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CONTINUOUS DEPENDENCE OF 2D LARGE SCALE PRIMITIVE EQUATIONS ON THE BOUNDARY CONDITIONS IN OCEANIC DYNAMICS

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Abstract. In this paper, we consider an initial boundary value problem for the two-dimensional primitive equations of large scale oceanic dynamics. Assuming that the depth of the ocean is a positive constant, we establish rigorous a priori bounds of the solution to problem. With the aid of these a priori bounds, the continuous dependence of the solution on changes in the boundary terms is obtained.

Keywords: a priori bounds; primitive equation; continuous dependence

MSC 2020: 35B40, 35Q30, 76D05

1. Introduction

The primitive equations are very useful models which often are used to study climate and weather prediction. In the 1990s, Lions, Temam and Wang (see [16]–[19]) first started mathematical study of the primitive equations of the atmosphere, the ocean and the coupled atmosphere-ocean. Assuming that all unknown functions are independent of the latitude y, Petcu et al. [23] obtained the two-dimensional primitive equations of the ocean from the three-dimensional primitive equations. The existence and uniqueness of strong solutions of the primitive equations were derived. In the following paper, Huang and Guo [9] considered the two-dimensional primitive equations of large scale oceanic motion. They obtained the existence and uniqueness of global strong solutions. Huang et al. [11] studied the two-dimensional primitive equations of large scale ocean in geophysics driven by degenerate noise. They proved

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the asymptotically strong Feller property of the probability transition semigroups. Due to the importance of primitive equations, there are many papers to study the problems; see, e.g., [8], [27], [12], [4], [26], [25], [5] and the references therein.

Recently, the structural stability of large-scale primitive equations has started to be considered. Li [14] obtained continuous dependence on the viscosity coefficient of the solution of three-dimensional viscous primitive equations of the ocean. By using the energy analysis methods, Li [13] proved that primitive equations of the coupled atmosphere-ocean are continuously dependent on the boundary parameters. The inspiration for the study came from the fluid equations. There are a lot of articles in the literature to study the stability of fluid equations (for example, see [20], [21], [24], [15], [6], [1], [2], [3]). In this paper we consider the following two-dimensional large-scale primitive equations (for details, see [9]):

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x_1}+w\frac{\partial u}{\partial x_2}-\frac{1}{\varepsilon'}v+\frac{1}{\varepsilon'}\frac{\partial p}{\partial x_1}=\gamma_1\Delta u,\\
&\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x_1}+w\frac{\partial v}{\partial x_2}+\frac{1}{\varepsilon'}u=\gamma_2\Delta v,\\
&\frac{\partial p}{\partial x_2}+\varrho=0,\\
&\frac{\partial u}{\partial x_1}+\frac{\partial w}{\partial x_2}=0,\\
&\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x_1}+w\frac{\partial T}{\partial x_2}=\gamma_3\Delta T,\\
&\varrho=\varrho_{\mathrm{ref}}(1-\beta_T(T-T_{\mathrm{ref}})).
\end{aligned}\tag{1.1}
$$

The domain is defined as

$$\Omega=(0,1)\times(-h,0),$$

where h is a positive constant. In (1.1) the unknown functions (u,v), w, ϱ, p, and T are the horizontal velocity field, and the vertical velocity, the density, the pressure, and the temperature, respectively, ε' is the Rossby number and $\gamma_i>0$ $(i=1,2,3)$ are the viscosity coefficients, ϱ_{ref}, T_{ref} are the reference values of the density and temperature, β_T is the expansion coefficient (constant), $\Delta=\partial_{x_1}^2+\partial_{x_2}^2$. We observe that in the case of ocean dynamics one has to add the diffusion-transport equation of the salinity to the system (1.1). The salinity equation is not present in (1.1), but it would raise little the additional difficulty to take into account the salinity.

The boundary of Ω is denoted by $\partial\Omega$ and can be partitioned into

$$
\begin{aligned}
\Gamma_0&=\{(x_1,x_2)\in\overline{\Omega}\colon\ 0<x_1<1,\ x_2=0\},\\
\Gamma_{-h}&=\{(x_1,x_2)\in\overline{\Omega}\colon\ 0<x_1<1,\ x_2=-h\},\\
\Gamma_s&=\{(x_1,x_2)\in\overline{\Omega}\colon\ x_1=0,\ \text{or } x_1=1,\ -h\leqslant x_2\leqslant 0\}.
\end{aligned}
$$

The system (1.1) also has the following boundary conditions:

$$
\begin{aligned}
(1.2)\quad & \frac{\partial u}{\partial x_2} = \alpha_1 g_1(x_1,t),\ \frac{\partial v}{\partial x_2} = \alpha_2 g_2(x_1,t),\ w=0,\ \frac{\partial T}{\partial x_2} = \beta\tau(x_1,t) \quad \text{on } \Gamma_0, \\
& \frac{\partial u}{\partial x_2} = \frac{\partial v}{\partial x_2} = 0,\ w=0,\ \frac{\partial T}{\partial x_2} = 0 \quad \text{on } \Gamma_{-h}, \\
& u = v = w = 0,\ \frac{\partial T}{\partial x_1} = 0 \quad \text{on } \Gamma_s,
\end{aligned}
$$

where $g_1(x_1,t)$, $g_2(x_1,t)$ are the wind stresses on the ocean surface, $\alpha_1, \alpha_2, \beta$ are positive constants, and $\tau(x_1,t)$ is the typical temperature distribution of the top surface of the ocean. The functions $g_1(x_1,t)$, $g_2(x_1,t)$, and $\tau(x_1,t)$ also satisfy the compatibility boundary conditions

$$
(1.3)\qquad g_1(0,t) = g_2(0,t) = \tau(0,t) = g_1(1,t) = g_2(1,t) = \tau(1,t) = 0.
$$

In addition, the initial conditions can be written as

$$
\begin{aligned}
(1.4)\qquad & u(x_1,x_2,0) = u_0(x_1,x_2), \quad v(x_1,x_2,0) = v_0(x_1,x_2), \\
& T(x_1,x_2,0) = T_0(x_1,x_2) \quad \text{in } \Omega.
\end{aligned}
$$

The aim of this paper is to show the continuous dependence of the solutions on changes in the boundary conditions of the problem (1.1)–(1.4). It is very important to know whether a small change in the equation can cause a large change in the solutions. While we take the advantage of mathematical analysis to study these equations, it is helpful for us to know their applicability in physics. Since some inevitable errors will appear in reality, the study of continuous dependence or convergence result becomes more and more significant. It is worth stressing that the ideas developed in this paper are helpful to study other type primitive equations with other kinds of boundary conditions.

The plan of the paper is the following. In the next section we give some preliminaries of the problem and some well-known inequalities which are used in the whole paper. We establish rigorous a priori bounds of the solutions in Section 3. Finally, we show how one may derive continuous dependence on the boundary conditions of our problem in Section 4.

2. Preliminaries of the problem

We formulate the equations (1.1)–(1.4). Realizing the boundary conditions (1.2), we integrate the equation $(1.1)_4$ from $-h$ to x_2 to obtain

$$w(x_1, x_2, t) = w(x_1, -h, t) - \int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t)\, \mathrm{d}\zeta = -\frac{\partial}{\partial x_1} \int_{-h}^{x_2} u(x_1, \zeta, t)\, \mathrm{d}\zeta \tag{2.1}$$

and

$$\int_{-h}^{0} \frac{\partial}{\partial x_1} u(x_1, \zeta, t)\, \mathrm{d}\zeta = -\frac{\partial}{\partial x_1} \int_{-h}^{0} u(x_1, \zeta, t)\, \mathrm{d}\zeta = 0. \tag{2.2}$$

By integrating $(1.1)_3$ and using $(1.1)_6$ we have

$$p(x_1, x_2, t) = p_s + \varrho_{\mathrm{ref}} \int_{x_2}^{0} (1 - \beta_T(T(x_1, \zeta, t) - T_{\mathrm{ref}}))\, \mathrm{d}\zeta, \tag{2.3}$$

where $p_s = p(x_1, 0, t)$ is the pressure on the surface of the ocean.

Inserting (2.1)–(2.3) into (1.1)–(1.4) and letting $\mu = \varrho_{\mathrm{ref}}\beta_T$, our problem can be rewritten as

$$\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t)\, \mathrm{d}\zeta\right) \frac{\partial u}{\partial x_2} - \frac{1}{\varepsilon'} v \\ &\qquad + \frac{1}{\varepsilon'}\frac{\partial p_s}{\partial x_1} - \frac{\mu}{\varepsilon'}\left(\int_{x_2}^{0} \frac{\partial}{\partial x_1} T(x_1, \zeta, t)\, \mathrm{d}\zeta\right) = \gamma_1 \Delta u, \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t)\, \mathrm{d}\zeta\right) \frac{\partial v}{\partial x_2} + \frac{1}{\varepsilon'} u = \gamma_2 \Delta v, \\ &\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t)\, \mathrm{d}\zeta\right) \frac{\partial T}{\partial x_2} = \gamma_3 \Delta T, \\ &\frac{\partial}{\partial x_1} \int_{-h}^{0} u(x_1, \zeta, t)\, \mathrm{d}\zeta = 0 \end{aligned} \tag{2.4}$$

with the initial-boundary conditions

$$\begin{aligned} &\frac{\partial u}{\partial x_2}\Big|_{x_2=0} = \alpha_1 g_1(x_1, t), \quad \frac{\partial v}{\partial x_2}\Big|_{x_2=0} = \alpha_2 g_2(x_1, t), \\ &\frac{\partial u}{\partial x_2}\Big|_{x_2=-h} = \frac{\partial v}{\partial x_2}\Big|_{x_2=-h} = 0, \quad (u, v)|_{\Gamma_s} = 0, \\ &\frac{\partial T}{\partial x_2}\Big|_{x_2=0} = \beta\tau(x_1, t), \ \frac{\partial T}{\partial x_2}\Big|_{x_2=-h} = 0, \quad \frac{\partial T}{\partial x_1}\Big|_{\Gamma_s} = 0, \end{aligned} \tag{2.5}$$

$$(u, v, T)|_{t=0} = (u_0, v_0, T_0). \tag{2.6}$$

In this paper, we also use some well-known inequalities. Now we list them here.

(1) If $\omega(x_1) \in C^1(0,1)$ and $\omega(0) = \omega(1) = 0$, then

$$\int_0^1 \omega^2 \, \mathrm{d}x_1 \leqslant \frac{1}{\pi^2} \int_0^1 \Big(\frac{\partial \omega}{\partial x_1}\Big)^2 \, \mathrm{d}x_1. \tag{2.7}$$

(2) If $\omega(x_2) \in C^1(-h,0)$ and $\omega(-h) = \omega(0) = 0$, then

$$\int_{-h}^0 \omega^2 \, \mathrm{d}x_2 \leqslant \frac{h^2}{\pi^2} \int_{-h}^0 \Big(\frac{\partial \omega}{\partial x_2}\Big)^2 \, \mathrm{d}x_2. \tag{2.8}$$

For proofs of this inequalities see [7], [22].

(3) If $\omega(x_1,x_2,t)$ is a sufficiently smooth function in $\Omega = (0,1) \times (-h,0)$ and $\omega(0,x_2,t) = \omega(1,x_2,t) = 0$, then

$$\begin{aligned}\Big(\int_\Omega \omega^4 \, \mathrm{d}A\Big)^{1/2} \leqslant C\Big[&\Big(\int_\Omega \omega^2 \, \mathrm{d}A\Big)^{1/2}\Big(\int_\Omega |\nabla\omega|^2 \, \mathrm{d}A\Big)^{1/2} \\ &+ \Big(\int_\Omega \omega^2 \, \mathrm{d}A\Big)^{1/4}\Big(\int_\Omega |\nabla\omega|^2 \, \mathrm{d}A\Big)^{3/4}\Big]\end{aligned} \tag{2.9}$$

or

$$\Big(\int_\Omega \omega^4 \, \mathrm{d}A\Big)^{1/2} \leqslant C\Big[\int_\Omega \omega^2 \, \mathrm{d}A + \delta \int_\Omega |\nabla\omega|^2 \, \mathrm{d}A\Big], \tag{2.10}$$

where $\nabla = (\partial_{x_1}, \partial_{x_2})$, C is a positive computable constant and δ is an arbitrary positive constant.

P r o o f of (3). By the Hölder inequality we write

$$\int_\Omega \omega^4 \, \mathrm{d}A \leqslant \int_{-h}^0 \Big(\int_0^1 \omega^6 \, \mathrm{d}x_1\Big)^{1/2}\Big(\int_0^1 \omega^2 \, \mathrm{d}x_1\Big)^{1/2} \mathrm{d}x_2. \tag{2.11}$$

Since $\omega(0,x_2,t) = \omega(1,x_2,t) = 0$, we have

$$\omega^3 = 3\int_0^{x_1} \omega^2(\xi,x_2,t)\frac{\partial\omega(\xi,x_2,t)}{\partial\xi} \, \mathrm{d}\xi = -3\int_{x_1}^1 \omega^2(\xi,x_2,t)\frac{\partial\omega(\xi,x_2,t)}{\partial\xi} \, \mathrm{d}\xi. \tag{2.12}$$

Therefore

$$|\omega|^3 \leqslant \frac{3}{2}\int_0^1 \omega^2(x_1,x_2,t)\Big|\frac{\partial\omega(x_1,x_2,t)}{\partial x_1}\Big| \, \mathrm{d}x_1. \tag{2.13}$$

Then we have

$$\Big(\int_0^1 \omega^6 \, \mathrm{d}x_1\Big)^{1/2} \leqslant \frac{3}{2}\Big(\int_0^1 \omega^2\Big|\frac{\partial\omega}{\partial x_1}\Big| \, \mathrm{d}x_1\Big). \tag{2.14}$$

Inserting (2.14) into (2.11), we get

$$\int_\Omega \omega^4\,\mathrm{d}A \leqslant \frac{3}{2}\int_{-h}^{0}\Big(\int_0^1 \omega^2\Big|\frac{\partial\omega}{\partial x_1}\Big|\,\mathrm{d}x_1\Big)\Big(\int_0^1 \omega^2\,\mathrm{d}x_1\Big)^{1/2}\,\mathrm{d}x_2 \tag{2.15}$$
$$\leqslant \frac{3}{2}\max_{-h\leqslant x_2\leqslant 0}\Big\{\Big(\int_0^1 \omega^2(x_2)\,\mathrm{d}x_1\Big)^{1/2}\Big\}\int_\Omega \omega^2\Big|\frac{\partial\omega}{\partial x_1}\Big|\,\mathrm{d}A.$$

Obviously, we have

$$\omega^2 = 2\int_{-h}^{x_2}\omega(x_1,\zeta,t)\frac{\partial\omega(x_1,\zeta,t)}{\partial\zeta}\,\mathrm{d}\zeta + \omega^2(x_1,-h,t) \tag{2.16}$$
$$= -2\int_{x_2}^{0}\omega(x_1,\zeta,t)\frac{\partial\omega(x_1,\zeta,t)}{\partial\zeta}\,\mathrm{d}\zeta + \omega^2(x_1,0,t),$$

so

$$\omega^2 \leqslant \int_{-h}^{0}|\omega|\Big|\frac{\partial\omega}{\partial x_2}\Big|\,\mathrm{d}x_2 + \frac{1}{2}[\omega^2(x_1,0,t)+\omega^2(x_1,-h,t)]. \tag{2.17}$$

To bound the last term of (2.17) we define a new known function $f(x_2)$ satisfying

$$f(0)>0,\ f(-h)<0,\ |f'(x_2)|\leqslant m_1,\ |f(x_2)|\leqslant m_2 \quad \text{for } -h\leqslant x_2\leqslant 0, \tag{2.18}$$

where m_1, m_2 are positive constants. For example, $f(x_2) = \frac{1}{2}m_1(x_2+\frac{1}{2}h)$, $m_1 h < 4m_2$, satisfies all the conditions in (2.18). Using the above estimates and employing, the divergence theorem allow us to write

$$\min\{f(0),\,-f(-h)\}[\omega^2(x_1,0,t)+\omega^2(x_1,-h,t)] \tag{2.19}$$
$$\leqslant f(0)\omega^2(x_1,0,t) - f(-h)\omega^2(x_1,-h,t)$$
$$= \int_{-h}^{0}\frac{\partial}{\partial x_2}(f\omega^2)\,\mathrm{d}x_2 = \int_{-h}^{0} f'(x_2)\omega^2\,\mathrm{d}x_2 + 2\int_{-h}^{0} f\omega\frac{\partial\omega}{\partial x_2}\,\mathrm{d}x_2$$
$$\leqslant m_1\int_{-h}^{0}\omega^2\,\mathrm{d}x_2 + 2m_2\int_{-h}^{0}|\omega|\Big|\frac{\partial\omega}{\partial x_2}\Big|\,\mathrm{d}x_2.$$

Inserting (2.19) into (2.17), we have

$$\omega^2 \leqslant m_3\int_{-h}^{0}\omega^2\,\mathrm{d}x_2 + m_4\int_{-h}^{0}|\omega|\Big|\frac{\partial\omega}{\partial x_2}\Big|\,\mathrm{d}x_2, \tag{2.20}$$

where

$$m_3 = \frac{m_1}{2\min\{f(0),-f(-h)\}},\quad m_4 = 1+\frac{m_2}{\min\{f(0),-f(-h)\}}. \tag{2.21}$$

Therefore,

$$\max_{-h\leqslant x_2\leqslant 0}\Big\{\Big(\int_0^1 \omega^2\,\mathrm{d}x_1\Big)^{1/2}\Big\} \leqslant \Big(m_3\int_\Omega \omega^2\,\mathrm{d}A + m_4\int_\Omega |\omega|\Big|\frac{\partial\omega}{\partial x_2}\Big|\,\mathrm{d}A\Big)^{1/2}. \tag{2.22}$$

Thus, from (2.15) and (2.22), by the Hölder inequality we have

$$\int_\Omega \omega^4\,\mathrm{d}A \leqslant \frac{3}{2}\Big[m_3\int_\Omega \omega^2\,\mathrm{d}A + m_4\Big(\int_\Omega \omega^2\,\mathrm{d}A\Big)^{1/2}\Big(\int_\Omega \Big|\frac{\partial\omega}{\partial x_2}\Big|^2\,\mathrm{d}A\Big)^{1/2}\Big]^{1/2} \times \Big(\int_\Omega \omega^4\,\mathrm{d}A\Big)^{1/2}\Big(\int_\Omega \Big|\frac{\partial\omega}{\partial x_1}\Big|^2\,\mathrm{d}A\Big)^{1/2}. \tag{2.23}$$

We have after simplification

$$\Big(\int_\Omega \omega^4\,\mathrm{d}A\Big)^{1/2} \leqslant C\Big[\Big(\int_\Omega \omega^2\,\mathrm{d}A\Big)^{1/2}\Big(\int_\Omega |\nabla\omega|^2\,\mathrm{d}A\Big)^{1/2} + \Big(\int_\Omega \omega^2\,\mathrm{d}A\Big)^{1/4}\Big(\int_\Omega |\nabla\omega|^2\,\mathrm{d}A\Big)^{3/4}\Big]. \tag{2.24}$$

□

3. A priori estimates

Now we derive some a priori estimates for the solutions of (2.4)–(2.6).

3.1. Estimates for $\|T\|_2^2$ and $\|\nabla T\|_2^2$. Taking the inner product of the equation $(2.4)_3$ with T in $L^2(\Omega)$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega T^2\,\mathrm{d}A + \gamma_3\int_\Omega |\nabla T|^2\,\mathrm{d}A = \gamma_3\int_0^1 \frac{\partial T}{\partial x_2}\Big|_{x_2=0} T(x_1,0,t)\,\mathrm{d}x_1 - \int_\Omega \Big[u\frac{\partial T}{\partial x_1} - \Big(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}u(x_1,\zeta,t)\,\mathrm{d}\zeta\Big)\frac{\partial T}{\partial x_2}\Big]T\,\mathrm{d}A. \tag{3.1}$$

We put

$$\frac{\partial H}{\partial x_2} = \frac{\beta(x_2+h)}{h}\tau(x_1,t). \tag{3.2}$$

Therefore,

$$\begin{aligned}\gamma_3\int_0^1 \frac{\partial T}{\partial x_2}\Big|_{x_2=0} T(x_1,0,t)\,\mathrm{d}x_1 &= \gamma_3\int_0^1 \frac{\partial H}{\partial x_2}\Big|_{x_2=0} T(x_1,0,t)\,\mathrm{d}x_1 = \gamma_3\int_\Omega \frac{\partial}{\partial x_2}\Big(\frac{\partial H}{\partial x_2}T\Big)\,\mathrm{d}A\\ &= \gamma_3\int_\Omega \Big(\frac{\partial^2 H}{\partial x_2^2}T\Big)\,\mathrm{d}A + \gamma_3\int_\Omega \Big(\frac{\partial H}{\partial x_2}\frac{\partial T}{\partial x_2}\Big)\,\mathrm{d}A\\ &\leqslant \frac{1}{2}\gamma_3\int_\Omega \Big(\frac{\partial^2 H}{\partial x_2^2}\Big)^2\,\mathrm{d}A + \frac{1}{2}\gamma_3\int_\Omega T^2\,\mathrm{d}A + \frac{1}{2}\gamma_3\int_\Omega \Big(\frac{\partial H}{\partial x_2}\Big)^2\,\mathrm{d}A + \frac{1}{2}\gamma_3\int_\Omega \Big(\frac{\partial T}{\partial x_2}\Big)^2\,\mathrm{d}A.\end{aligned} \tag{3.3}$$

Integrating by parts we have

$$-\int_\Omega \left[u\frac{\partial T}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1}u(x_1,\zeta,t)\,\mathrm{d}\zeta\right)\frac{\partial T}{\partial x_2}\right]T\,\mathrm{d}A = 0. \tag{3.4}$$

By the above results we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega T^2\,\mathrm{d}A + \gamma_3\int_\Omega |\nabla T|^2\,\mathrm{d}A \leqslant \gamma_3\int_\Omega T^2\,\mathrm{d}A + a_1(t), \tag{3.5}$$

where

$$a_1(t) = \gamma_3\int_\Omega \left(\frac{\partial^2 H}{\partial x_2^2}\right)^2\mathrm{d}A + \gamma_3\int_\Omega\left(\frac{\partial H}{\partial x_2}\right)^2\mathrm{d}A. \tag{3.6}$$

Using the inequality (3.5) and the Gronwall inequality, we get

$$\int_\Omega T^2\,\mathrm{d}A \leqslant \int_\Omega T_0^2\,\mathrm{d}A\cdot \mathrm{e}^{\gamma_3 t} + \int_0^t a_1(\eta)\mathrm{e}^{\gamma_3(t-\eta)}\,\mathrm{d}\eta \doteq a_2(t). \tag{3.7}$$

Moreover, we have

$$\int_0^t\int_\Omega |\nabla T|^2\,\mathrm{d}A\,\mathrm{d}\eta \leqslant F_1(t), \tag{3.8}$$

where

$$F_1(t) = \int_0^t\left[\frac{1}{\gamma_3}a_1(\eta) + a_2(\eta)\right]\mathrm{d}\eta + \frac{1}{\gamma_3}\int_\Omega T_0^2\,\mathrm{d}A. \tag{3.9}$$

3.2. Estimates for $\|u\|_2^2$ and $\|v\|_2^2$. Taking the inner product of the equation $(2.4)_1$ with u in $L^2(\Omega)$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\int_\Omega u^2\,\mathrm{d}A + \gamma_1\int_\Omega |\nabla u|^2\,\mathrm{d}A \\ &= \gamma_1\int_0^1 \frac{\partial u}{\partial x_2}\Big|_{x_2=0} u(x_1,0,t)\,\mathrm{d}x_1 - \frac{1}{\varepsilon'}\int_\Omega uv\,\mathrm{d}A \\ &\quad - \int_\Omega\left[u\frac{\partial u}{\partial x_1} - \left(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}u(x_1,\zeta,t)\,\mathrm{d}\zeta\right)\frac{\partial u}{\partial x_2}\right]u\,\mathrm{d}A \\ &\quad - \frac{1}{\varepsilon'}\int_\Omega \frac{\partial p_s}{\partial x_1}u\,\mathrm{d}A + \frac{\mu}{\varepsilon'}\int_\Omega\left(\int_{x_2}^0\frac{\partial}{\partial x_1}T(x_1,\zeta,t)\,\mathrm{d}\zeta\right)u\,\mathrm{d}A.\end{aligned} \tag{3.10}$$

We define a function $S_1(x_1,x_2,t)$ as

$$\frac{\partial S_1}{\partial x_2} = \frac{\alpha_1(x_2+h)}{h}g_1(x_1,t). \tag{3.11}$$

Obviously, S_1 fulfills the same boundary conditions as u. Therefore,

(3.12)

$$\begin{aligned}
&\gamma_1 \int_0^1 \frac{\partial u}{\partial x_2}\Big|_{x_2=0} u(x_1, 0, t)\,\mathrm{d}x_1 \\
&\quad = \gamma_1 \int_0^1 \frac{\partial S_1}{\partial x_2}\Big|_{x_2=0} u(x_1, 0, t)\,\mathrm{d}x_1 = \gamma_1 \int_\Omega \frac{\partial}{\partial x_2}\Big(\frac{\partial S_1}{\partial x_2} u\Big)\,\mathrm{d}A \\
&\quad = \gamma_1 \int_\Omega \frac{\partial^2 S_1}{\partial x_2^2} u\,\mathrm{d}A + \gamma_1 \int_\Omega \frac{\partial S_1}{\partial x_2}\frac{\partial u}{\partial x_2}\,\mathrm{d}A \\
&\quad \leqslant \frac{1}{2}\gamma_1 \int_\Omega \Big(\frac{\partial^2 S_1}{\partial x_2^2}\Big)^2\,\mathrm{d}A + \frac{1}{2}\gamma_1 \int_\Omega u^2\,\mathrm{d}A + \frac{1}{2}\gamma_1 \int_\Omega \Big(\frac{\partial S_1}{\partial x_2}\Big)^2\,\mathrm{d}A + \frac{1}{2}\gamma_1 \int_\Omega \Big(\frac{\partial u}{\partial x_2}\Big)^2\,\mathrm{d}A.
\end{aligned}$$

An integration leads to

$$(3.13)\qquad -\int_\Omega \Big[u\frac{\partial u}{\partial x_1} - \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t)\,\mathrm{d}\zeta\Big)\frac{\partial u}{\partial x_2}\Big] u\,\mathrm{d}A = 0.$$

Integrating by parts and realizing (2.2), we get

$$\begin{aligned}
(3.14)\qquad -\frac{1}{\varepsilon'}\int_\Omega \frac{\partial p_s}{\partial x_1} u\,\mathrm{d}A &= -\frac{1}{\varepsilon'}\int_0^1 \frac{\partial p_s}{\partial x_1}\Big(\int_{-h}^0 u\,\mathrm{d}x_2\Big)\,\mathrm{d}x_1 \\
&= \frac{1}{\varepsilon'}\int_0^1 p_s\Big(\int_{-h}^0 \frac{\partial u}{\partial x_1}\,\mathrm{d}x_2\Big)\,\mathrm{d}x_1 = 0.
\end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
(3.15)\quad \frac{\mu}{\varepsilon'}\int_\Omega\Big(\int_{x_2}^0 \frac{\partial}{\partial x_1} T(x_1, \zeta, t)\,\mathrm{d}\zeta\Big) u\,\mathrm{d}A &\leqslant \frac{h\mu}{\varepsilon'}\Big[\int_\Omega\Big(\frac{\partial T}{\partial x_1}\Big)^2\,\mathrm{d}A\Big]^{1/2}\Big[\int_\Omega u^2\,\mathrm{d}A\Big]^{1/2} \\
&\leqslant \frac{h\mu}{2\varepsilon'}\int_\Omega\Big(\frac{\partial T}{\partial x_1}\Big)^2\,\mathrm{d}A + \frac{h\mu}{2\varepsilon'}\int_\Omega u^2\,\mathrm{d}A.
\end{aligned}$$

By the above results we get

$$\begin{aligned}
(3.16)\quad &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2\,\mathrm{d}A + \gamma_1\int_\Omega |\nabla u|^2\,\mathrm{d}A \\
&\qquad \leqslant -\frac{2}{\varepsilon'}\int_\Omega uv\,\mathrm{d}A + \Big(\gamma_1 + \frac{h\mu}{\varepsilon'}\Big)\int_\Omega u^2\,\mathrm{d}A \\
&\qquad\quad + \frac{h\mu}{\varepsilon'}\int_\Omega\Big(\frac{\partial T}{\partial x_1}\Big)^2\,\mathrm{d}A + \gamma_1\int_\Omega\Big(\frac{\partial^2 S_1}{\partial x_2^2}\Big)^2\,\mathrm{d}A + \gamma_1\int_\Omega\Big(\frac{\partial S_1}{\partial x_2}\Big)^2\,\mathrm{d}A.
\end{aligned}$$

Similarly, we can have from $(2.4)_2$

$$\begin{aligned}
(3.17)\quad &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^2\,\mathrm{d}A + \gamma_2\int_\Omega |\nabla v|^2\,\mathrm{d}A \\
&\qquad \leqslant \frac{2}{\varepsilon'}\int_\Omega uv\,\mathrm{d}A + \gamma_2\int_\Omega v^2\,\mathrm{d}A + \gamma_2\int_\Omega\Big(\frac{\partial^2 S_2}{\partial x_2^2}\Big)^2\,\mathrm{d}A + \gamma_2\int_\Omega\Big(\frac{\partial S_2}{\partial x_2}\Big)^2\,\mathrm{d}A,
\end{aligned}$$

where

$$\frac{\partial S_2}{\partial x_2} = \frac{\alpha_1(x_2+h)}{h} g_2(x_1,t). \tag{3.18}$$

Combing (3.16) and (3.17), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega u^2\,\mathrm{d}A + \int_\Omega v^2\,\mathrm{d}A\Big) + \gamma_1\int_\Omega |\nabla u|^2\,\mathrm{d}A + \gamma_2\int_\Omega |\nabla v|^2\,\mathrm{d}A \\ \leqslant m_1\Big(\int_\Omega u^2\,\mathrm{d}A + \int_\Omega v^2\,\mathrm{d}A\Big) + \frac{h\mu}{\varepsilon'}\int_\Omega\Big(\frac{\partial T}{\partial x_1}\Big)^2\,\mathrm{d}A + a_3(t),\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}a_3(t) = \gamma_1\int_\Omega\Big(\frac{\partial^2 S_1}{\partial x_2^2}\Big)^2\,\mathrm{d}A + \gamma_1\int_\Omega\Big(\frac{\partial S_1}{\partial x_2}\Big)^2\,\mathrm{d}A \\ + \gamma_2\int_\Omega\Big(\frac{\partial^2 S_2}{\partial x_2^2}\Big)^2\,\mathrm{d}A + \gamma_2\int_\Omega\Big(\frac{\partial S_2}{\partial x_2}\Big)^2\,\mathrm{d}A.\end{aligned} \tag{3.20}$$

By the Gronwall inequality and (3.8) we get

$$\int_\Omega u^2\,\mathrm{d}A + \int_\Omega v^2\,\mathrm{d}A \leqslant F_2(t), \tag{3.21}$$

where

$$F_2(t) = \frac{h\mu}{\varepsilon'}\mathrm{e}^t F_1(t) + \int_0^t a_3(\eta)\mathrm{e}^{m_1(t-\eta)}\,\mathrm{d}\eta + \mathrm{e}^t\Big(\int_\Omega u_0^2\,\mathrm{d}A + \int_\Omega v_0^2\,\mathrm{d}A\Big). \tag{3.22}$$

Moreover, we have

$$\gamma_1\int_0^t\int_\Omega |\nabla u|^2\,\mathrm{d}A\,\mathrm{d}\eta + \gamma_2\int_0^t\int_\Omega |\nabla v|^2\,\mathrm{d}A\,\mathrm{d}\eta \leqslant F_3(t), \tag{3.23}$$

where

$$F_3(t) = m_1\int_0^t F_2(\eta)\,\mathrm{d}\eta + \frac{h\mu}{\varepsilon'}F_1(t) + \int_0^t a_3(\eta)\,\mathrm{d}\eta + \int_\Omega u_0^2\,\mathrm{d}A + \int_\Omega v_0^2\,\mathrm{d}A. \tag{3.24}$$

3.3. Estimate for $|T|$. We multiply $(2.4)_3$ by T^{p-1} and integrate by parts to find

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega T^p\,\mathrm{d}A + \frac{p-1}{p^2}\gamma_3\int_\Omega |\nabla T^{p/2}|^2\,\mathrm{d}A \\ = \gamma_3\int_0^1 \frac{\partial T}{\partial x_2}T^{p-1}\,\mathrm{d}x_1 - \int_\Omega\Big[u\frac{\partial T}{\partial x_1} - \Big(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}u(x_1,\zeta,t)\,\mathrm{d}\zeta\Big)\frac{\partial T}{\partial x_2}\Big]T^{p-1}\,\mathrm{d}A.\end{aligned} \tag{3.25}$$

After integrating by parts the second term of (3.25) and realizing the boundary condition (2.5), we get

$$-\int_\Omega\Big[u\frac{\partial T}{\partial x_1}-\Big(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}u(x_1,\zeta,t)\,\mathrm{d}\zeta\Big)\frac{\partial T}{\partial x_2}\Big]T^{p-1}\,\mathrm{d}A=0. \tag{3.26}$$

By the Hölder inequality and the Cauchy-Schwarz inequality we have

(3.27)

$$\begin{aligned}\gamma_3\int_0^1\frac{\partial T}{\partial x_2}T^{p-1}\,\mathrm{d}x&=\gamma_3\int_0^1\frac{\partial H}{\partial x_2}T^{p-1}\,\mathrm{d}x_1=\gamma_3\int_\Omega\frac{\partial}{\partial x_2}\Big(\frac{\partial H}{\partial x_2}T^{p-1}\Big)\,\mathrm{d}A\\&=\gamma_3\int_\Omega\frac{\partial^2 H}{\partial x_2^2}T^{p-1}\,\mathrm{d}A+(p-1)\gamma_3\int_\Omega\frac{\partial H}{\partial x_2}T^{p-2}\frac{\partial T}{\partial x_2}\,\mathrm{d}A\\&\leqslant\frac{\gamma_3}{p}\int_\Omega\Big(\frac{\partial^2 H}{\partial x_2^2}\Big)^p\,\mathrm{d}A+\frac{(p-1)\gamma_3}{p}\int_\Omega T^p\,\mathrm{d}A\\&\quad+\frac{8(p-1)\gamma_3}{p^2}\int_\Omega\Big(\frac{\partial H}{\partial x_2}\Big)^p\,\mathrm{d}A+\frac{4(p-1)(p-2)\gamma_3}{p^2}\int_\Omega T^p\,\mathrm{d}A\\&\quad+\frac{(p-1)\gamma_3}{p}\int_\Omega|\nabla T^{p/2}|^2\,\mathrm{d}A.\end{aligned}$$

Therefore, we have from (3.25)–(3.27)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega T^p\,\mathrm{d}A\leqslant{}&\frac{(p-1)(5p-8)\gamma_3}{p}\int_\Omega T^p\,\mathrm{d}A\\&+\frac{8(p-1)\gamma_3}{p}\int_\Omega\Big(\frac{\partial H}{\partial x_2}\Big)^p\,\mathrm{d}A+\gamma_3\int_\Omega\Big(\frac{\partial^2 H}{\partial x_2^2}\Big)^p\,\mathrm{d}A\end{aligned} \tag{3.28}$$

By the Gronwall inequality we have

$$\begin{aligned}\int_\Omega T^p\,\mathrm{d}A\leqslant{}&\int_\Omega T_0^p\,\mathrm{d}A\cdot\mathrm{e}^{(p-1)(5p-8)\gamma_3 t/p}\\&+\int_0^t\mathrm{e}^{(p-1)(5p-8)\gamma_3(t-\eta)/p}\\&\times\Big[\frac{8(p-1)\gamma_3}{p}\int_\Omega\Big(\frac{\partial H}{\partial x_2}\Big)^p\,\mathrm{d}A+\gamma_3\int_\Omega\Big(\frac{\partial^2 H}{\partial x_2^2}\Big)^p\,\mathrm{d}A\Big]\,\mathrm{d}\eta.\end{aligned}$$

Therefore,

$$\begin{aligned}\Big(\int_\Omega T^p\,\mathrm{d}A\Big)^{1/p}\leqslant{}&\Big\{\int_\Omega T_0^p\,\mathrm{d}A\cdot\mathrm{e}^{(p-1)(5p-8)\gamma_3 t/p}+\int_0^t\mathrm{e}^{(p-1)(5p-8)\gamma_3(t-\eta)/p}\\&\times\Big[\frac{8(p-1)\gamma_3}{p}\int_\Omega\Big(\frac{\partial H}{\partial x_2}\Big)^p\,\mathrm{d}A+\gamma_3\int_\Omega\Big(\frac{\partial^2 H}{\partial x_2^2}\Big)^p\,\mathrm{d}A\Big]\,\mathrm{d}\eta\Big\}^{1/p}.\end{aligned} \tag{3.29}$$

Letting now $p \to \infty$ in (3.29) we obtain

$$\sup_{\Omega} |T| \leqslant T_m, \tag{3.30}$$

where

$$T_m = \sup_{\Omega} \{\|\tau\|_\infty,\ \|T_0\|_\infty\}.$$

3.4. Estimate for $\|\nabla \partial u/\partial x_2\|_2^2$. By $(2.4)_1$ we have

$$\begin{aligned}(3.31)\quad \int_0^t \int_\Omega \frac{\partial}{\partial x_2}\Big\{ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} - \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1,\zeta,t)\,\mathrm{d}\zeta\Big)\frac{\partial u}{\partial x_2} \\ - \frac{1}{\varepsilon'} v + \frac{1}{\varepsilon'}\frac{\partial p_s}{\partial x_1} - \frac{\mu}{\varepsilon'}\Big(\int_{x_2}^0 \frac{\partial}{\partial x_1} T(x_1,\zeta,t)\,\mathrm{d}\zeta\Big) - \gamma_1 \Delta u\Big\}\frac{\partial u}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta = 0.\end{aligned}$$

Integrating by parts, we have

(3.32)

$$\begin{aligned}\frac{1}{2}\int_\Omega \Big(\frac{\partial u}{\partial x_2}\Big)^2 \mathrm{d}A + \gamma_1 \int_0^t \int_\Omega \Big|\nabla \frac{\partial u}{\partial x_2}\Big|^2 \mathrm{d}A\,\mathrm{d}\eta \\ = \gamma_1 \int_0^t \int_0^1 \frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_2^2}\,\mathrm{d}x_1\,\mathrm{d}\eta + \frac{1}{2}\int_\Omega \Big(\frac{\partial u_0}{\partial x_2}\Big)^2 \mathrm{d}A \\ - \int_0^t \int_\Omega \Big[u\frac{\partial^2 u}{\partial x_1 x_2} - \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1,\zeta,\eta)\,\mathrm{d}\zeta\Big)\frac{\partial^2 u}{\partial x_2^2}\Big]\frac{\partial u}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta \\ + \frac{1}{\varepsilon'}\int_0^t \int_\Omega \frac{\partial v}{\partial x_2}\frac{\partial u}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta - \frac{\mu}{\varepsilon'}\int_0^t \int_\Omega \frac{\partial T}{\partial x_1}\frac{\partial u}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta.\end{aligned}$$

Similarly, by $(2.4)_2$

$$\begin{aligned}(3.33)\quad \frac{1}{2}\int_\Omega \Big(\frac{\partial v}{\partial x_2}\Big)^2 \mathrm{d}A + \gamma_2 \int_\Omega \Big|\nabla \frac{\partial v}{\partial x_2}\Big|^2 \mathrm{d}A\,\mathrm{d}\eta \\ = \gamma_2 \int_0^t \int_0^1 \frac{\partial v}{\partial x_2}\frac{\partial^2 v}{\partial x_2^2}\,\mathrm{d}x_1\,\mathrm{d}\eta \\ - \frac{1}{\varepsilon'}\int_\Omega \frac{\partial v}{\partial x_2}\frac{\partial u}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta + \frac{1}{2}\int_\Omega \Big(\frac{\partial v_0}{\partial x_2}\Big)^2 \mathrm{d}A \\ - \int_\Omega \Big[u\frac{\partial^2 v}{\partial x_1 x_2} - \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1,\zeta,t)\,\mathrm{d}\zeta\Big)\frac{\partial^2 v}{\partial x_2^2}\Big]\frac{\partial v}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta.\end{aligned}$$

Upon integrating by parts, we get

$$-\int_0^t \int_\Omega \Big[u\frac{\partial^2 u}{\partial x_1 x_2} - \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1,\zeta,\eta)\,\mathrm{d}\zeta\Big)\frac{\partial^2 u}{\partial x_2^2}\Big]\frac{\partial u}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta = 0. \tag{3.34}$$

By (3.8), (3.23) and the Hölder inequality we have
(3.35)

$$-\frac{\mu}{\varepsilon'}\int_0^t\int_\Omega \frac{\partial T}{\partial x_1}\frac{\partial u}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta \leqslant \frac{\mu}{\varepsilon'}\Big(\int_0^t\int_\Omega\Big(\frac{\partial T}{\partial x_1}\Big)^2\mathrm{d}A\,\mathrm{d}\eta\Big)^{1/2}\Big(\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial x_2}\Big)^2\mathrm{d}A\,\mathrm{d}\eta\Big)^{1/2}$$
$$\leqslant \frac{\mu}{\varepsilon'}\sqrt{\frac{F_1(t)F_3(t)}{\gamma_1}}.$$

By (3.11) and using the divergence theorem,

(3.36)
$$\gamma_1\int_0^t\int_0^1 \frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_2^2}\,\mathrm{d}x_1\,\mathrm{d}\eta$$
$$= \gamma_1\int_0^t\int_\Omega \frac{\partial}{\partial x_2}\Big(\frac{\partial S_1}{\partial x_2}\frac{\partial^2 u}{\partial x_2^2}\Big)\,\mathrm{d}A\,\mathrm{d}\eta$$
$$= \gamma_1\int_0^t\int_\Omega \frac{\partial^2 S_1}{\partial x_2^2}\frac{\partial^2 u}{\partial x_2^2}\,\mathrm{d}A\,\mathrm{d}\eta + \gamma_1\int_0^t\int_\Omega \frac{\partial S_1}{\partial x_2}\frac{\partial^3 u}{\partial x_2^3}\,\mathrm{d}A\,\mathrm{d}\eta$$
$$\leqslant \frac{\gamma_1}{2}\int_0^t\int_\Omega\Big(\frac{\partial^2 S_1}{\partial x_2^2}\Big)^2\mathrm{d}A\,\mathrm{d}\eta + \frac{\gamma_1}{2}\int_0^t\int_\Omega\Big(\frac{\partial^2 u}{\partial x_2^2}\Big)^2\mathrm{d}A\,\mathrm{d}\eta$$
$$+ \int_\Omega \frac{\partial S_1}{\partial x_2}\frac{\partial u}{\partial x_2}\,\mathrm{d}A + \int_0^t\int_\Omega \frac{\partial (S_1)_t}{\partial x_2}\frac{\partial u}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta$$
$$+ \frac{1}{\varepsilon'}\int_0^t\int_\Omega \frac{\partial S_1}{\partial x_2}\frac{\partial v}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta$$
$$+ \int_0^t\int_\Omega \frac{\partial S_1}{\partial x_2}\Big[u\frac{\partial^2 u}{\partial x_1 x_2} - \Big(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}u(x_1,\zeta,\eta)\,\mathrm{d}\zeta\Big)\frac{\partial^2 u}{\partial x_2^2}\Big]\,\mathrm{d}A\,\mathrm{d}\eta$$
$$+ \frac{\mu}{\varepsilon'}\int_0^t\int_\Omega \frac{\partial S_1}{\partial x_2}\frac{\partial T}{\partial x_1}\,\mathrm{d}A\,\mathrm{d}\eta + \gamma_1\int_0^t\int_\Omega \frac{\partial^2 S_1}{\partial x_1 x_2}\frac{\partial^2 u}{\partial x_1 x_2}\,\mathrm{d}A\,\mathrm{d}\eta.$$

By the Cauchy-Schwarz inequality we get
(3.37)

$$\int_\Omega \frac{\partial S_1}{\partial x_2}\frac{\partial u}{\partial x_2}\,\mathrm{d}A \leqslant \int_\Omega\Big(\frac{\partial S_1}{\partial x_2}\Big)^2\mathrm{d}A + \frac{1}{4}\int_\Omega\Big(\frac{\partial u}{\partial x_2}\Big)^2\mathrm{d}A,$$
$$\int_0^t\int_\Omega \frac{\partial (S_1)_t}{\partial x_2}\frac{\partial u}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta \leqslant \frac{1}{2}\int_0^t\int_\Omega\Big(\frac{\partial (S_1)_t}{\partial x_2}\Big)^2\mathrm{d}A\,\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial x_2}\Big)^2\mathrm{d}A\,\mathrm{d}\eta,$$
$$\frac{1}{\varepsilon'}\int_0^t\int_\Omega \frac{\partial S_1}{\partial x_2}\frac{\partial v}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta \leqslant \frac{1}{2\varepsilon'}\int_0^t\int_\Omega\Big(\frac{\partial S_1}{\partial x_2}\Big)^2\mathrm{d}A\,\mathrm{d}\eta + \frac{1}{2\varepsilon'}\int_0^t\int_\Omega\Big(\frac{\partial v}{\partial x_2}\Big)^2\mathrm{d}A\,\mathrm{d}\eta,$$
$$\frac{\mu}{\varepsilon'}\int_0^t\int_\Omega \frac{\partial S_1}{\partial x_2}\frac{\partial T}{\partial x_1}\,\mathrm{d}A\,\mathrm{d}\eta \leqslant \frac{\mu}{2\varepsilon'}\int_0^t\int_\Omega\Big(\frac{\partial S_1}{\partial x_2}\Big)^2\mathrm{d}A\,\mathrm{d}\eta + \frac{\mu}{2\varepsilon'}\int_0^t\int_\Omega\Big(\frac{\partial T}{\partial x_1}\Big)^2\mathrm{d}A\,\mathrm{d}\eta,$$
$$\gamma_1\int_0^t\int_\Omega \frac{\partial^2 S_1}{\partial x_1 x_2}\frac{\partial^2 u}{\partial x_1 x_2}\,\mathrm{d}A\,\mathrm{d}\eta \leqslant \frac{\gamma_1}{2}\int_0^t\int_\Omega\Big(\frac{\partial^2 S_1}{\partial x_1 x_2}\Big)^2\mathrm{d}A\,\mathrm{d}\eta + \frac{\gamma_1}{2}\int_0^t\int_\Omega\Big(\frac{\partial^2 u}{\partial x_1 x_2}\Big)^2\mathrm{d}A\,\mathrm{d}\eta$$

and by (2.10) with $\delta = 1$ we arrive at

(3.38)

$$\int_0^t \int_\Omega \frac{\partial S_1}{\partial x_2}\Big[u\frac{\partial^2 u}{\partial x_1 x_2} - \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1,\zeta,\eta)\,\mathrm{d}\zeta\Big)\frac{\partial^2 u}{\partial x_2^2}\Big]\,\mathrm{d}A\,\mathrm{d}\eta$$

$$\leqslant \Big[\int_0^t \int_\Omega \Big(\frac{\partial^2 u}{\partial x_1 x_2}\Big)^2 \mathrm{d}A\,\mathrm{d}\eta\Big]^{1/2} \Big[\int_0^t \Big(\int_\Omega u^4\,\mathrm{d}A\Big)^{1/2}\Big(\int_\Omega \Big|\frac{\partial S_1}{\partial x_2}\Big|^4 \mathrm{d}A\Big)^{1/2} \mathrm{d}\eta\Big]^{1/2}$$

$$+ h\Big[\int_0^t \int_\Omega \Big(\frac{\partial^2 u}{\partial x_2^2}\Big)^2 \mathrm{d}A\,\mathrm{d}\eta\Big]^{1/2} \Big[\int_0^t \Big(\int_\Omega \Big|\frac{\partial u}{\partial x_1}\Big|^4 \mathrm{d}A\Big)^{1/2}\Big(\int_\Omega \Big|\frac{\partial S_1}{\partial x_2}\Big|^4 \mathrm{d}A\Big)^{1/2} \mathrm{d}\eta\Big]^{1/2}$$

$$\leqslant \frac{2C}{\gamma_1} \max_t \Big\{\int_\Omega \Big|\frac{\partial S_1}{\partial x_2}\Big|^4 \mathrm{d}A\Big\} \int_0^t \Big(\int_\Omega u^2\,\mathrm{d}A + \int_\Omega |\nabla u|^2\,\mathrm{d}A\Big)\,\mathrm{d}\eta$$

$$+ \frac{\gamma_1}{8}\int_0^t \int_\Omega \Big(\frac{\partial^2 u}{\partial x_1 x_2}\Big)^2 \mathrm{d}A\,\mathrm{d}\eta$$

$$+ Ch \max_t \Big\{\Big(\int_\Omega \Big|\frac{\partial S_1}{\partial x_2}\Big|^4 \mathrm{d}A\Big)^{1/2}\Big\} \Big[\int_0^t \int_\Omega \Big(\frac{\partial^2 u}{\partial x_2^2}\Big)^2 \mathrm{d}A\,\mathrm{d}\eta\Big]^{1/2}$$

$$\times \Big[\int_0^t \Big(\int_\Omega \Big|\frac{\partial u}{\partial x_1}\Big|^2 \mathrm{d}A + \int_\Omega \Big|\frac{\partial^2 u}{\partial x_1^2}\Big|^2 \mathrm{d}A + \int_\Omega \Big|\frac{\partial^2 u}{\partial x_1 x_2}\Big|^2 \mathrm{d}A\Big)\,\mathrm{d}\eta\Big]^{1/2}$$

$$\leqslant \frac{2C}{\gamma_1} \max_t \Big\{\int_\Omega \Big|\frac{\partial S_1}{\partial x_2}\Big|^4 \mathrm{d}A\Big\} \int_0^t \Big(\int_\Omega u^2\,\mathrm{d}A + \int_\Omega |\nabla u|^2\,\mathrm{d}A\Big)\,\mathrm{d}\eta$$

$$+ \frac{\gamma_1}{8}\int_0^t \int_\Omega \Big(\frac{\partial^2 u}{\partial x_1 x_2}\Big)^2 \mathrm{d}A\,\mathrm{d}\eta$$

$$+ \sqrt{C}h \max_t \Big\{\Big(\int_\Omega \Big|\frac{\partial S_1}{\partial x_2}\Big|^4 \mathrm{d}A\Big)^{1/2}\Big\} \Big[\int_0^t \int_\Omega \Big(\frac{\partial^2 u}{\partial x_2^2}\Big)^2 \mathrm{d}A\,\mathrm{d}\eta\Big]^{1/2}$$

$$\times \Big[\int_0^t \Big(\int_\Omega \Big|\frac{\partial u}{\partial x_1}\Big|^2 \mathrm{d}A + \int_\Omega \Big|\frac{\partial^2 u}{\partial x_1^2}\Big|^2 \mathrm{d}A\Big)\,\mathrm{d}\eta\Big]^{1/2}$$

$$+ \frac{2Ch^2}{\gamma_1} \max_t \Big\{\int_\Omega \Big|\frac{\partial S_1}{\partial x_2}\Big|^4 \mathrm{d}A\Big\} \Big[\int_0^t \int_\Omega \Big(\frac{\partial^2 u}{\partial x_2^2}\Big)^2 \mathrm{d}A\,\mathrm{d}\eta\Big] + \frac{\gamma_1}{8}\int_0^t \int_\Omega \Big|\frac{\partial^2 u}{\partial x_1 x_2}\Big|^2 \mathrm{d}A\,\mathrm{d}\eta.$$

Inserting the above bounds into (3.32) and using the inequalities (3.21) and (3.23), we write (3.32) after simplification as

$$\int_\Omega \Big(\frac{\partial u}{\partial x_2}\Big)^2 \mathrm{d}A + \gamma_1 \int_0^t \int_\Omega \Big|\nabla \frac{\partial u}{\partial x_2}\Big|^2 \mathrm{d}A\,\mathrm{d}\eta \leqslant \frac{1}{\varepsilon'}\int_0^t \int_\Omega \frac{\partial v}{\partial x_2}\frac{\partial u}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta + a_4(t), \tag{3.40}$$

where $a_4(t)$ is a positive computable function. In a similar way we can write from (3.33)

$$\int_\Omega \Big(\frac{\partial v}{\partial x_2}\Big)^2 \mathrm{d}A + \gamma_2 \int_0^t \int_\Omega \Big|\nabla \frac{\partial v}{\partial x_2}\Big|^2 \mathrm{d}A\,\mathrm{d}\eta \leqslant -\frac{1}{\varepsilon'}\int_0^t \int_\Omega \frac{\partial v}{\partial x_2}\frac{\partial u}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta + a_5(t). \tag{3.41}$$

Combining (3.39) and (3.40), we have

$$\int_\Omega\Big(\frac{\partial u}{\partial x_2}\Big)^2\,\mathrm{d}A+\int_\Omega\Big(\frac{\partial v}{\partial x_2}\Big)^2\,\mathrm{d}A+\gamma_1\int_0^t\int_\Omega\Big|\nabla\frac{\partial u}{\partial x_2}\Big|^2\,\mathrm{d}A\,\mathrm{d}\eta + \gamma_2\int_0^t\int_\Omega\Big|\nabla\frac{\partial v}{\partial x_2}\Big|^2\,\mathrm{d}A\,\mathrm{d}\eta\leqslant F_5(t). \tag{3.42}$$

Using (2.10) with $\delta=1$, we have

$$\begin{aligned}\int_0^t\int_\Omega\Big(\frac{\partial u}{\partial x_2}\Big)^4\,\mathrm{d}A\,\mathrm{d}\eta &\leqslant C\bigg(\int_0^t\int_\Omega\Big[\Big(\frac{\partial u}{\partial x_2}\Big)^2\,\mathrm{d}A+\Big|\nabla\frac{\partial u}{\partial x_2}\Big|^2\Big]\,\mathrm{d}A\,\mathrm{d}\eta\bigg)^2\\ &\leqslant C\bigg(\int_0^t F_5(\eta)d\eta+\frac{1}{\gamma_1}F_5(t)\bigg)^2\doteq F_6(t).\end{aligned}\tag{3.43}$$

4. Continuous dependence on the boundary conditions

Suppose that (u^*,v^*,T^*,p_s^*) also are the solutions of (2.4)–(2.6) with different boundary conditions g_1^*, g_2^* and τ^*. Let

$$\begin{aligned}&\tilde u=u-u^*,\quad \tilde v=v-v^*,\quad \widetilde T=T-T^*,\quad \pi_s=p_s-p_s^*,\\ &\tilde g_1=g_1-g_1^*,\quad \tilde g_2=g_2-g_2^*,\quad \tilde\tau=\tau-\tau^*,\end{aligned}\tag{4.1}$$

then $(\tilde u,\tilde v,\pi_s)$ satisfies

$$\begin{aligned}&\frac{\partial\tilde u}{\partial t}+\tilde u\frac{\partial u}{\partial x_1}-\bigg(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}\tilde u(x_1,\zeta,t)\,\mathrm{d}\zeta\bigg)\frac{\partial u}{\partial x_2}\\ &\quad+u^*\frac{\partial\tilde u}{\partial x_1}-\bigg(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}u^*(x_1,\zeta,t)\,\mathrm{d}\zeta\bigg)\frac{\partial\tilde u}{\partial x_2}\\ &\quad-\frac{1}{\varepsilon'}\tilde v+\frac{1}{\varepsilon'}\frac{\partial\pi_s}{\partial x_1}-\frac{\mu}{\varepsilon'}\bigg(\int_{x_2}^{0}\frac{\partial}{\partial x_1}\widetilde T(x_1,\zeta,t)\,\mathrm{d}\zeta\bigg)=\gamma_1\Delta\tilde u,\\ &\frac{\partial\tilde v}{\partial t}+\tilde u\frac{\partial v}{\partial x_1}-\bigg(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}\tilde u(x_1,\zeta,t)\,\mathrm{d}\zeta\bigg)\frac{\partial v}{\partial x_2}\\ &\quad+u^*\frac{\partial\tilde v}{\partial x_1}-\bigg(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}u^*(x_1,\zeta,t)\,\mathrm{d}\zeta\bigg)\frac{\partial\tilde v}{\partial x_2}-\frac{1}{\varepsilon'}\tilde u=\gamma_2\Delta\tilde v,\\ &\frac{\partial\widetilde T}{\partial t}+\tilde u\frac{\partial T}{\partial x_1}-\bigg(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}\tilde u(x_1,\zeta,t)\,\mathrm{d}\zeta\bigg)\frac{\partial T}{\partial x_2}\\ &\quad+u^*\frac{\partial\widetilde T}{\partial x_1}-\bigg(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}u^*(x_1,\zeta,t)\,\mathrm{d}\zeta\bigg)\frac{\partial\widetilde T}{\partial x_2}=\gamma_3\Delta\widetilde T,\\ &\frac{\partial}{\partial x_1}\int_{-h}^{0}\tilde u(x_1,\zeta,t)\,\mathrm{d}\zeta=0\end{aligned}\tag{4.2}$$

with the initial-boundary conditions

$$\frac{\partial \tilde u}{\partial x_2}\Big|_{x_2=0} = \alpha_1 \tilde g_1, \quad \frac{\partial \tilde v}{\partial x_2}\Big|_{x_2=0} = \alpha_2 \tilde g_2, \quad \frac{\partial \tilde u}{\partial x_2}\Big|_{x_2=-h} = \frac{\partial \tilde v}{\partial x_2}\Big|_{x_2=-h} = 0,$$
$$(\tilde u, \tilde v)|_{\Gamma_s} = 0, \quad \frac{\partial \widetilde T}{\partial x_2}\Big|_{x_2=0} = \beta \tilde\tau, \quad \frac{\partial \widetilde T}{\partial x_2}\Big|_{x_2=-h} = 0, \quad \frac{\partial \widetilde T}{\partial x_1}\Big|_{\Gamma_s} = 0 \tag{4.3}$$

and the initial conditions

$$(\tilde u, \tilde v, \widetilde T)|_{t=0} = (0,0,0). \tag{4.4}$$

We have the following theorem.

Theorem 4.1. *If* $T_0, u_0, v_0 \in L^2(\Omega)$, $g_1, g_2 \in H^1(\Omega)$, $\tau \in L^\infty(\Omega)$, *then*

$$(u, v, T, p_s) \to (u^*, v^*, T^*, p_s^*) \text{ as } (g_1, g_2, \tau) \to (g_1^*, g_2^*, \tau^*).$$

Furthermore, the inequality

$$\begin{aligned} &\int_\Omega \tilde u^2 \,\mathrm{d}A + \int_\Omega \tilde v^2 \,\mathrm{d}A + \theta \int_\Omega \widetilde T^2 \,\mathrm{d}A + \frac{\gamma_1}{2}\int_0^t \int_\Omega |\nabla \tilde u|^2 \,\mathrm{d}A \,\mathrm{d}\eta \\ &\quad + \gamma_2 \int_0^t \int_\Omega |\nabla \tilde v|^2 \,\mathrm{d}A \,\mathrm{d}\eta + \theta\gamma_3 \int_0^t \int_\Omega |\nabla \widetilde T|^2 \,\mathrm{d}A \,\mathrm{d}\eta \\ &\leqslant b_6(t) \int_0^t \mathrm{e}^{\int_s^t b_6(\eta)\,\mathrm{d}\eta} \int_0^s \int_0^1 [\alpha_1 \tilde g_1^2 + \alpha_2 \tilde g_2^2 + \theta\beta\tilde\tau^2] \,\mathrm{d}x_1 \,\mathrm{d}\eta \,\mathrm{d}s \\ &\quad + \int_0^t \int_0^1 [\alpha_1 \tilde g_1^2 + \alpha_2 \theta \tilde g_2^2 + \beta \tilde\tau^2] \,\mathrm{d}x_1 \,\mathrm{d}\eta, \end{aligned} \tag{4.5}$$

where θ *is a positive constant and* $b_6(t)$ *is a positive function, holds.*

P r o o f. Now taking the inner product of the equation $(4.2)_1$ with $\tilde u$, we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega \tilde u^2 \,\mathrm{d}A + \gamma_1 \int_0^t \int_\Omega |\nabla \tilde u|^2 \,\mathrm{d}A \,\mathrm{d}\eta \\ &\quad = \frac{1}{\varepsilon'}\int_0^t \int_\Omega \tilde u \tilde v \,\mathrm{d}A \,\mathrm{d}\eta - \frac{1}{\varepsilon'}\int_0^t \int_\Omega \frac{\partial \pi_s}{\partial x_1} \tilde u \,\mathrm{d}A \,\mathrm{d}\eta \\ &\quad + \gamma_1 \int_0^t \int_0^1 \frac{\partial \tilde u}{\partial x_2} \tilde u \,\mathrm{d}x_1 \,\mathrm{d}\eta + \frac{\mu}{\varepsilon'} \int_0^t \int_\Omega \Big(\int_{x_2}^0 \frac{\partial}{\partial x_1} \widetilde T(x_1, \zeta, \eta) \,\mathrm{d}\zeta \Big) \tilde u \,\mathrm{d}A \,\mathrm{d}\eta \\ &\quad - \int_0^t \int_\Omega \Big[\tilde u \frac{\partial u}{\partial x_1} - \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde u(x_1, \zeta, \eta) \,\mathrm{d}\zeta \Big) \frac{\partial u}{\partial x_2} \Big] \tilde u \,\mathrm{d}A \,\mathrm{d}\eta \\ &\quad - \int_0^t \int_\Omega \Big[u^* \frac{\partial \tilde u}{\partial x_1} - \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u^*(x_1, \zeta, \eta) \,\mathrm{d}\zeta \Big) \frac{\partial \tilde u}{\partial x_2} \Big] \tilde u \,\mathrm{d}A \,\mathrm{d}\eta. \end{aligned} \tag{4.6}$$

After integrating by parts, we obtain

$$(4.7)\qquad -\int_0^t\int_\Omega\left[u^*\frac{\partial\tilde u}{\partial x_1}-\left(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}u^*(x_1,\zeta,\eta)\,\mathrm{d}\zeta\right)\frac{\partial\tilde u}{\partial x_2}\right]\tilde u\,\mathrm{d}A\,\mathrm{d}\eta=0,$$

$$\begin{aligned}(4.8)\qquad &-\frac{1}{\varepsilon'}\int_0^t\int_\Omega\frac{\partial\pi_s}{\partial x_1}\tilde u\,\mathrm{d}A\,\mathrm{d}\eta\\ &=-\frac{1}{\varepsilon'}\int_0^t\int_0^1\frac{\partial\pi_s}{\partial x_1}\left(\int_{-h}^0\tilde u(x_1,x_2,\eta)\,\mathrm{d}x_2\right)\mathrm{d}x_1\,\mathrm{d}\eta=0.\end{aligned}$$

By the Hölder inequality, (2.7), (3.23), (3.42) and the A-G mean inequality, we have

$$\begin{aligned}(4.9)\qquad &-\int_0^t\int_\Omega\left[\tilde u\frac{\partial u}{\partial x_1}-\left(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}\tilde u(x_1,\zeta,\eta)\,\mathrm{d}\zeta\right)\frac{\partial u}{\partial x_2}\right]\tilde u\,\mathrm{d}A\,\mathrm{d}\eta\\
&\leqslant\left[\int_0^t\int_\Omega\left(\frac{\partial u}{\partial x_1}\right)^2\mathrm{d}A\right]^{1/2}\left[\int_0^t\int_\Omega\tilde u^4\,\mathrm{d}A\,\mathrm{d}\eta\right]^{1/2}\\
&\quad+\left[\int_0^t\int_\Omega\left(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}\tilde u(x_1,\zeta,\eta)\,\mathrm{d}\zeta\right)^2\mathrm{d}A\,\mathrm{d}\eta\right]^{1/2}\\
&\quad\times\left[\int_0^t\int_\Omega\left(\frac{\partial u}{\partial x_2}\right)^4\mathrm{d}A\,\mathrm{d}\eta\right]^{1/4}\left[\int_0^t\int_\Omega\tilde u^4\,\mathrm{d}A\,\mathrm{d}\eta\right]^{1/4}\\
&\leqslant\sqrt{\frac{F_3(t)}{\gamma_1}}C\left[\int_0^t\int_\Omega\tilde u^2\,\mathrm{d}A\,\mathrm{d}\eta+\delta_1\int_0^t\int_\Omega|\nabla\tilde u|^2\,\mathrm{d}A\,\mathrm{d}\eta\right]\\
&\quad+\frac{\sqrt{C}h}{\pi}\sqrt[4]{F_6(t)}\left[\int_0^t\int_\Omega\tilde u^2\,\mathrm{d}A\,\mathrm{d}\eta\right]^{1/2}\\
&\quad\times\left[\int_0^t\int_\Omega\tilde u^2\,\mathrm{d}A\,\mathrm{d}\eta+\delta_1\int_0^t\int_\Omega|\nabla\tilde u|^2\,\mathrm{d}A\,\mathrm{d}\eta\right]^{1/2}\\
&\leqslant b_1(t)\int_0^t\int_\Omega\tilde u^2\,\mathrm{d}A\,\mathrm{d}\eta+b_2(t)\delta_1\int_0^t\int_\Omega|\nabla\tilde u|^2\,\mathrm{d}A\,\mathrm{d}\eta\end{aligned}$$

for computable functions $b_1(t),b_2(t)$ and a positive arbitrary constant δ_1.

By the Cauchy-Schwarz inequality again

$$\begin{aligned}(4.10)\qquad &\frac{\mu}{\varepsilon'}\int_0^t\int_\Omega\left(\int_{x_2}^0\frac{\partial}{\partial x_1}\widetilde T(x_1,\zeta,\eta)\,\mathrm{d}\zeta\right)\tilde u\,\mathrm{d}A\,\mathrm{d}\eta\\
&=-\frac{\mu}{\varepsilon'}\int_0^t\int_\Omega\left(\int_{x_2}^0\widetilde T(x_1,\zeta,\eta)\,\mathrm{d}\zeta\right)\frac{\partial\tilde u}{\partial x_1}\,\mathrm{d}A\,\mathrm{d}\eta\\
&\leqslant\frac{h^2\mu^2}{\gamma_1\varepsilon'^2}\int_0^t\int_\Omega\widetilde T^2\,\mathrm{d}A\,\mathrm{d}\eta+\frac{\gamma_1}{4}\int_0^t\int_\Omega\left(\frac{\partial\tilde u}{\partial x_1}\right)^2\mathrm{d}A\,\mathrm{d}\eta\end{aligned}$$

and

$$\text{(4.11)} \qquad \gamma_1 \int_0^t \int_0^1 \frac{\partial \tilde{u}}{\partial x_2} \tilde{u} \,\mathrm{d}x_1 \,\mathrm{d}\eta = \gamma_1 \alpha_1 \int_0^t \int_0^1 \tilde{g}_1(x_1,\eta) \tilde{u}(x_1,0,\eta) \,\mathrm{d}x_1 \,\mathrm{d}\eta$$
$$\leqslant \frac{\gamma_1\alpha_1}{2} \int_0^t \int_0^1 \tilde{g}_1^2(x_1,\eta) \,\mathrm{d}x_1 \,\mathrm{d}\eta + \frac{\gamma_1\alpha_1}{2} \int_0^t \int_0^1 \tilde{u}^2(x_1,0,\eta) \,\mathrm{d}x_1 \,\mathrm{d}\eta.$$

Using a similar method in (2.20), we have
(4.12)

$$\gamma_1 \int_0^t \int_0^1 \frac{\partial \tilde{u}}{\partial x_2} \tilde{u} \,\mathrm{d}x_1 \,\mathrm{d}\eta \leqslant \frac{\gamma_1\alpha_1}{2} \int_0^t \int_0^1 \tilde{g}_1^2(x_1,\eta) \,\mathrm{d}x_1 \,\mathrm{d}\eta + \frac{m_3\gamma_1\alpha_1}{2} \int_0^t \int_\Omega \tilde{u}^2 \,\mathrm{d}A \,\mathrm{d}\eta$$
$$+ \frac{m_4\gamma_1\alpha_1}{2} \int_0^t \int_\Omega |\tilde{u}| \Big|\frac{\partial \tilde{u}}{\partial x_1}\Big| \,\mathrm{d}A \,\mathrm{d}\eta$$
$$\leqslant \frac{\gamma_1\alpha_1}{2} \int_0^t \int_0^1 \tilde{g}_1^2 \,\mathrm{d}x_1 \,\mathrm{d}\eta + \frac{m_4\gamma_1\alpha_1}{4}\delta_1 \int_0^t \int_\Omega |\nabla \tilde{u}|^2 \,\mathrm{d}A \,\mathrm{d}\eta$$
$$+ \frac{(2m_3 + m_4\delta_1^{-1})\gamma_1\alpha_1}{4} \int_0^t \int_\Omega \tilde{u}^2 \,\mathrm{d}A \,\mathrm{d}\eta.$$

Inserting (4.7)–(4.12) into (4.6) and choosing $\delta_1 = \gamma_1/(4b_2(t) + m_4\gamma_1\alpha_1)$, we have

$$\text{(4.13)} \qquad \frac{1}{2} \int_\Omega \tilde{u}^2 \,\mathrm{d}A + \frac{1}{2}\gamma_1 \int_0^t \int_\Omega |\nabla \tilde{u}|^2 \,\mathrm{d}A \,\mathrm{d}\eta$$
$$\leqslant \frac{1}{\varepsilon'} \int_0^t \int_\Omega \tilde{u}\tilde{v} \,\mathrm{d}A \,\mathrm{d}\eta + \frac{\gamma_1\alpha_1}{2} \int_0^t \int_0^1 \tilde{g}_1^2(x_1,t) \,\mathrm{d}x_1 \,\mathrm{d}\eta$$
$$+ \frac{h^2\mu^2}{\gamma_1\varepsilon'^2} \int_0^t \int_\Omega \widetilde{T}^2 \,\mathrm{d}A \,\mathrm{d}\eta + b_3(t) \int_0^t \int_\Omega \tilde{u}^2 \,\mathrm{d}A \,\mathrm{d}\eta,$$

where

$$\text{(4.14)} \qquad b_3(t) = b_1(t) + \frac{(2m_3 + m_4\delta_1^{-1})\gamma_1\alpha_1}{4}.$$

Now taking the inner product of the equation $(4.2)_2$ with $\tilde{v}$ in $L^2(\Omega)$, we have

$$\text{(4.15)} \qquad \frac{1}{2} \int_\Omega \tilde{v}^2 \,\mathrm{d}A + \gamma_2 \int_0^t \int_\Omega |\nabla \tilde{v}|^2 \,\mathrm{d}A \,\mathrm{d}\eta$$
$$= -\frac{1}{\varepsilon'} \int_0^t \int_\Omega \tilde{u}\tilde{v} \,\mathrm{d}A \,\mathrm{d}\eta + \gamma_2 \int_0^t \int_0^1 \frac{\partial \tilde{v}}{\partial x_2} \tilde{v} \,\mathrm{d}x_1 \,\mathrm{d}\eta$$
$$- \int_0^t \int_\Omega \Big[\tilde{u}\frac{\partial v}{\partial x_1} - \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1}\tilde{u}(x_1,\zeta,\eta) \,\mathrm{d}\zeta\Big)\frac{\partial v}{\partial x_2}\Big] \tilde{u} \,\mathrm{d}A \,\mathrm{d}\eta$$
$$- \int_0^t \int_\Omega \Big[u^*\frac{\partial \tilde{v}}{\partial x_1} - \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1}u^*(x_1,\zeta,\eta) \,\mathrm{d}\zeta\Big)\frac{\partial \tilde{v}}{\partial x_2}\Big] \tilde{u} \,\mathrm{d}A \,\mathrm{d}\eta.$$

In the same way as before we arrive at

(4.16)
$$\frac{1}{2}\int_\Omega \tilde v^2\,\mathrm{d}A+\frac{1}{2}\gamma_2\int_0^t\int_\Omega|\nabla\tilde v|^2\,\mathrm{d}A\,\mathrm{d}\eta \leqslant -\frac{1}{\varepsilon'}\int_0^t\int_\Omega \tilde u\tilde v\,\mathrm{d}A\,\mathrm{d}\eta+\frac{\gamma_2\alpha_2}{2}\int_0^t\int_0^1 \tilde g_2^2(x_1,\eta)\,\mathrm{d}x_1\,\mathrm{d}\eta+b_4(t)\int_0^t\int_\Omega \tilde v^2\,\mathrm{d}A\,\mathrm{d}\eta$$

for a computable positive function $b_4(t)$. A combination of (4.13) and (4.16) leads to

$$\begin{aligned}\int_\Omega \tilde u^2\,\mathrm{d}A+\int_\Omega \tilde v^2\,\mathrm{d}A+\gamma_1\int_0^t\int_\Omega|\nabla\tilde u|^2\,\mathrm{d}A\,\mathrm{d}\eta+\gamma_2\int_0^t\int_\Omega|\nabla\tilde v|^2\,\mathrm{d}A\,\mathrm{d}\eta \\ \leqslant \frac{h^2\mu^2}{\gamma_1\varepsilon'^2}\int_0^t\int_\Omega \widetilde T^2\,\mathrm{d}A\,\mathrm{d}\eta+b_5(t)\Big(\int_0^t\int_\Omega \tilde u^2\,\mathrm{d}A\,\mathrm{d}\eta+\int_0^t\int_\Omega \tilde v^2\,\mathrm{d}A\,\mathrm{d}\eta\Big) \\ +\gamma_1\alpha_1\int_0^t\int_0^1 \tilde g_1^2(x_1,\eta)\,\mathrm{d}x_1\,\mathrm{d}\eta+\gamma_2\alpha_2\int_0^t\int_0^1 \tilde g_2^2(x_1,\eta)\,\mathrm{d}x_1\,\mathrm{d}\eta,\end{aligned}\tag{4.17}$$

where $b_5(t)=2(b_3(t)+b_4(t))$. We take the inner product of the equation $(4.2)_3$ with $\widetilde T$ to obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega \widetilde T^2\,\mathrm{d}A+\gamma_3\int_0^t\int_\Omega|\nabla\widetilde T|^2\,\mathrm{d}A\,\mathrm{d}\eta \\ =\gamma_3\int_0^t\int_0^1\frac{\partial\widetilde T}{\partial x_2}\widetilde T\,\mathrm{d}x_1\,\mathrm{d}\eta \\ -\int_0^t\int_\Omega\Big[\tilde u\frac{\partial T}{\partial x_1}-\Big(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}\tilde u(x_1,\zeta,\eta)\,\mathrm{d}\zeta\Big)\frac{\partial T}{\partial x_2}\Big]\widetilde T\,\mathrm{d}A\,\mathrm{d}\eta \\ -\int_0^t\int_\Omega\Big[u^*\frac{\partial\widetilde T}{\partial x_1}-\Big(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}u^*(x_1,\zeta,\eta)\,\mathrm{d}\zeta\Big)\frac{\partial\widetilde T}{\partial x_2}\Big]\widetilde T\,\mathrm{d}A\,\mathrm{d}\eta.\end{aligned}\tag{4.18}$$

Similarly to the computations in (4.11) and (4.12) we have

$$\begin{aligned}\gamma_3\int_0^t\int_0^1\frac{\partial\widetilde T}{\partial x_2}\widetilde T\,\mathrm{d}x_1\,\mathrm{d}\eta\leqslant\frac{\gamma_3\beta}{2}\int_0^t\int_0^1\tilde\tau^2\,\mathrm{d}x_1\,\mathrm{d}\eta \\ +\frac{(2m_3+m_4\delta_2^{-1})\gamma_3\beta}{4}\int_0^t\int_\Omega\widetilde T^2\,\mathrm{d}A\,\mathrm{d}\eta \\ +\frac{m_4\gamma_3\beta}{4}\delta_2\int_0^t\int_\Omega|\nabla\widetilde T|^2\,\mathrm{d}A\,\mathrm{d}\eta\end{aligned}\tag{4.19}$$

for an arbitrary positive constant δ_2. Integrating by parts, we obtain

$$-\int_0^t\int_\Omega\Big[u^*\frac{\partial\widetilde T}{\partial x_1}-\Big(\int_{-h}^{x_2}\frac{\partial}{\partial x_1}u^*(x_1,\zeta,\eta)\,\mathrm{d}\zeta\Big)\frac{\partial\widetilde T}{\partial x_2}\Big]\widetilde T\,\mathrm{d}A\,\mathrm{d}\eta=0.\tag{4.20}$$

By integration by parts, the Hölder inequality, (3.30), (2.7)–(2.8), we get for $\delta_3 > 0$

$$
\begin{aligned}
(4.21)\quad & -\int_0^t \int_\Omega \Big[\tilde{u}\frac{\partial T}{\partial x_1} - \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1}\tilde{u}(x_1,\zeta,\eta)\,\mathrm{d}\zeta\Big)\frac{\partial T}{\partial x_2}\Big]\widetilde{T}\,\mathrm{d}A\,\mathrm{d}\eta \\
&= \int_0^t \int_\Omega \tilde{u}T\frac{\partial \widetilde{T}}{\partial x_1}\,\mathrm{d}A\,\mathrm{d}\eta - \int_0^t \int_\Omega \Big(\int_{-h}^{x_2} \frac{\partial}{\partial x_1}\tilde{u}(x_1,\zeta,\eta)\,\mathrm{d}\zeta\Big)T\frac{\partial \widetilde{T}}{\partial x_2}\,\mathrm{d}A\,\mathrm{d}\eta \\
&\leqslant \frac{T_m}{\pi}\Big(\int_0^t \int_\Omega \Big(\frac{\partial \widetilde{T}}{\partial x_1}\Big)^2\,\mathrm{d}A\,\mathrm{d}\eta\Big)^{1/2}\Big(\int_0^t \int_\Omega \Big(\frac{\partial \tilde{u}}{\partial x_1}\Big)^2\,\mathrm{d}A\,\mathrm{d}\eta\Big)^{1/2} \\
&\quad + \frac{h}{\pi}T_m\Big(\int_0^t \int_\Omega \Big(\frac{\partial \tilde{u}}{\partial x_1}\Big)^2\,\mathrm{d}A\,\mathrm{d}\eta\Big)^{1/2}\Big(\int_0^t \int_\Omega \Big(\frac{\partial \widetilde{T}}{\partial x_2}\Big)^2\,\mathrm{d}A\,\mathrm{d}\eta\Big)^{1/2} \\
&\leqslant \frac{1+h}{2\pi}\delta_3 T_m \int_0^t \int_\Omega |\nabla \widetilde{T}|^2\,\mathrm{d}A\,\mathrm{d}\eta + \frac{1+h}{2\delta_3\pi}T_m \int_0^t \int_\Omega \Big(\frac{\partial \tilde{u}}{\partial x_1}\Big)^2\,\mathrm{d}A\,\mathrm{d}\eta.
\end{aligned}
$$

Inserting (4.19)–(4.21) into (4.18) and choosing $\delta_2 = 1/(m_4\beta)$, $\delta_3 = \frac{1}{2}\gamma_3(T_m(1+h))$, we get

$$
\begin{aligned}
(4.22)\quad & \int_\Omega \widetilde{T}^2\,\mathrm{d}A + \gamma_3 \int_0^t \int_\Omega |\nabla \widetilde{T}|^2\,\mathrm{d}A\,\mathrm{d}\eta \\
&\leqslant \gamma_3\beta \int_0^t \int_0^1 (\tilde{\tau})^2\,\mathrm{d}x_1\,\mathrm{d}\eta + \frac{1+h}{\delta_3\pi}T_m \int_0^t \int_\Omega \Big(\frac{\partial \tilde{u}}{\partial x_1}\Big)^2\,\mathrm{d}A\,\mathrm{d}\eta \\
&\quad + \frac{(2m_3 + m_4\delta_2^{-1})\gamma_3\beta}{2}\int_0^t \int_\Omega \widetilde{T}^2\,\mathrm{d}A\,\mathrm{d}\eta.
\end{aligned}
$$

Combining (4.17) and (4.22), we have for a positive constant $\theta = \frac{1}{2}\delta_3\pi\gamma_1((1+h)T_m)$,

(4.23)

$$
\begin{aligned}
& \int_\Omega \tilde{u}^2\,\mathrm{d}A + \int_\Omega \tilde{v}^2\,\mathrm{d}A + \theta\int_\Omega \widetilde{T}^2\,\mathrm{d}A + \frac{\gamma_1}{2}\int_0^t \int_\Omega |\nabla \tilde{u}|^2\,\mathrm{d}A\,\mathrm{d}\eta \\
&\quad + \gamma_2 \int_0^t \int_\Omega |\nabla \tilde{v}|^2\,\mathrm{d}A\,\mathrm{d}\eta + \theta\gamma_3 \int_0^t \int_\Omega |\nabla \widetilde{T}|^2\,\mathrm{d}A\,\mathrm{d}\eta \\
&\leqslant b_6(t)\Big(\int_0^t \int_\Omega \tilde{u}^2\,\mathrm{d}A\,\mathrm{d}\eta + \int_0^t \int_\Omega \tilde{v}^2\,\mathrm{d}A\,\mathrm{d}\eta + \theta\int_0^t \int_\Omega \widetilde{T}^2\,\mathrm{d}A\,\mathrm{d}\eta\Big) \\
&\quad + \gamma_1\alpha_1 \int_0^t \int_0^1 \tilde{g}_1^2\,\mathrm{d}x_1\,\mathrm{d}\eta + \gamma_2\alpha_2 \int_0^t \int_0^1 \tilde{g}_2^2\,\mathrm{d}x_1\,\mathrm{d}\eta + \theta\gamma_3\beta \int_0^t \int_0^1 \tilde{\tau}^2\,\mathrm{d}x_1\,\mathrm{d}\eta,
\end{aligned}
$$

where $b_6(t)$ is a positive computable function. Applying the Gronwall inequality to (4.23), we get

$$
\begin{aligned}
(4.24)\quad & \int_0^t \int_\Omega \tilde{u}^2\,\mathrm{d}A\,\mathrm{d}\eta + \int_0^t \int_\Omega \tilde{v}^2\,\mathrm{d}A\,\mathrm{d}\eta + \theta\int_0^t \int_\Omega \widetilde{T}^2\,\mathrm{d}A\,\mathrm{d}\eta \\
&\leqslant \int_0^t \mathrm{e}^{\int_s^t b_6(\eta)\,\mathrm{d}\eta}\int_0^s \int_0^1 [\gamma_1\alpha_1\tilde{g}_1^2 + \gamma_2\alpha_2\tilde{g}_2^2 + \theta\gamma_3\beta\tilde{\tau}^2]\,\mathrm{d}x_1\,\mathrm{d}\eta\,\mathrm{d}s.
\end{aligned}
$$

Finally, if we insert (4.24) into (4.23), the proof of Theorem 4.1 is completed. □

Conclusions. In this paper, we have obtained the continuous dependence of the two-dimensional large-scale primitive equations in oceanic dynamics, where the depth of the ocean is assumed to be a positive constant. When the depth of the ocean is positive but not always constant, Huang and Guo [10] have obtained the existence and uniqueness of global strong solution for the problem. The study of the continuous dependence of the primitive equations in this case may be more interesting.

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