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CONTROLLABLE AND TOLERABLE GENERALIZED EIGENVECTORS OF INTERVAL MAX-PLUS MATRICES

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By max-plus algebra we mean the set of reals $\mathbb{R}$ equipped with the operations $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$ for $a, b \in \mathbb{R}$. A vector x is said to be a generalized eigenvector of max-plus matrices A, $B \in \mathbb{R}(m, n)$ if $A \otimes x = \lambda \otimes B \otimes x$ for some $\lambda \in \mathbb{R}$. The investigation of properties of generalized eigenvectors is important for the applications. The values of vector or matrix inputs in practice are usually not exact numbers and they can be rather considered as values in some intervals. In this paper the properties of matrices and vectors with inexact (interval) entries are studied and complete solutions of the controllable, the tolerable and the strong generalized eigenproblem in max-plus algebra are presented.

As a consequence of the obtained results, efficient algorithms for checking equivalent conditions are introduced.

Keywords: interval generalized eigenvector, fuzzy matrix

Classification: 15A80, 15A18, 08A72

1. INTRODUCTION

Max-plus algebra (the addition and the multiplication are formally replaced by operations of maximum and plus) can be used in a range of practical problems related to scheduling, optimization, modeling of discrete dynamic systems, graph theory, knowledge engineering, cluster analysis.

The research of max-plus algebra can be motivated by max-plus multi-processor interaction systems [3, 4, 10]. In these generalized systems we have m entities $E_1, \ldots, E_m$ (processors, servers, machines, etc.) producing entity outputs $O_1, \ldots, O_n$ (data, products, etc) working in stages whereby each entity is contributing to the completion of each entity output and working for all outputs simultaneously. In the algebraic model of their interactive work, entry $x_i(k)$ of a vector $x(k)$, represents the state of entity E_i after some stage k, and the entry a_{ij} of a matrix A encodes the influence of the work of entity E_j in the previous stage on the work of entity E_i in the current stage to complete the partial entity output O_i. Summing up all the influence effects multiplied by the results of

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previous stages, we have $\bigoplus_j a_{ij} \otimes x_j(k)$. The summation is often interpreted as waiting till all works of the system are finished and all the necessary influence constraints are satisfied. Moreover, similarly as in [4], suppose that m other entities $F_1, \dots, F_m$ prepare partial entity outputs for entity outputs $U_1, \dots, U_n$, whereby b_{ij} and y_j, alike as above, encode the influence of the work and the state of the corresponding entity, respectively. If the entities are linked then it may be required that $y_j = \lambda \otimes x_j$, where λ is some constant from $\mathbb{R}$. Consider a synchronization problem: to find λ and states of all $2m$ entities so that each pair (O_i, U_i) is completed at the same state. Algebraically, we have to solve the generalized eigenproblem, $A \otimes x = \lambda \otimes B \otimes x$.

The aim of this paper is to characterize a generalized eigenvector and present equivalent conditions for vectors with inexact (interval) entries. Moreover, this paper describes polynomial algorithms recognizing whether a given interval vector is a generalized eigenvector of a given interval matrices, for three types of interval generalized eigenvectors (the controllable, the tolerable and the strong generalized eigenvectors).

Let us give more details on the organization of the paper and on the obtained results. The next section will be occupied by definitions and notation of generalized eigenproblem, leading to the discussion of conditions for the existence of a generalized eigenvector. Sections 3, 4 deals with definitions of various versions of interval generalized eigenvectors. Sections 5, 6, 7 are devoted to the characterization of the equivalent conditions for the controllable, the tolerable and the strong generalized eigenvectors. Based on the results we also analyze the computational complexity of checking the conditions obtained in Theorem 5.3, Theorem 6.4 and Theorem 7.3.

Let us conclude with a brief overview of the works on max-plus algebra to which this paper is related. The concept of a generalized eigenproblem was studied for the first time independently in [2] and [7]. Some characteristics for the spectrum of the system $A \otimes x = \lambda \otimes B \otimes x$ were presented in [8, 22]. In particular, see [2] for some necessary or sufficient conditions for a solvability of the system. A special method for finding a generalized eigenvalue is given in [3]. The iteration method for the search whole spectrum is presented in [8]. In the paper [22] is shown that any union of closed intervals is the spectrum for some generalized eigenproblem. The paper [4] is devoted to a generalized eigenproblem for the special matrix B.

2. PRELIMINARIES

Denote the set of real numbers by $\mathbb{R}$ and the set of all natural numbers by $\mathbb{N}$. The symbol $\overline{\mathbb{R}}$ will stand for $\mathbb{R} \cup \{-\infty\}$. For two elements $a, b \in \overline{\mathbb{R}}$ we set $a \oplus b = \max(a; b)$ and $a \otimes b = a + b$. Throughout the paper we denote $-\infty$, the neutral element with respect to $\oplus$, by ε and the neutral element 0 with respect to $\otimes$, by e. Suppose that $n \geq 1, m \geq 1$ are given integers. The set of $n \times m$ matrices over $\overline{\mathbb{R}}$ is denoted by $\overline{\mathbb{R}}(n, m)$, specially the set of $n \times 1$ vectors over $\overline{\mathbb{R}}$ is denoted by $\overline{\mathbb{R}}(n)$. The triple $(\overline{\mathbb{R}}, \oplus, \otimes)$ is called *max-plus algebra*. The operations $\oplus, \otimes$ are extended to the matrix-vector algebra over $\overline{\mathbb{R}}$ by the direct analogy to the conventional linear algebra. If each entry of a matrix $A \in \overline{\mathbb{R}}(n, n)$ (a vector $x \in \overline{\mathbb{R}}(n)$) is equal to ε we shall denote this as $A = \varepsilon$ $(x = \varepsilon)$.

For $A \in \mathbb{R}(m, n)$, $C \in \mathbb{R}(m, n)$ we write $A \leq C$ if $a_{ij} \leq c_{ij}$ holds true for all i, $j \in N$. Similarly, for $x = (x_1, \dots, x_n)^T \in \mathbb{R}(n)$ and $y = (y_1, \dots, y_n)^T \in \mathbb{R}(n)$ we write $x \leq y$ if $x_i \leq y_i$ for each $i \in N$.

Let $N = \{1, 2, \ldots, n\}$ and C_n be the set of all cyclic permutations defined on non-empty subsets of N. For a cyclic permutation $\sigma = (i_1, i_2, \ldots, i_l) \in C_n$ and for $A \in \overline{\mathbb{R}}(n,n)$ we denote the length of σ by $l(\sigma)$ and define

$$w_A(\sigma) = a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_l i_1}, \quad \mu_A(\sigma) = \frac{w_A(\sigma)}{l(\sigma)}.$$

The *eigenproblem* in max-plus algebra is formulated as follows: Given $A \in \overline{\mathbb{R}}(n,n)$, find $x \in \overline{\mathbb{R}}(n)$ and $\lambda \in \mathbb{R}$ satisfying

$$A \otimes x = \lambda \otimes x.$$

A *digraph* is a pair $D = (V, E)$, where V, called the node set, is a finite set, and E, called the edge set, is a subset of $V \times V$. A digraph $D' = (V', E')$ is a subdigraph of the digraph D (for brevity $D' \subseteq D$), if $V' \subseteq V$ and $E' \subseteq E$.

A sequence $p = (v_1, \ldots, v_k)$ of nodes in D is called a path (in D) if $k = 1$ or $k > 1$ and $(v_i, v_{i+1}) \in E$ for all $i = 1, \ldots, k-1$. A path $(v_1, \ldots, v_k)$ is called a cycle if $v_1 = v_k$ and $k > 1$, and it is called an elementary path (cycle) if, moreover, $v_i \neq v_j$ for $i, j = 1, ..., k,\ i \neq j$ $(i, j = 1, ..., k-1,\ i \neq j)$. The number k is the length of the path p (cycle c) and is denoted by $l(p)$ $(l(c))$. By a *strongly connected component* of a digraph $D = (V, E)$ we mean a subdigraph $\mathcal{K} = (V_{\mathcal{K}}, E_{\mathcal{K}})$ generated by a non-empty subset $V_{\mathcal{K}} \subseteq V$ such that any two distinct nodes $i, j \in V_{\mathcal{K}}$ are contained in a common cycle, $E_{\mathcal{K}} = E \cap (V_{\mathcal{K}} \times V_{\mathcal{K}})$ and $V_{\mathcal{K}}$ is the maximal subset with this property. A strongly connected component $\mathcal{K}$ of a digraph is called non-trivial, if there is a cycle of positive length in $\mathcal{K}$.

The symbol $D_A = (V, E)$ stands for a complete, edge-weighted digraph associated with A. The node set of D_A is N, the edge set E of D_A is the set $\{(i,j);\ a_{ij} > \varepsilon\}$ and the weight of any edge (i, j) is a_{ij}. Throughout the paper, by a cycle in the digraph we mean an elementary cycle or a loop, and by path we mean a nontrivial elementary path, i. e., an elementary path containing at least one edge. Evidently, we will use the same notation, as well as the concept of weight, both for cycles and cyclic permutations.

The matrix A is called irreducible if D_A is strongly connected, reducible otherwise.

Theorem 2.1. (Cuninghame-Green [6]) Each square irreducibile matrix has exactly one eigenvalue (denoted as $\lambda(A)$). This unique eigenvalue is equal to the maximal average weight of cycles in D_A $(\lambda(A) = \max_{\sigma \in C_n} \mu_A(\sigma))$.

A cycle $\sigma \in C_n$ is *critical*, if $\mu_A(\sigma) = \lambda(A)$, a node in D_A is called *critical* if it is contained in at least one critical cycle; N_A^c stands for the set of all critical nodes in D_A and E_A^c denotes the set of all edges of all critical cycles in D_A. If $i, j \in N_A^c$ belong to the same critical cycle then i and j are called equivalent otherwise they are called nonequivalent.

The critical digraph of A is the digraph $C(A)$ with the set of nodes N_A^c and the set of edges E_A^c.

Theorem 2.2. (Cuninghame-Green [6]) Let $A \in \overline{\mathbb{R}}(n,n)$ and $\alpha \in \mathbb{R}$. Then $\lambda(\alpha \otimes A) = \alpha \otimes \lambda(A)$.

The problem of finding the eigenvalue $\lambda(A)$ is also called the *maximum cycle mean problem* whereby various algorithms for solving this problem are known, that of Karp [12] having the worst-case performance $O(n^3)$ and Howard's algorithm [11] of unproven computational complexity showing excellent algorithmic performance.

For $B \in \overline{\mathbb{R}}(n,n)$ we denote by $\Delta(B)$ the matrix $B \oplus B^2 \oplus \ldots \oplus B^n$ where B^s stands for the s-fold iterated product $B \otimes B \otimes \ldots \otimes B$.
Let $A_\lambda = \lambda(A)^{-1} \otimes A$. (The upper index -1 denotes the inverse element of $\lambda(A)$ in the sense of the group operation $\otimes$). It was shown in [6] that the matrix $\Delta(A_\lambda)$ contains at least one column, the diagonal element of which is e. Every such column is an eigenvector of the matrix A, it is called a *fundamental eigenvector* of the matrix A. The set of all fundamental eigenvectors is denoted by F_A. We say that two fundamental eigenvectors g_i and g_j are equivalent if $g_i = \alpha \otimes g_j$ for some $\alpha \in \mathbb{R}$ and nonequivalent otherwise.

Theorem 2.3. (Cuninghame-Green [6]) Let $g_1, g_2, \ldots, g_n$ denote the columns of the matrix $\Delta(A_\lambda)$. Then

(i) $j \in N_A^c$ if and only if $g_j \in F_A$;

(ii) g_i, g_j are equivalent members of F_A if and only if the nodes i, j are contained in a common critical cycle.

For a given irreducible matrix $A \in \overline{\mathbb{R}}(n,n)$, the *eigenspace* $V(A, \lambda(A))$ is defined as the set of all eigenvectors of A with associated eigenvalue $\lambda(A)$, i. e.,

$$V(A, \lambda(A)) = \{x \in \overline{\mathbb{R}}(n);\ A \otimes x = \lambda(A) \otimes x\}.$$

Theorem 2.4. (Cuninghame-Green [6]) Let $g_1, g_2, \ldots, g_n$ denote the columns of the matrix $\Delta(A_\lambda)$. Then

$$V(A, \lambda(A)) = \left\{ \bigoplus_{j \in N_A^{c,*}} \alpha_j \otimes g_j;\ \alpha_j \in \overline{\mathbb{R}},\ j \in N_A^{c,*} \right\},$$

where $N_A^{c,*}$ is any maximal set of nonequivalent eigennodes of A.

Notice that $V(A, \lambda(A))$ creates a max cone, i. e., $\alpha \otimes u \oplus \beta \otimes v \in V(A, \lambda(A))$ for all $\alpha \in \mathbb{R}$ and $u, v \in V(A, \lambda(A))$.

Definition 2.5. Let $N_A^{c,*} = \{j_1, \ldots, j_k\}$ be any maximal set of nonequivalent eigennodes of A. Define the generating matrix of $V(A, \lambda(A))$ as the matrix resulting from stacking the columns $(\Delta(A_\lambda))_{\cdot j_1}, \ldots, (\Delta(A_\lambda))_{\cdot j_k}$ together:

$$G_{A,\lambda} = [(\Delta(A_\lambda))_{\cdot j_1}, \ldots, (\Delta(A_\lambda))_{\cdot j_k}] \tag{1}$$

3. INTERVAL EIGENVECTORS

In this section we will consider interval versions of the eigenvectors and define the greatest eigenvector which exists for bounded case in contrast with the unlimited case, the

greatest eigenvector does not exist.
Similarly to [13] – [21], we define interval vector with bounds $\underline{x}, \overline{x} \in \mathbb{R}(n)$ as follows

$$\boldsymbol{X} = [\underline{x}, \overline{x}] = \{\, x \in \mathbb{R}(n);\ \underline{x} \le x \le \overline{x}\,\}.$$

For a given $A \in \overline{\mathbb{R}}(n,n)$ and $\boldsymbol{X} \subseteq \mathbb{R}(n)$ define the greatest interval eigenvector $x^{\oplus}(A, \boldsymbol{X})$ corresponding to a matrix A and an interval vector $\boldsymbol{X}$ as

$$x^{\oplus}(A, \boldsymbol{X}) = \bigoplus_{x \in V(A,\lambda(A)) \cap \boldsymbol{X}} x.$$

If $\boldsymbol{X} = \mathbb{R}(n)$ then the max cone $V(A, \lambda(A))$ is not bounded and hence $x^{\oplus}(A, \boldsymbol{X})$ does not exist. This is in a contrast with the case that $\boldsymbol{X} \subset \mathbb{R}(n)$ for which $x^{\oplus}(A, \boldsymbol{X})$ exists if only if $V(A) \cap \boldsymbol{X} \neq \emptyset$.

Theorem 3.1. (Zimmermann [24]) Suppose given $A \in \overline{\mathbb{R}}(m,n)$ and $b \in \mathbb{R}(m)$. Then the system $A \otimes x = b$ is solvable if and only if $x^*(A,b) \in \mathbb{R}(n)$ is its solution, where $x_j^*(A,b) = \min_{i \in M}\{b_i - a_{ij}\}$ for $j \in N$.

Theorem 3.2. (Cechlárová [5]) Suppose given $B, C \in \overline{\mathbb{R}}(m,n)$ and $b, c \in \mathbb{R}(m)$. Then the system of inequalities

$$B \otimes x \le b,$$
$$C \otimes x \ge c$$

has a solution if and only if $C \otimes x^*(B,b) \ge c$.

According to the last theorem we can formulate the following lemma.

Lemma 3.3. Suppose given $A \in \overline{\mathbb{R}}(n,n)$ and $\boldsymbol{X}$. Then the system of inequalities

$$G_{A,\lambda} \otimes x \le \overline{x},$$
$$G_{A,\lambda} \otimes x \ge \underline{x}$$

is solvable if and only if $G_{A,\lambda} \otimes x^*(G_{A,\lambda}, \overline{x}) \ge \underline{x}$.

Proof. Suppose that $y \in \boldsymbol{X}$ is a solution of the system $(G_{A,\lambda} \otimes x \le \overline{x}) \wedge (G_{A,\lambda} \otimes x \ge \underline{x})$. Then $y \le x^*(G_{A,\lambda}, \overline{x})$ and hence we obtain

$$G_{A,\lambda} \otimes x^*(G_{A,\lambda}, \overline{x}) \ge G_{A,\lambda} \otimes y \ge \underline{x} \Rightarrow G_{A,\lambda} \otimes x^*(G_{A,\lambda}, \overline{x}) \ge \underline{x}.$$

The reverse implication trivially follows. □

Notice that the vector $G_{A,\lambda} \otimes x^*(G_{A,\lambda}, \overline{x})$ is the greatest eigenvector of A lying in $\boldsymbol{X}$ if it exists and is a max-plus linear combination of fundamental nonequivalent eigenvectors. Then we can state that

$$x^{\oplus}(A, \boldsymbol{X}) = G_{A,\lambda} \otimes x^*(G_{A,\lambda}, \overline{x}),$$

where $x^{\oplus}(A, \boldsymbol{X})$ can be computed in $O(n^3)$ elementary operations.

Suppose that matrices A, $B \in \overline{\mathbb{R}}(m, n)$ are given. The generalized eigenproblem for the couple (A, B) is the task of finding $x \in \overline{\mathbb{R}}(n), x \neq \varepsilon$ (generalized eigenvector) and $\lambda \in \overline{\mathbb{R}}$ (generalized eigenvalue or just eigenvalue) such that

$$A \otimes x = \lambda \otimes B \otimes x. \tag{2}$$

We denote the set of all generalized eigenvectors of (A, B) corresponding to $\lambda \in \overline{\mathbb{R}}$ and the set of generalized eigenvalues, called the spectrum of (A, B), by

$$V(A, B, \lambda) = \{x \in \overline{\mathbb{R}}(n);\ A \otimes x = \lambda \otimes B \otimes x\},$$

$$\Lambda(A, B) = \{\lambda \in \overline{\mathbb{R}};\ V(A, B, \lambda) \neq \emptyset\},$$

respectively. The next statement provides useful information about the spectrum.

Lemma 3.4. (Butkovič [3]) $\Lambda(A, B) \subseteq [\max_{i \in M} \min_{j \in N}(a_{ij} - b_{ij}), \min_{i \in M} \max_{j \in N}(a_{ij} - b_{ij})]$ holds for any $A, B \in \mathbb{R}(m, n)$.

Define the greatest generalized interval eigenvector $x^{\oplus}(A, B, \lambda, \boldsymbol{X})$ corresponding to matrices (A, B), eigenvalue λ and an interval vector $\boldsymbol{X}$ as

$$x^{\oplus}(A, B, \lambda, \boldsymbol{X}) = \bigoplus_{x \in V(A,B,\lambda) \cap \boldsymbol{X}} x.$$

The task to find the greatest generalized interval eigenvector corresponding to (A, B), eigenvalue λ and an interval vector $\boldsymbol{X}$ if it exists, is hard problem since the solvability of the system $A \otimes x = B \otimes x$ was generally shown to be polynomially equivalent to solving a mean-payoff game [1], for which efficient pseudopolynomial algorithms and an alternating method for computing $A \otimes x = B \otimes x$ ([3]) exist, but existence of a polynomial algorithm has been a long-standing open question.

4. CLASSIFICATION OF INTERVAL GENERALIZED EIGENVECTORS

Analogously as above define interval matrices with bounds $\underline{A}, \overline{A} \in \mathbb{R}(m, n)$ and $\underline{B}, \overline{B} \in \mathbb{R}(m, n)$ as follows

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \left\{ A \in \mathbb{R}(m, n);\ \underline{A} \leq A \leq \overline{A} \right\},$$

$$\boldsymbol{B} = [\underline{B}, \overline{B}] = \left\{ B \in \mathbb{R}(m, n);\ \underline{B} \leq B \leq \overline{B} \right\},$$

respectively. We consider the following three types of interval generalized eigenvectors.

Definition 4.1. If $\boldsymbol{A}$, $\boldsymbol{B}$ and interval vector $\boldsymbol{X}$ are given, then $\boldsymbol{X}$ is called

- *a controllable generalized eigenvector* of $(\boldsymbol{A}, \boldsymbol{B})$
  if $(\exists \lambda \in \mathbb{R})(\exists A \in \boldsymbol{A})(\forall B \in \boldsymbol{B})(\forall x \in \boldsymbol{X})\ x \in V(A, B, \lambda)$,

- *a tolerable generalized eigenvector* of $(\boldsymbol{A}, \boldsymbol{B})$ if $(\exists\lambda \in \mathbb{R})(\forall A \in \boldsymbol{A})(\exists B \in \boldsymbol{B})(\forall x \in \boldsymbol{X})\ x \in V(A, B, \lambda)$,
- *a strong generalized eigenvector* of $(\boldsymbol{A}, \boldsymbol{B})$ if $(\exists\lambda \in \mathbb{R})(\forall A \in \boldsymbol{A})(\forall B \in \boldsymbol{B})(\forall x \in \boldsymbol{X})\ x \in V(A, B, \lambda)$.

For given indices $i \in M, j \in N$ we define matrix $\tilde{A}^{(ij)} \in \mathbb{R}(m, n)$ and vector $\tilde{x}^{(i)}$ by putting for every $k \in M, l \in N$

$$\tilde{a}_{kl}^{(ij)} = \begin{cases} \overline{a}_{ij}, & \text{for } \ k = i,\, l = j \\ \underline{a}_{kl}, & \text{otherwise} \end{cases}, \qquad \tilde{x}_k^{(i)} = \begin{cases} \overline{x}_i, & \text{for } \ k = i \\ \underline{x}_k, & \text{otherwise} \end{cases}.$$

Lemma 4.2. Suppose given $x \in \mathbb{R}(n)$ and $A \in \mathbb{R}(m, n)$. Then

(i) $x \in \boldsymbol{X}$ if and only if $x = \bigoplus_{i \in N} \gamma_i \otimes \tilde{x}^{(i)}$ for some values $\gamma_i \in \mathbb{R}$ with $\underline{x}_i - \overline{x}_i \le \gamma_i \le 0$,

(ii) $A \in \boldsymbol{A}$ if and only if $A = \bigoplus_{i \in M, j \in N} \alpha_{ij} \otimes \tilde{A}^{(ij)}$ for some values $\alpha_{ij} \in \mathbb{R}$ with $\underline{a}_{ij} - \overline{a}_{ij} \le \alpha_{ij} \le 0$.

Proof. For the proof of assertion (i), let us suppose that $x \in \boldsymbol{X}$, i.e. the inequalities $\underline{x}_i \le x_i \le \overline{x}_i$ hold for every $i \in N$. This implies $\underline{x}_i - \overline{x}_i \le x_i - \overline{x}_i \le 0$. Denoting $\gamma_i = x_i - \overline{x}_i$ we get $\gamma_i \otimes \overline{x}_i = \gamma_i + \overline{x}_i = x_i$ and $\gamma_i \otimes \underline{x}_j = \gamma_i + \underline{x}_j = x_i - \overline{x}_i + \underline{x}_j \le \underline{x}_j$ for every $i \neq j$. Since $\underline{x}_j \le x_j$, we obtain that

$$\bigoplus_{i \in N} \gamma_i \otimes \tilde{x}_j^{(i)} = \gamma_j \otimes \tilde{x}_j^{(j)} \oplus \bigoplus_{i \in N,\ i \neq j} \gamma_i \otimes \tilde{x}_j^{(i)} = \gamma_j \otimes \overline{x}_j \oplus \bigoplus_{i \in N,\ j \neq i} \gamma_i \otimes \underline{x}_j = x_j$$

for all $j \in N$. Thus x can be expressed as a max-linear combination of $\tilde{x}^{(i)}$ for $i \in N$.

The proof of assertion (ii) is analogous. □

5. CONTROLLABLE GENERALIZED EIGENVECTOR

Theorem 5.1. Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$. Then $\boldsymbol{X}$ is controllable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if there exist $\lambda \in \mathbb{R}$ and $A \in \boldsymbol{A}$ such that $(\forall B \in \boldsymbol{B})(\forall k \in N)[A \otimes \tilde{x}^{(k)} = \lambda \otimes B \otimes \tilde{x}^{(k)}]$.

Proof. Assume that there are $\lambda \in \mathbb{R}$ and $A \in \boldsymbol{A}$ such that $(\forall B \in \boldsymbol{B})(\forall k \in N)[A \otimes \tilde{x}^{(k)} = \lambda \otimes B \otimes \tilde{x}^{(k)}]$. Then by Lemma 4.2(i) for arbitrary $x \in \boldsymbol{X}$ we get

$$A \otimes x = A \otimes \bigoplus_{k=1}^{n} \gamma_k \otimes \tilde{x}^{(k)} = \bigoplus_{k=1}^{n} \gamma_k \otimes (A \otimes \tilde{x}^{(k)})$$

$$= \bigoplus_{k=1}^{n} \gamma_k \otimes (\lambda \otimes B \otimes \tilde{x}^{(k)}) = \lambda \otimes B \otimes x.$$

The converse implication trivially follows. □

Theorem 5.2. Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$. Then $\boldsymbol{X}$ is controllable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if there exist $\lambda \in \mathbb{R}$ and $A \in \boldsymbol{A}$ such that $A \otimes \tilde{x}^{(k)} = \lambda \otimes \underline{B} \otimes \tilde{x}^{(k)}$ and $A \otimes \tilde{x}^{(k)} = \lambda \otimes \overline{B} \otimes \tilde{x}^{(k)}$ for all $k \in N$.

Proof. Assume that there are $\lambda \in \mathbb{R}$ and $A \in \boldsymbol{A}$ such that $A \otimes \tilde{x}^{(k)} = \lambda \otimes \underline{B} \otimes \tilde{x}^{(k)}$ and $A \otimes \tilde{x}^{(k)} = \lambda \otimes \overline{B} \otimes \tilde{x}^{(k)}$ for all $k \in N$. Then for arbitrary $x \in \boldsymbol{X}$ we get

$$A \otimes x = A \otimes \bigoplus_{k=1}^{n} \gamma_k \otimes \tilde{x}^{(k)} = \bigoplus_{k=1}^{n} \gamma_k \otimes (A \otimes \tilde{x}^{(k)})$$

$$= \bigoplus_{k=1}^{n} \gamma_k \otimes (\lambda \otimes \underline{B} \otimes \tilde{x}^{(k)}) = \lambda \otimes \underline{B} \otimes \bigoplus_{k=1}^{n} \gamma_k \otimes \tilde{x}^{(k)} = \lambda \otimes \underline{B} \otimes x$$

and

$$A \otimes x = A \otimes \bigoplus_{k=1}^{n} \gamma_k \otimes \tilde{x}^{(k)} = \bigoplus_{k=1}^{n} \gamma_k \otimes (A \otimes \tilde{x}^{(k)})$$

$$= \bigoplus_{k=1}^{n} \gamma_k \otimes (\lambda \otimes \overline{B} \otimes \tilde{x}^{(k)}) = \lambda \otimes \overline{B} \otimes \bigoplus_{k=1}^{n} \gamma_k \otimes \tilde{x}^{(k)} = \lambda \otimes \overline{B} \otimes x.$$

By monotonicity of the operations $\oplus$, $\otimes$ we obtain

$$A \otimes x = \lambda \otimes \underline{B} \otimes x \leq \lambda \otimes B \otimes x \leq \lambda \otimes \overline{B} \otimes x = A \otimes x,$$

i.e. the equality $\lambda \otimes B \otimes x = A \otimes x$ holds for each $B \in \boldsymbol{B}$ and each $x \in \boldsymbol{X}$.

The converse implication trivially follows. □

To recognize the existence of $A \in \boldsymbol{A}$ in Theorem 5.2, we define the vector $\tilde{C} \in \mathbb{R}(2mn)$ and the matrix $\tilde{D} \in \mathbb{R}(2mn, mn)$ as follows:

$$\tilde{C} = \begin{pmatrix} \underline{B} \otimes \tilde{x}^{(1)} \\ \vdots \\ \underline{B} \otimes \tilde{x}^{(n)} \\ \overline{B} \otimes \tilde{x}^{(1)} \\ \vdots \\ \overline{B} \otimes \tilde{x}^{(n)} \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} \tilde{A}^{(11)} \otimes \tilde{x}^{(1)} & \tilde{A}^{(12)} \otimes \tilde{x}^{(1)} & \dots & \tilde{A}^{(mn)} \otimes \tilde{x}^{(1)} \\ \vdots & & & \\ \tilde{A}^{(11)} \otimes \tilde{x}^{(n)} & \tilde{A}^{(12)} \otimes \tilde{x}^{(n)} & \dots & \tilde{A}^{(mn)} \otimes \tilde{x}^{(n)} \\ \tilde{A}^{(11)} \otimes \tilde{x}^{(1)} & \tilde{A}^{(12)} \otimes \tilde{x}^{(1)} & \dots & \tilde{A}^{(mn)} \otimes \tilde{x}^{(1)} \\ \vdots & & & \\ \tilde{A}^{(11)} \otimes \tilde{x}^{(n)} & \tilde{A}^{(12)} \otimes \tilde{x}^{(n)} & \dots & \tilde{A}^{(mn)} \otimes \tilde{x}^{(n)} \end{pmatrix}. \quad (3)$$

Consider the following max-plus linear system

$$\tilde{D} \otimes y = \lambda \otimes \tilde{C} \quad (4)$$

where the vector $y \in \mathbb{R}(mn)$ consists of the variables $y_{ij} \in \mathbb{R}$.

Theorem 5.3. Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$. Then $\boldsymbol{X}$ is controllable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if there is $\lambda \in \mathbb{R}$ such that the max-plus linear system $\tilde{D} \otimes y = \lambda \otimes \tilde{C}$ has a solution y satisfying the condition $\underline{a}_{ij} - \overline{a}_{ij} \leq y_{ij} \leq 0$, for every $i \in M, j \in N$.

P r o o f. Suppose that y is a solution of the linear system $\tilde{D} \otimes y = \lambda \otimes \tilde{C}$ satisfying the condition $\underline{a}_{ij} - \overline{a}_{ij} \leq y_{ij} \leq 0$, for every $i \in M, j \in N$. Then the matrix $A \in \mathbb{R}(m,n)$ defined as the max-plus linear combination

$$A = \bigoplus_{i \in M, j \in N} y_{ij} \otimes \tilde{A}^{(ij)} \tag{5}$$

belongs to the interval matrix $[\underline{A}, \overline{A}]$ in view of Lemma 4.2(ii).

Moreover, from the equality $\tilde{D} \otimes y = \lambda \otimes \tilde{C}$, we have the following block equations for every fixed $i \in N$

$$\bigoplus_{k \in M, l \in N} \left(\tilde{A}^{(kl)} \otimes \tilde{x}^{(i)} \right) \otimes y_{kl} = \lambda \otimes \underline{B} \otimes \tilde{x}^{(i)}$$
$$\Leftrightarrow \bigoplus_{k \in M, l \in N} \left(y_{kl} \otimes \tilde{A}^{(kl)} \right) \otimes \tilde{x}^{(i)} = \lambda \otimes \underline{B} \otimes \tilde{x}^{(i)} \Leftrightarrow A \otimes \tilde{x}^{(i)} = \lambda \otimes \underline{B} \otimes \tilde{x}^{(i)}$$

and

$$\bigoplus_{k \in M, l \in N} \left(\tilde{A}^{(kl)} \otimes \tilde{x}^{(i)} \right) \otimes y_{kl} = \lambda \otimes \overline{B} \otimes \tilde{x}^{(i)}$$
$$\Leftrightarrow \bigoplus_{k \in M, l \in N} \left(y_{kl} \otimes \tilde{A}^{(kl)} \right) \otimes \tilde{x}^{(i)} = \lambda \otimes \overline{B} \otimes \tilde{x}^{(i)} \Leftrightarrow A \otimes \tilde{x}^{(i)} = \lambda \otimes \overline{B} \otimes \tilde{x}^{(i)}.$$

Thus, in view of Theorem 5.2, $\boldsymbol{X}$ is controllable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$.

For the converse implication, let us assume that $\boldsymbol{X}$ is controllable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$, i.e. that there exist $\lambda \in \mathbb{R}$ and $A \in \boldsymbol{A}$ such that for each $B \in [\underline{B}, \overline{B}])$ and each $x \in [\underline{x}, \overline{x}])$ the equality $A \otimes x = \lambda \otimes B \otimes x$ holds true. By Lemma 4.2(ii), there exist coefficients $\alpha_{ij} \in \mathbb{R}$, $i \in M, j \in N$ such that $A = \bigoplus_{i \in M, j \in N} \alpha_{ij} \otimes \tilde{A}^{(ij)}$ and $\underline{a}_{ij} - \overline{a}_{ij} \leq \alpha_{ij} \leq 0$. It is easy to verify that $y \in \mathbb{R}(mn, 1)$, where $y_{ij} = \alpha_{ij}$ for every $i \in M, j \in N$, satisfy the conditions of the equality $\tilde{D} \otimes y = \lambda \otimes \tilde{C}$. □

Theorem 5.3 reduces the recognition problem whether $\boldsymbol{X}$ is controllable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ to the solvability problem of the system $\tilde{D} \otimes y = \lambda \otimes \tilde{C}$ with $\underline{a}_{ij} - \overline{a}_{ij} \leq y_{ij} \leq 0$.

Theorem 5.4. (Gavalec et al. [9]) Suppose given $C \in \mathbb{R}(r,t)$, $b \in \mathbb{R}(r)$ and $\underline{y}, \overline{y} \in \mathbb{R}(t)$. The problem of recognizing the solvability of bounded parametric max-plus linear system $C \otimes y = \lambda \otimes b$ with bounds $\underline{y} \leq y \leq \overline{y}$, for some value of parameter $\lambda \in \mathbb{R}$, can be solved in $O(rt)$ time.

Theorem 5.5. The recognition problem whether a given interval vector $\boldsymbol{X}$ is controllable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ is solvable in $O(m^2 n^3)$ time.

P r o o f. According to Theorem 5.3, the recognition problem on controllable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ is equivalent to recognizing whether the max-plus linear system $\tilde{D} \otimes y = \lambda \otimes \tilde{C}$ with $\underline{a}_{ij} - \overline{a}_{ij} \leq y_{ij} \leq 0$ is solvable for fixed $\lambda \in \mathbb{R}$. The computation of

$\tilde{C}$ needs $O(mn^2)$ time and the computation of $\tilde{D}$ requires to compute products $A^{(ij)} \otimes \tilde{x}^{(k)}$ for all $i \in M, j, k \in N$, while each of them needs $O(mn)$ time. Therefore, the computation of $\tilde{D}$ is done in $O(m^2n^3)$ steps. By Theorem 5.4 the computation of a parametric system $A \otimes y = \lambda \otimes b$ needs $O(rt)$ time, where $A \in \mathbb{R}(r,t)$ and $b \in \mathbb{R}(r)$ ([9]). Finally, when all entries have been computed, then the solvability of $\tilde{D} \otimes y = \lambda \otimes \tilde{C}$ with $\underline{a}_{ij} - \overline{a}_{ij} \le y_{ij} \le 0$ can be recognized in $O(mn^2) + O(m^2n^3) = O(m^2n^3)$ time. □

Example 5.6. Consider interval matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and interval vector $\boldsymbol{X}$ which have the forms

$$\underline{A} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}, \ \overline{A} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 2 \end{pmatrix},$$

$$\underline{B} = \begin{pmatrix} -2 & 1 \\ -1 & 0 \\ -1 & 1 \end{pmatrix}, \ \overline{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$\underline{x} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \ \overline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Task: Check whether $\boldsymbol{X}$ is controllable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$.

Solution: Denote $\tilde{X} = (\tilde{x}^{(1)}, \tilde{x}^{(2)})$. Then for

$$\tilde{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \tilde{x}^{(2)} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

and

$$\tilde{A}^{(11)} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}, \ \tilde{A}^{(21)} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \ \tilde{A}^{(31)} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 2 & 2 \end{pmatrix},$$

$$\tilde{A}^{(12)} = \tilde{A}^{(22)} = \tilde{A}^{(32)} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \\ 1 & 2 \end{pmatrix},$$

we have

$$\tilde{A}^{(ij)} \otimes \tilde{X} = \begin{pmatrix} 3 & 3 \\ 2 & 2 \\ 3 & 3 \end{pmatrix} \text{ for any } i \in \{1,2,3\}, \ j \in \{1,2\}$$

and

$$\underline{B} \otimes \tilde{X} = \overline{B} \otimes \tilde{X} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

The system $\tilde{D} \otimes y = \lambda \otimes \tilde{C}$ has the following form

$$\begin{pmatrix} 3&3&3&3&3&3\\ 2&2&2&2&2&2\\ 3&3&3&3&3&3\\ 3&3&3&3&3&3\\ 2&2&2&2&2&2\\ 3&3&3&3&3&3\\ 3&3&3&3&3&3\\ 2&2&2&2&2&2\\ 3&3&3&3&3&3\\ 3&3&3&3&3&3\\ 2&2&2&2&2&2\\ 3&3&3&3&3&3 \end{pmatrix} \otimes \begin{pmatrix} y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6 \end{pmatrix} = \lambda \otimes \begin{pmatrix} 2\\1\\2\\2\\1\\2\\2\\1\\2\\2\\1\\2 \end{pmatrix} \tag{6}$$

whereby $\underline{y}, \overline{y}$ are created from the entries of $\underline{A}, \overline{A}$ as follows $\underline{y} = (0-1, 2-2, 0-1, 1-1, 1-2, 2-2)^T = (-1,0,-1,0,-1,0)^T$ and $\overline{y} = (0,0,0,0,0,0)^T$. By Theorem A.5 of [9] system $\tilde{D} \otimes y = \lambda \otimes \tilde{C}$ is solvable if and only if $\tilde{D} \otimes y^*(\lambda_{\min}) = \lambda_{\min} \otimes \tilde{C}$ is solvable, where

$$m_j = \min_{1 \le i \le 12} (\tilde{c}_i - \tilde{d}_{ij}) \quad for \quad 1 \le j \le 6, \tag{7}$$

$$\lambda_{\min} = \max_{1 \le j \le 6} (\underline{y}_j - m_j), \tag{8}$$

$$y_j^*(\lambda_{\min}) = \begin{cases} \lambda_{\min} + m_j & \text{if } \lambda_{\min} + m_j \le \overline{y}_j, \\ \overline{y}_j & \text{otherwise.} \end{cases} \tag{9}$$

It is easy to show that $\lambda_{\min} = 1$ and vector $y^*(\lambda_{\min}) = (0,0,0,0,0,0)^T$ is the solution of the system. Now we can create the matrix A as follows:

$$A = 0 \otimes A^{(11)} \oplus 0 \otimes A^{(12)} \oplus 0 \otimes A^{(21)} \oplus 0 \otimes A^{(22)} \oplus 0 \otimes A^{(31)} \oplus 0 \otimes A^{(32)} = \overline{A}.$$

Thus, we conclude that $\boldsymbol{X}$ is controllable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$.

6. TOLERABLE GENERALIZED EIGENVECTOR

Lemma 6.1. Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$. Then $\boldsymbol{X}$ is a tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if

$$(\exists \lambda \in \mathbb{R})(\forall A \in \boldsymbol{A})(\exists B \in \boldsymbol{B})(\forall k \in N)[A \otimes \tilde{x}^{(k)} = \lambda \otimes B \otimes \tilde{x}^{(k)}].$$

Proof. Suppose that there is $\lambda \in \mathbb{R}$ such that $(\forall A \in \boldsymbol{A})(\exists B \in \boldsymbol{B})[A \otimes \tilde{x}^{(k)} = \lambda \otimes B \otimes \tilde{x}^{(k)}]$ holds for all $k \in N$ and $x \in \mathbb{R}(n)$ is an arbitrary vector in $\boldsymbol{X}$. Then in view of Lemma 4.2(i) we get $x = \bigoplus_{k \in N} \gamma_k \otimes \tilde{x}^{(k)}$. Therefore,

$$A \otimes x = A \otimes \bigoplus_{k \in N} \gamma_k \otimes \tilde{x}^{(k)} = \bigoplus_{k \in N} \gamma_k \otimes (A \otimes \tilde{x}^{(k)})$$

$$= \bigoplus_{k \in N} \gamma_k \otimes (\lambda \otimes B \otimes \tilde{x}^{(k)}) = \lambda \otimes B \otimes \bigoplus_{k \in N} \gamma_k \otimes \tilde{x}^{(k)} = \lambda \otimes B \otimes x.$$

The converse implication is trivial. □

Theorem 6.2. Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{X}$. Then $\boldsymbol{X}$ is a tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if

$$(\exists\lambda \in \mathbb{R})(\forall(k,l) \in M \times N)(\exists B \in \boldsymbol{B})(\forall x \in \boldsymbol{X})[\tilde{A}^{(kl)} \otimes x = \lambda \otimes B \otimes x].$$

Proof. We use the proof by contrapositive. Suppose that for each $\lambda \in \mathbb{R}$ there is $A \in \boldsymbol{A}$ such that for any $B \in \boldsymbol{B}$ there are $x \in \boldsymbol{X}$ and $i \in N$ such that $(A \otimes x)_i \neq \lambda \otimes (B \otimes x)_i$. We shall prove that for each $\lambda \in \mathbb{R}$ there is $(k,l) \in M \times N$ such that for any $B \in \boldsymbol{B}$ there are $x \in \boldsymbol{X}$ and $i \in N$ such that $(\tilde{A}^{(kl)} \otimes x)_i \neq \lambda \otimes (B \otimes x)_i$.
Consider two cases:
Case 1: $(A \otimes x)_i > \lambda \otimes (B \otimes x)_i$. Suppose that $a_{is} \otimes x_s = \bigoplus_{j \in N} a_{ij} \otimes x_j$. Then we have

$$(\tilde{A}^{(is)} \otimes x)_i = \overline{a}_{is} \otimes x_s \oplus \bigoplus_{j \neq s} \underline{a}_{ij} \otimes x_j$$

$$\geq a_{is} \otimes x_s = \bigoplus_{j \in N} a_{ij} \otimes x_j = (A \otimes x)_i > \lambda \otimes (B \otimes x)_i.$$

Case 2: $(A \otimes x)_i < \lambda \otimes (B \otimes x)_i$. If $a_{is} \otimes x_s = \bigoplus_{j \in N} a_{ij} \otimes x_j$ then for $r \neq i$ we have

$$(\tilde{A}^{(rs)} \otimes x)_i = \bigoplus_{j \in N} \underline{a}_{ij} \otimes x_j \leq \bigoplus_{j \in N} a_{ij} \otimes x_j = (A \otimes x)_i < \lambda \otimes (B \otimes x)_i.$$

In both cases we have obtained a contradiction.
The converse implication trivially follows. □

Theorem 6.3. Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{X}$. Then $\boldsymbol{X}$ is a tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if

$$(\exists\lambda \in \mathbb{R})(\forall(k,l) \in M \times N)(\exists B(k,l) \in \boldsymbol{B})(\forall r \in N)\tilde{A}^{(kl)} \otimes \tilde{x}^{(r)} = \lambda \otimes B(k,l) \otimes \tilde{x}^{(r)}.$$

Proof. The assertion follows from Lemma 6.1 and Theorem 6.2. □

Let $k \in M, l, r \in N$ be given and $B(k,l) \in \boldsymbol{B}$. Then in view of Lemma 4.2(ii) we get $B(k,l) = \bigoplus_{i \in M, j \in N} \beta_{ij}^{kl} \otimes \tilde{B}^{(ij)}$ for $\underline{b}_{ij} - \overline{b}_{ij} \leq \beta_{ij} \leq 0$ and hence

$$\tilde{A}^{(kl)} \otimes \tilde{x}^{(r)} = \lambda \otimes B(k,l) \otimes \tilde{x}^{(r)}$$

$$\Leftrightarrow (-\lambda) \otimes \tilde{A}^{(kl)} \otimes \tilde{x}^{(r)} = B(k,l) \otimes \tilde{x}^{(r)}$$

$$\Leftrightarrow (-\lambda) \otimes \tilde{A}^{(kl)} \otimes \tilde{x}^{(r)} = \bigoplus_{i \in M, j \in N} \beta_{ij}^{kl} \otimes \tilde{B}^{(ij)} \otimes \tilde{x}^{(r)}$$

$$\Leftrightarrow (-\lambda) \otimes \tilde{A}^{(kl)} \otimes \tilde{x}^{(r)} = \bigoplus_{i \in M, j \in N} (\tilde{B}^{(ij)} \otimes \tilde{x}^{(r)}) \otimes \beta_{ij}^{kl}.$$

To recognize the existence of $B(k,l) \in \boldsymbol{B}$ for each $k \in M$, $l \in N$ in Theorem 6.3, we define the matrices $\tilde{C} \in \mathbb{R}(m^2n^2, 1)$, $D(r) \in \mathbb{R}(m, mn)$ and the block-diagonal matrix $\tilde{D} \in \mathbb{R}(m^2n^2, m^2n^3)$ as follows:

$$\tilde{C} = \begin{pmatrix} \tilde{A}^{(11)} \otimes \tilde{x}^{(1)} \\ \tilde{A}^{(12)} \otimes \tilde{x}^{(1)} \\ \vdots \\ \tilde{A}^{(mn)} \otimes \tilde{x}^{(1)} \\ \tilde{A}^{(11)} \otimes \tilde{x}^{(2)} \\ \tilde{A}^{(12)} \otimes \tilde{x}^{(2)} \\ \vdots \\ \tilde{A}^{(mn)} \otimes \tilde{x}^{(2)} \\ \vdots \\ \tilde{A}^{(11)} \otimes \tilde{x}^{(n)} \\ \vdots \\ \tilde{A}^{(mn)} \otimes \tilde{x}^{(n)} \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} D(1) & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \dots & \varepsilon \\ \vdots & & & & & & \\ \varepsilon & \varepsilon & D(1) & \varepsilon & \varepsilon & \dots & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & D(2) & \varepsilon & \dots & \varepsilon \\ \vdots & & & & & & \\ \varepsilon & \varepsilon & \dots & \varepsilon & D(2) & \dots & \varepsilon \\ \vdots & & & & & & \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \dots & D(n) & \varepsilon \\ \vdots & & & & & & \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \dots & \varepsilon & D(n) \end{pmatrix}, \tag{10}$$

where

$$D(r) = \begin{pmatrix} \tilde{B}^{(11)} \otimes \tilde{x}^{(r)} & \dots & \tilde{B}^{(1n)} \otimes \tilde{x}^{(r)} & \tilde{B}^{(21)} \otimes \tilde{x}^{(r)} \dots & \tilde{B}^{(mn)} \otimes \tilde{x}^{(r)} \end{pmatrix}. \tag{11}$$

Consider the following max-plus linear system

$$\tilde{D} \otimes y = (-\lambda) \otimes \tilde{C} \tag{12}$$

where the vector $y \in \mathbb{R}(m^2n^3)$ consists of the variables $y_{ij}^{kl} \in \mathbb{R}$.

Theorem 6.4. Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$. Then $\boldsymbol{X}$ is tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if there is $\lambda \in \mathbb{R}$ such that the max-plus linear system $\tilde{D} \otimes y = \lambda \otimes \tilde{C}$ has a solution y satisfying the condition $\underline{b}_{ij} - \overline{b}_{ij} \leq y_{ij}^{kl} \leq 0$, for every $(i,j),(k,l) \in M \times N$.

Proof. Suppose that y is a solution of the linear system $\tilde{D} \otimes y = (-\lambda) \otimes \tilde{C}$ satisfying the condition $\underline{b}_{ij} - \overline{b}_{ij} \leq y_{ij}^{kl} \leq 0$, for every $(i,j),(k,l) \in M \times N$. Then the matrix $B(k,l) \in \mathbb{R}(m,n)$ defined as the max-plus linear combination

$$B(k,l) = \bigoplus_{i \in M, j \in N} y_{ij}^{kl} \otimes \tilde{B}^{(ij)} \tag{13}$$

belongs to the interval matrix $[\underline{B}, \overline{B}]$ in view of Lemma 4.2(ii).

Moreover, from the equality $\tilde{D} \otimes y = (-\lambda) \otimes \tilde{C}$, we have the following block equations for every fixed $k, r \in M, l \in N$

$$\bigoplus_{i\in M, j\in N} \left(\tilde{B}^{(ij)} \otimes \tilde{x}^{(r)}\right) \otimes y_{ij}^{kl} = (-\lambda) \otimes \tilde{A}^{(k,l)} \otimes \tilde{x}^{(r)}$$
$$\Leftrightarrow \bigoplus_{i\in M, j\in N} \left(y_{ij}^{kl} \otimes \tilde{B}^{(ij)}\right) \otimes \tilde{x}^{(r)} = (-\lambda) \otimes \tilde{A}^{(k,l)} \otimes \tilde{x}^{(r)}$$
$$\Leftrightarrow B(k,l) \otimes \tilde{x}^{(r)} = (-\lambda) \otimes \tilde{A}^{(k,l)} \otimes \tilde{x}^{(r)}$$

Thus, in view of Theorem 6.3, $\boldsymbol{X}$ is tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$.

For the converse implication, let us assume that $\boldsymbol{X}$ is tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$. By Theorem 6.3 there exists $\lambda \in \mathbb{R}$ such that

$$(\forall k \in M; l \in N)(\exists B(k,l) \in \boldsymbol{B})(\forall r \in N)\tilde{A}^{(kl)} \otimes \tilde{x}^{(r)} = \lambda \otimes B(k,l) \otimes \tilde{x}^{(r)}$$

and according to Lemma 4.2(ii), there exist coefficients $\beta_{ij}^{kl} \in \mathbb{R}$, $(i,j),(k,l) \in M \times N$ such that $B(k,l) = \bigoplus_{i\in M, j\in N} \beta_{ij}^{kl} \otimes \tilde{B}^{(ij)}$ and $\underline{b}_{ij} - \overline{b}_{ij} \leq \beta_{ij}^{kl} \leq 0$. It is easy to verify that $y \in \mathbb{R}(m^2n^3, 1)$, where $y_{ij}^{kl} = \beta_{ij}^{kl}$ satisfy the conditions of the equality $\tilde{D} \otimes y = (-\lambda) \otimes \tilde{C}$. □

Theorem 6.4 reduces the recognition problem whether $\boldsymbol{X}$ is tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ to the solvability problem of the system $\tilde{D} \otimes y = (-\lambda) \otimes \tilde{C}$ with $\underline{b}_{ij} - \overline{b}_{ij} \leq y_{ij}^{kl} \leq 0$.

Theorem 6.5. Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$. The recognition problem whether a given interval vector $\boldsymbol{X}$ is controllable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ is solvable in $O(m^4n^5)$ time.

Proof. According to Theorem 6.4, the recognition problem on tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ is equivalent to recognizing whether the max-plus linear system $\tilde{D} \otimes y = (-\lambda) \otimes \tilde{C}$ with $\underline{b}_{ij} - \overline{b}_{ij} \leq y_{ij}^{kl} \leq 0$ is solvable for fixed $\lambda \in \mathbb{R}$. The computation of $\tilde{C}$ needs $O(m^2n^3)$ time and the computation of $\tilde{D}$ requires to compute products $D(r)$ for all $r \in N$, while each of them needs $O(m^2n^2)$ time. Therefore, the computation of $\tilde{D}$ is done in $O(m^3n^4)$ steps. By Theorem 5.4 the computation of a parametric system $A \otimes y = \lambda \otimes b$ needs $O(rt)$ time, where $A \in \mathbb{R}(r,t)$ and $b \in \mathbb{R}(r)$ ([9]). Finally, when all entries have been computed, then the solvability of $\tilde{D} \otimes y = (-\lambda) \otimes \tilde{C}$ with $\underline{b}_{ij} - \overline{b}_{ij} \leq y_{ij}^{kl} \leq 0$ can be recognized in $O(m^2n^3) + O(m^3n^4) + O(m^4n^5) = O(m^4n^5)$ time. □

7. STRONG GENERALIZED EIGENVECTOR

Lemma 7.1. Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$. Then $\boldsymbol{X}$ is a strong generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if $(\exists \lambda \in \mathbb{R})(\forall A \in \boldsymbol{A})(\forall B \in \boldsymbol{B})(\forall k \in N)[A \otimes \tilde{x}^{(k)} = \lambda \otimes B \otimes \tilde{x}^{(k)}]$.

Proof. Suppose that there is $\lambda \in \mathbb{R}$ such that $(\forall A \in \boldsymbol{A})(\forall B \in \boldsymbol{B})[A \otimes \tilde{x}^{(k)} = \lambda \otimes B \otimes \tilde{x}^{(k)}]$ holds for all $k \in N$ and $x \in \mathbb{R}(n)$ is an arbitrary vector in $\boldsymbol{X}$. Then in view of Lemma 4.2(i) we get $x = \bigoplus_{k\in N} \gamma_k \otimes \tilde{x}^{(k)}$. Therefore,

$$A \otimes x = A \otimes \bigoplus_{k\in N} \gamma_k \otimes \tilde{x}^{(k)} = \bigoplus_{k\in N} \gamma_k \otimes (A \otimes \tilde{x}^{(k)})$$

$$= \bigoplus_{k\in N} \gamma_k \otimes (\lambda \otimes B \otimes \tilde{x}^{(k)}) = \lambda \otimes B \otimes \bigoplus_{k\in N} \gamma_k \otimes \tilde{x}^{(k)} = \lambda \otimes B \otimes x.$$

The converse implication is trivial. □

Theorem 7.2. Suppose given $\boldsymbol{A} = [\underline{A}, \overline{A}]$, $\boldsymbol{B} = [\underline{B}, \overline{B}]$ and $\boldsymbol{X}$. Then $\boldsymbol{X}$ is a strong generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if there is $\lambda \in \mathbb{R}$ such that $\underline{A} \otimes x = \lambda \otimes \overline{B} \otimes x$ and $\overline{A} \otimes x = \lambda \otimes \underline{B} \otimes x$.

Proof. Suppose that $\underline{A} \otimes x = \lambda \otimes \overline{B} \otimes x$, $\overline{A} \otimes x = \lambda \otimes \underline{B} \otimes x$ and $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$ are arbitrary but fixed matrices. Then the inequalities $\lambda \otimes \overline{B} \otimes x = \underline{A} \otimes x \le A \otimes x \le \overline{A} \otimes x = \lambda \otimes \underline{B} \otimes x$ and the monotonicity of the operations imply the following $A \otimes x = \lambda \otimes B \otimes x$.

The reverse implications trivially follows. □

Theorem 7.3. Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$. Then $\boldsymbol{X}$ is a strong generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if there is $\lambda \in \mathbb{R}$ such that $\underline{A} \otimes \tilde{x}^{(t)} = \lambda \otimes \overline{B} \otimes \tilde{x}^{(t)}$ and $\overline{A} \otimes \tilde{x}^{(t)} = \lambda \otimes \underline{B} \otimes \tilde{x}^{(t)}$ for any $t \in N$.

Proof. The equivalence follows from Lemma 7.1 and Theorem 7.2. □

Theorem 7.3 reduces the recognition problem whether $\boldsymbol{X}$ is a strong generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ to the solvability of max-plus linear systems.

Theorem 7.4. The recognition problem whether a given interval vector $\boldsymbol{X}$ is a strong generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ is solvable in $O(mn^2)$ time.

Proof. According to Theorem 7.3, the recognition problem of strong generalized eigenproblem of $(\boldsymbol{A}, \boldsymbol{B})$ is equivalent to recognizing whether the systems $\underline{A} \otimes \tilde{x}^{(t)} = \lambda \otimes \overline{B} \otimes \tilde{x}^{(t)}$ and $\overline{A} \otimes \tilde{x}^{(t)} = \lambda \otimes \underline{B} \otimes \tilde{x}^{(t)}$ for each $t \in N$ are solvable for some $\lambda \in \mathbb{R}$. The computation of these systems requires to compute products $\underline{A} \otimes \tilde{x}^{(t)}$, $\lambda \otimes \overline{B} \otimes \tilde{x}^{(t)}$, $\overline{A} \otimes \tilde{x}^{(t)}$, $\lambda \otimes \underline{B} \otimes \tilde{x}^{(t)}$ for any $t \in N$, while each of them needs $O(mn)$ time. Therefore, the computation is done in $4n \cdot O(mn) = O(mn^2)$ time for a given $\lambda \in \mathbb{R}$, whereby λ can be computed from the first equality of the system $\underline{A} \otimes \tilde{x}^{(t)} = \lambda \otimes \overline{B} \otimes \tilde{x}^{(t)}$. Finally, when all entries have been computed, then the solvability of $\underline{A} \otimes \tilde{x}^{(t)} = \lambda \otimes \overline{B} \otimes \tilde{x}^{(t)}$ and $\overline{A} \otimes \tilde{x}^{(t)} = \lambda \otimes \underline{B} \otimes \tilde{x}^{(t)}$ for each $t \in N$ can be recognized in $O(mn^2)$ time. □

8. CONCLUSION

In this paper we have presented a generalized eigenproblem and equivalent conditions for three types of an interval generalized eigenproblem in max-plus algebra. All results have been formally analyzed with a target to suggest the computational complexity of checking the obtained equivalent conditions. These results are illustrated by a numerical example.

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