly better than the ML estimators for small sample sizes and for large sample sizes their behaviour is approximately similar. The Bayes estimators of the shape and scale parameters θ and λ perform better than the corresponding ML estimators in terms of biases and MSEs. However, the ML estimator of censoring parameter β performs better than the corresponding Bayes estimators. It is further seen that the Bayes estimators of θ and λ based on reference prior perform slightly better than the corresponding Bayes estimators based on Jeffreys prior in terms of biases and MSEs. However, the Bayes estimator of the censoring parameter β based on Jeffreys prior performs better than the corresponding Bayes estimator based on reference prior. When comparing the estimators based on the lengths of confidence/credible intervals and the corresponding coverage percentages, it is noted that the coverage percentages depend on the corresponding lengths in the sense that greater the length, the better the coverage percentage. It is very difficult to draw a general rule regarding the lengths and coverage percentages as Bayes estimators and ML estimators have different behaviour. However, it can be said that ML estimators have lager confidence interval lengths with higher coverage and Bayes estimators have smaller credible interval lengths with smaller coverage.

The same model is analysed in Ajmal et al. [2]; however, the paper suffers from the following two major faults pointed out by the peers. First, the data analysed by the authors clearly do not satisfy the assumptions of the model, namely proportionality of censoring; this violation is eminent in Chemotherapy and Radiation group where censored observations are fully concentrated on the right tale. Second, the authors investigated the appropriateness of the model using the goodness-of-fit test proposed by Hollander and Proschan [12]. This test certainly does not test the fit with proportionality assumption (that most likely does not apply). But nor this test can be employed to verify assumption on the Weibull distribution of the lifetimes only, because the data are used twice (once when fitting the model and once in the test) while the test is designed with the model distribution fixed, not estimated. Hence resulting p-values are wrong.

6. REAL DATA ANALYSIS

In this section, we analyze a real data set from [9]. The data belongs to Group IV of the Primary Biliary Cirrhosis (PBC) liver study conducted by Mayo Clinic. The event of interest is the time to death of PBC patients. The data on the survival times (in days) of 36 patients who had the highest category of bilirubin are 400, 77, 859, 71, 1037, 1427, 733, 334, 41, 51, 549, 1170, 890, 1413, 853, 216, 1882^+, 1067^+, 131, 223, 1827, 2540, 1297, 264, 797, 930, 1329^+, 264, 1350, 1191, 130, 943, 974, 790, 1765^+, 1320^+. The observations with '+' indicate censored times. For computational ease, each observation is divided by 1000. To move further, we first apply the goodness-of-fit test proposed by Henze [11] to check whether the data at hand can be analyze or not by assuming the proportional hazards model of random censorship. The test is based on the number of runs (R_n), sample size and the number of uncensored observations (N_n). The p-value of the test is given by

$$p(j,l) = P[R_n \le j | \min(N_n, n - N_n) = l],$$

where

$$P[R_n = 2s | N_n = k] = \frac{2\binom{k-1}{s-1}\binom{n-k-1}{s-1}}{\binom{n}{k}}$$

and

$$P[R_n = 2s+1 | N_n = k] = \frac{\binom{k-1}{s-1}\binom{n-k-1}{s-1} + \binom{k-1}{s}\binom{n-k-1}{s-1}}{\binom{n}{k}},$$

see [11] for further detail. In our case $n = 36$, $n - N_n = 5$, $R_n = 9$ and

$$p(9,5) = P[R_n \le 9 | (n - N_n) = 5] = \frac{20n(n^3 - 15n^2 + 80n - 150)}{n(n-1)(n-2)(n-3)(n-4)} = 0.4766.$$

Based on the observed p-value, we can say that the proportional hazards model holds. Now to fit the proposed model we compute the ML estimates and the Bayes estimates under Jeffreys and reference priors and the results are reported in Table 7. To further investigate the goodness of fit of the Weibull random censorship model, we plot the Kaplan–Meier empirical survival estimates and the fitted survival function in Figure 1. The figure shows a good agreement between the Kaplan–Meier survival curve and the fitted Weibull survival functions.

Method	n	$\hat{\theta}$	Length	$\hat{\lambda}$	Length	$\hat{\beta}$	Length
ML	20	0.5400	0.3736	1.0890	1.3937	1.1327	2.0669
		(0.0129)	(95)	(0.1705)	(95)	(0.3948)	(93)
Jeffreys		0.5304	0.3202	1.0651	1.1885	1.262	2.0558
		(0.0115)	(90)	(0.1434)	(92)	(0.5913)	(92)
Reference		0.5301	0.3182	1.0385	1.1784	1.3559	2.2638
		(0.0111)	(90)	(0.1327)	(91)	(0.8504)	(92)
ML	40	0.5179	0.2518	1.0341	0.9263	1.0687	1.3452
		(0.0048)	(95)	(0.0651)	(95)	(0.1312)	(93)
Jeffreys		0.509	0.2013	1.0223	0.8058	1.1255	1.2615
		(0.0043)	(87)	(0.0604)	(92)	(0.1638)	(91)
Reference		0.5087	0.2011	1.0063	0.7937	1.1557	1.309
		(0.0043)	(86)	(0.0574)	(91)	(0.1814)	(92)
ML	60	0.5125	0.2031	1.0298	0.7479	1.0319	0.0551
		(0.0029)	(96)	(0.0405)	(95)	(0.0821)	(95)
Jeffreys		0.5045	0.1518	1.0202	0.6306	1.0681	0.9495
		(0.0028)	(84)	(0.0379)	(90)	(0.0923)	(88)
Reference		0.5032	0.1522	1.0069	0.6237	1.0901	0.9717
		(0.0026)	(82)	(0.0353)	(89)	(0.107)	(90)

Tab. 1. Average estimates and corresponding MSEs (in parentheses), average 95% confidence/credible interval lengths and corresponding coverage percentages (in parentheses) when $\theta = 0.5$, $\lambda = 1$ and $\beta = 1$.

Method	n	$\hat{\theta}$	Length	$\hat{\lambda}$	Length	$\hat{\beta}$	Length
ML	20	1.6280	1.1276	1.0890	1.3969	1.1231	2.0401
		(0.1128)	(95)	(0.1686)	(94)	(0.3423)	(95)
Jeffreys		1.5920	0.9541	1.0592	1.1909	1.2512	2.0514
		(0.0963)	(89)	(0.1268)	(91)	(0.5251)	(93)
Reference		1.5944	0.9560	1.0360	1.1841	1.3410	2.2380
		(0.0938)	(89)	(0.1297)	(90)	(0.7025)	(94)
ML	40	1.5541	0.7555	1.0380	0.9258	1.0575	1.3319
		(0.0398)	(96)	(0.0650)	(95)	(0.1359)	(92)
Jeffreys		1.5303	0.6120	1.0208	0.7972	1.1155	1.2599
		(0.0380)	(88)	(0.0573)	(91)	(0.1696)	(91)
Reference		1.5290	0.6070	1.0100	0.8011	1.1453	1.3025
		(0.0363)	(88)	(0.0571)	(90)	(0.1885)	(92)
ML	60	1.5453	0.6118	1.0318	0.7502	1.0340	1.0571
		(0.0284)	(96)	(0.0404)	(95)	(0.0811)	(96)
Jeffreys		1.5163	0.4512	1.0196	0.6330	1.0678	0.9514
		(0.0268)	(82)	(0.0372)	(90)	(0.0958)	(86)
Reference		1.5154	0.4520	1.0095	0.6322	1.0929	0.9779
		(0.0256)	(80)	(0.0365)	(90)	(0.1097)	(90)

Tab. 2. Average estimates and corresponding MSEs (in parentheses), average 95% confidence/credible interval lengths and corresponding coverage percentages (in parentheses) when $\theta = 1.5$, $\lambda = 1$ and $\beta = 1$.

Method	n	$\hat{\theta}$	Length	$\hat{\lambda}$	Length	$\hat{\beta}$	Length
ML	20	2.6929	1.8601	1.0943	1.3889	1.0985	1.9936
		(0.2911)	(96)	(0.1642)	(95)	(0.3211)	(95)
Jeffreys		2.6479	1.6051	1.0738	1.1996	1.2242	1.9967
		(0.2570)	(91)	(0.1440)	(91)	(0.4795)	(93)
Reference		2.6463	1.6036	1.0414	1.1895	1.3106	2.1673
		(0.2501)	(92)	(0.1293)	(90)	(0.6270)	(93)
ML	40	2.5927	01.2606	1.0467	0.9383	1.0701	1.3487
		(0.1257)	(94)	(0.0671)	(94)	(0.1437)	(93)
Jeffreys		2.5479	0.9969	1.0325	0.8090	1.1271	1.2668
		(0.1148)	(85)	(0.0607)	(90)	(0.1746)	(91)
Reference		2.5441	0.9970	1.0148	0.8041	1.1610	1.3244
		(0.1152)	(85)	(0.0565)	(90)	(0.2010)	(93)
ML	60	2.5554	1.0133	1.0333	0.7497	1.0240	1.0458
		(0.0778)	(94)	(0.0396)	(95)	(0.0713)	(95)
Jeffreys		2.5031	0.7416	1.0219	0.6322	1.0546	0.9441
		(0.0742)	(82)	(0.0380)	(90)	(0.0813)	(89)
Reference		2.5014	0.7497	1.0115	0.6373	1.0757	0.9698
		(0.0682)	(80)	(0.0352)	(92)	(0.0850)	(91)

Tab. 3. Average estimates and corresponding MSEs (in parentheses), average 95% confidence/credible interval lengths and corresponding coverage percentages (in parentheses) when $\theta = 2.5$, $\lambda = 1$ and $\beta = 1$.

Method	n	$\hat{\theta}$	Length	$\hat{\lambda}$	Length	$\hat{\beta}$	Length
ML	20	0.5391	0.3735	1.0689	1.1595	0.5364	0.9995
		(0.0114)	(96)	(0.1216)	(94)	(0.0789)	(94)
Jeffreys		0.5358	0.3486	1.0448	1.0608	0.6002	1.0014
		(0.0111)	(92)	(0.1063)	(92)	(0.1047)	(92)
Reference		0.5353	0.3494	1.0323	1.0626	1.6289	1.0584
		(0.0110)	(92)	(0.1081)	(92)	(0.1272)	(92)
ML	40	0.5176	0.2519	1.0346	0.7908	0.5251	0.6889
		(0.0047)	(95)	(0.0474)	(94)	(0.0322)	(95)
Jeffreys		0.5160	0.2385	1.0225	0.7424	0.5558	0.6763
		(0.0047)	(92)	(0.0461)	(92)	(0.0383)	(94)
Reference		0.5155	0.2385	1.0152	0.7450	0.5664	0.6907
		(0.0046)	(93)	(0.0431)	(93)	(0.0410)	(94)
ML	60	0.5109	0.2027	1.0197	0.6340	0.5133	0.5506
		(0.0028)	(96)	(0.0265)	(96)	(0.0191)	(94)
Jeffreys		0.5096	0.1939	1.0121	0.6063	0.5332	0.5380
		(0.0028)	(94)	(0.0256)	(94)	(0.0213)	(96)
Reference		0.5094	0.1940	1.0077	0.6039	0.5401	0.5479
		(0.0028)	(94)	(0.0250)	(94)	(0.0224)	(96)

Tab. 4. Average estimates and corresponding MSEs (in parentheses), average 95% confidence/credible interval lengths and corresponding coverage percentages (in parentheses) when $\theta = 0.5$, $\lambda = 1$ and $\beta = 0.5$.

Method	n	$\hat{\theta}$	Length	$\hat{\lambda}$	Length	$\hat{\beta}$	Length
ML	20	1.0807	0.7482	1.0595	1.1500	0.5449	1.0130
		(0.0481)	(95)	(0.1093)	(94)	(0.0799)	(94)
Jeffreys		1.0738	0.7012	1.0383	1.0585	0.60945	1.0170
		(0.0473)	(91)	(0.1012)	(91)	(0.1063)	(93)
Reference		1.0751	0.6994	1.0223	1.0498	0.6371	1.0725
		(0.0480)	(91)	(0.0966)	(91)	(0.1273)	(93)
ML	40	1.0305	0.5008	1.0313	0.7845	0.5110	0.6728
		(0.0180)	(95)	(0.0379)	(95)	(0.0277)	(96)
Jeffreys		1.0276	0.4784	1.0197	0.7422	0.5402	0.6623
		(0.0184)	(93)	(0.0359)	(94)	(0.0320)	(95)
Reference		1.0267	0.4768	1.0135	0.7444	0.5511	0.6788
		(0.0180)	(93)	(0.0360)	(94)	(0.0344)	(94)
ML	60	1.0157	0.4032	1.0183	0.6329	0.5132	0.5508
		(0.0118)	(94)	(0.0274)	(94)	(0.0219)	(94)
Jeffreys		1.0129	0.3850	1.0112	0.6047	0.5328	0.5371
		(0.0117)	(94)	(0.0264)	(92)	(0.0241)	(94)
Reference		1.0127	0.3837	1.0074	0.6036	0.5389	0.5486
		(0.0117)	(92)	(0.0266)	(92)	(0.0255)	(94)

Tab. 5. Average estimates and corresponding MSEs (in parentheses), average 95% confidence/credible interval lengths and corresponding coverage percentages (in parentheses) when $\theta = 1$, $\lambda = 1$ and $\beta = 0.5$.

Method	n	$\hat{\theta}$	Length	$\hat{\lambda}$	Length	$\hat{\beta}$	Length
ML	20	1.6196	1.1204	1.0718	1.1713	0.5551	1.0281
		(0.1081)	(96)	(0.1203)	(94)	(0.0763)	(94)
Jeffreys		1.6131	1.0465	1.0488	1.0750	0.6191	1.0257
		(0.1200)	(92)	(0.1083)	(92)	(0.1014)	(94)
Reference		1.6150	1.0468	1.0314	1.0628	0.6484	1.0826
		(0.1170)	(92)	(0.1027)	(91)	(0.1217)	(94)
ML	40	1.5609	0.7603	1.0406	0.7927	0.5137	0.6762
		(0.0457)	(95)	(0.0474)	(95)	(0.0319)	(94)
Jeffreys		1.0276	0.4784	1.0197	0.7422	0.5402	0.6623
		(0.0184)	(93)	(0.0359)	(94)	(0.0320)	(95)
Reference		1.5558	0.7164	1.0204	0.7450	0.5553	0.6806
		(0.0463)	(92)	(0.0429)	(93)	(0.0397)	(93)
ML	60	1.5395	0.6100	1.0225	0.6364	0.5172	0.5542
		(0.0251)	(96)	(0.0302)	(94)	(0.0209)	(94)
Jeffreys		1.5374	0.5827	1.0145	0.6073	0.5376	0.5450
		(0.0247)	(92)	(0.0285)	(92)	(0.0232)	(94)
Reference		1.5349	0.5836	1.0094	0.6081	0.5442	0.5523
		(0.0247)	(94)	(0.0282)	(92)	(0.0246)	(94)

Tab. 6. Average estimates and corresponding MSEs (in parentheses), average 95% confidence/credible interval lengths and corresponding coverage percentages (in parentheses) when $\theta = 1.5$, $\lambda = 1$ and $\beta = 0.5$.

Method	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\lambda}'$
ML	1.296	0.946	0.161	0.000
	5	5	3	122
Jeffreys	1.298	0.929	0.181	0.000
	6	3	1	229
Reference	1.298	0.934	0.180	0.000
	1	5	0	226

Tab. 7. The Bayesian and classical estimates of the parameteres for the real data.

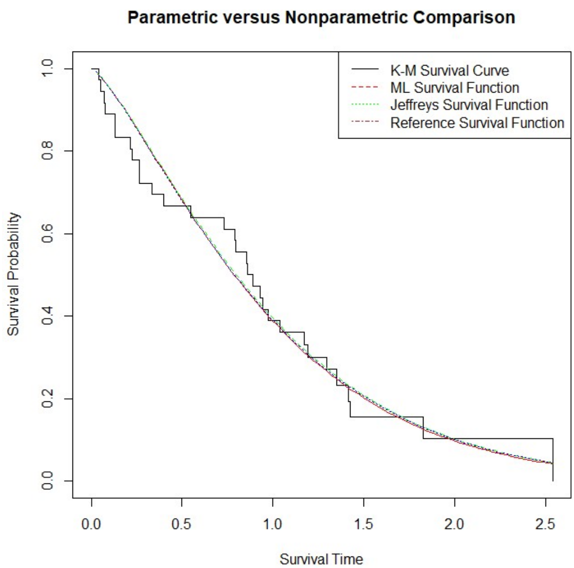

Fig. 1. Kaplan–Meier survival curve and the fitted survival functions for the real data set.

APPENDIX

R code

Generate sample of size n as

```
for (i in 1: n)
{
x[i] < − rweibull(1, shape=theta, scale=1/lambda ˆ (1/theta))
t[i] < − rweibull(1, shape=theta, scale=1/(beta*lambda) (1/theta))
d[i] < − if(x[i]¡t[i]) 1 else 0
y[i] < − d[i]*x[i]+(1-d[i])*t[i]
}
sd < − sum(d)
```

```
sly < − sum(-log(y))
For ML estimation, use R function to obtain MLEs
fn < − function(p) - (n*log(p[1]) + n*log(p[2])+(n-sum(d))
*log(p[3])- p[2]*(1+p[3])*sum(y p[1])+(p[1]-1)*sum(log(y)))
out < − nlm(fn, p = c(1.5, 0.8, 0.7), hessian = TRUE)
MLEs < − out estimate
For importance samples, use the algorithm
library(GB2)
for (k in 1:1000)
{
p3[k] < − rbeta(1, shape1=(b+sd-0.5), shape2=(n-sd+0.5))
p1[k] < − rgamma(1, shape=n, scale=1/sly)
w[k] < − sum(y p1[k])
p2[k] < −rgamma(1, shape=n, scale=1/((1+p3[k])*w[k]))
wf[k] < − (w[k]) -n
}
Obtain the Bayes estimates from
Btheta < − sum(p1*wf)/sum(wf)
Blambda < − sum(p2*wf)/sum(wf)
Bbeta < − sum(p3*wf)/sum(wf)
```

End of R code

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