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DISTRIBUTIVITY OF ORDINAL SUM IMPLICATIONS OVER OVERLAP AND GROUPING FUNCTIONS

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In 2015, a new class of fuzzy implications, called ordinal sum implications, was proposed by Su et al. They then discussed the distributivity of such ordinal sum implications with respect to t-norms and t-conorms. In this paper, we continue the study of distributivity of such ordinal sum implications over two newly-born classes of aggregation operators, namely overlap and grouping functions, respectively. The main results of this paper are characterizations of the overlap and/or grouping function solutions to the four usual distributive equations of ordinal sum fuzzy implications. And then sufficient and necessary conditions for ordinal sum implications distributing over overlap and grouping functions are given.

Keywords: distributivity, fuzzy implication functions, ordinal sum, overlap functions, grouping functions

Classification: 03B52, 03E72

1. INTRODUCTION

1.1. Brief overview on overlap and grouping functions

In recent years, two kinds of binary aggregation functions, called overlap and grouping functions respectively, were introduced by Bustince et al. [8, 9]. Those two functions arise from problems in image processing, decision making and classification [6, 9, 19, 20, 21, 23] based on fuzzy preference relations, where the associativity property is not strongly required in reality, and thus it's not necessary to consider t-norms and t-conorms as models of operations. In image processing, for example, in 2007, scholars such as Bustince et al. used the so-called restricted equivalent function to calculate the threshold value of images [6]. In decision making, in 2017, Elkano et al. [21] gave a consensus method via penalty functions for decision making in ensembles of fuzzy rule based classification systems and introduced a method for constructing confidence and support measures from overlap functions. In classification, in 2015, Elkano et al. [20] adapted the inference system of fuzzy association rule classification model replacing the product triangular norm with n-dimensional overlap function for high-dimensional problems. This enables us to obtain more sufficient output from the one-to-one and one-to-more pattern subsequent clustering of the basic classifier.

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On the other hand, overlap and grouping functions are also developing rapidly in theory and many profound results have been obtained, such as the construction of the corresponding fuzzy implications [10, 13, 16, 17, 40], the properties of migrativity, homogeneity, idempotency and limiting [4, 7, 14, 50, 51], the modularity equation between overlap (grouping) functions and other aggregation functions [42, 48, 49], the additive and multiplicative generator pairs [15, 32], the n-dimensional extension concepts of overlap functions [22] and general overlap functions [12], the notions of interval overlap [36] and grouping functions, and the notion of interval-valued ordered weight averaging (OWA) operators with interval weights derived from interval-valued overlap functions [5].

1.2. Significance and development of distributivity of fuzzy implications over aggregation functions

The solution of functional equations is one of the oldest research topics in the field of mathematical analysis, and many great mathematicians including Euler, Cauchy and Abel, have been concerned with solving functional equations [1, 26]. Distribution equation is a kind of functional equations which have been widely concerned in the literature.

The distributivity equations of fuzzy implications over two aggregation functions have become a focus of research. In classical logic, the distributivity of one binary operator over another determines the basic structure of the algebra imposed by these operators. Generally, there are four classical distributive equations involving implications as follows [2]:

$$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r) \tag{1}$$

$$(p \wedge q) \to r \equiv (p \to r) \vee (q \to r) \tag{2}$$

$$p \to (q \wedge r) \equiv (p \to q) \wedge (p \to r) \tag{3}$$

$$p \to (q \vee r) \equiv (p \to q) \vee (p \to r) \tag{4}$$

Note that the above equivalences are tautologies in classical logic. A new direction of studies focused on distributivity between implication functions and aggregation functions like t-norms (t-conorms) that is important in the framework of logical connectives. On the one hand, there is a great deal of literature studying and exploring the generalizations of these distributivity equations to fuzzy connectives, especially involving fuzzy implications. On the other hand, it is well known that fuzzy systems can approximate any continuous function to any desired precision. However, it usually takes a large rule base to achieve accuracy. In order to avoid the explosion of combinatorial rules, many researchers studied distributive equations. This idea was first proposed by Combs and Andrews [11] based on the logical validity of (2) in classical propositional logic. And then, in 2002, Trillas and Alsina [41] studied the general form $I(T(p,q),r) = S(I(p,r), I(q,r))$ of (2). Later, in [3, 24], Baczyński and Jayaram investigated the distributivity of f-implication over t-norms and t-conorms, implication over nilpotent or strict t-conorm, respectively. Recently, Zhou [47] characterized the t-norm and t-conorm solutions and continuous Archimedean t-conorm solutions for k-generated implications. Since then, lots of scholars have studied various distributive equations [27, 29, 38, 39, 45, 46].

The above are finding t-norm and t-conorm solutions to equations (1) − (4) for given fuzzy implications, some other studies found the fuzzy implication solutions to equations (1) − (4) for given t-norms and t-conorms. For example, Xie et al. [43] studied the equation $I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$ of (4), where S_1 and S_2 are two continuous t-conorms given as ordinal sums, they characterized all solutions of I when I is restricted to fuzzy implications continuous on $(0, 1] \times [0, 1]$. Qin et al. [30] characterized the fuzzy implication solutions which are continuous everywhere except at (0, 0), to (3) when T_1 is a continuous t-norm and T_2 is a continuous Archimedean t-norm. Along this method, others are already illustrated in the following works [31, 43, 44].

1.3. Motivation of our research

Fuzzy implication is one of main logical connectives of fuzzy set theory, which plays an important role in many branches of fuzzy mathematics. The research on distributive equations of fuzzy implications over other logical connectives is a hot topic with important theoretical significance and application value. And so many researches have done a lot of research to this subject and achieved fruitful results. Such an active research topic deserves further exploration. In 2015, a new class of fuzzy implications, called ordinal sum implications, was proposed by Su et al. [37]. Moreover, they studied some basic properties of this new class of fuzzy implications, such as neutral property, exchange principle, consequent boundary and so on. So far, although many researchers have studied various distributivity equations as mentioned above, few have studied the distributivity of ordinal sum implication over overlap and grouping functions, only Qiao and Hu [33, 35] have studied general implication solutions related to overlap and grouping functions. Therefore, in this paper, as a supplement of this research topic from this point of view, we study the distributivity of ordinal sum implications over overlap and grouping functions. More precisely, we characterize the structure of overlap and grouping functions in the following equations for a given ordinal sum implication:

$$I(G(x, y), z) = O(I(x, z), I(y, z)) \tag{5}$$

$$I(O(x, y), z) = G(I(x, z), I(y, z)) \tag{6}$$

$$I(x, O_1(y, z) = O_2(I(x, y), I(x, z)) \tag{7}$$

$$I(x, G_1(y, z) = G_2(I(x, y), I(x, z)) \tag{8}$$

for all $x, y, z \in [0, 1]$, where I is an ordinal sum implication, O, O_1, O_2 are overlap functions and G, G_1, G_2 are grouping functions.

The rest of this paper is organized as follows. Section 2 presents the basic concepts that are necessary to understand the paper, including the concepts related to fuzzy implication functions, overlap and grouping functions. Section 3 is the main part of this paper, where we investigate the sufficient and necessary conditions under which ordinal sum implications satisfy the equations (5) − (8) with respect to overlap and grouping functions respectively. Section 4 concludes the paper with final remarks and further works.

2. PRELIMINARIES

In this section, we will assume that the reader is familiar with the theory of t-norms, t-conorms and aggregation operators as well as all necessary results [2, 25]. We recall here only some fundamental concepts and results which shall be used in the paper. We begin with the definition of fuzzy implication function.

Definition 2.1. (Baczyński and Jayaram [2]) A binary function $I : [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:

(I1) I is non-increasing in its first variable;

(I2) I is non-decreasing in its second variable;

(I3) $I(0,0) = 1$;

(I4) $I(1,1) = 1$;

(I5) $I(1,0) = 0$.

Let I be a fuzzy implication, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$. Then we recall some important properties of fuzzy implications.

Definition 2.2. (Baczyński and Jayaram [2]) Let I be a fuzzy implication. Then I is said to satisfy:

(OP) The ordering property, if, for all $x, y \in [0,1]$, $I(x,y) = 1$ if and only if $x \leq y$.

(NP) The left neutrality property, if $I(1,y) = y$ for all $y \in [0,1]$.

(IP) The identity principle if $I(x,x) = 1$ for all $x \in [0,1]$.

(EP) The exchange principle, if $I(x, I(y,z)) = I(y, I(x,z))$ for all $x, y, z \in [0,1]$.

(CB) The consequent boundary, if $I(x,y) \geq y$ for all $x, y \in [0,1]$.

Example 2.3. (Baczyński and Jayaram [2]) Define a function $I : [0,1]^2 \to [0,1]$ as follows:

(1) $\forall x, y \in [0,1], I_{LK}(x,y) = \min\{1 - x + y, 1\}$.

(2) $\forall x, y \in [0,1]$, $I_{GG}(x,y) = \begin{cases} 1, & x \leq y, \\ \frac{y}{x}, & x > y. \end{cases}$

(3) $\forall x, y \in [0,1]$, $I_{RS}(x,y) = \begin{cases} 1, & x \leq y, \\ 0, & x > y. \end{cases}$

(4) $\forall x, y \in [0,1]$, $I_{WB}(x,y) = \begin{cases} 1, & x < 1, \\ y, & x = 1. \end{cases}$

(5) $\forall x, y \in [0,1]$, $I_{GD}(x,y) = \begin{cases} 1, & x \leq y, \\ y, & x > y. \end{cases}$

Then $I_{LK}, I_{GG}, I_{RS}, I_{WB}, I_{GD}$ are all fuzzy implications.

Definition 2.4. (Baczyński and Jayaram [2]) A function $N : [0,1] \to [0,1]$ is called a fuzzy negation if it satisfies the following conditions:

(N1) $N(0) = 1$ and $N(1) = 0$,

(N2) $N(y) \le N(x)$ for all $x, y \in [0,1]$ with $x \le y$.

Definition 2.5. (Su et al. [37]) Let $(I_k)_{k\in A}$ be a family of fuzzy implications, and $([a_k, b_k])_{k\in A}$ be a family of pairwise disjoint close subintervals of $[0,1]$ with $b_k > a_k > 0$ for all $k \in A$. Define the binary function I: $[0,1]^2 \to [0,1]$ by

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & x, y \in [a_k, b_k], \\ I_{GD}(x,y), & \text{otherwise.} \end{cases} \tag{9}$$

We call I given by (9) an ordinal sum of fuzzy implications $(I_k)_{k\in A}$, denoted by $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$.

Theorem 2.6. (Su et al. [37]) Let $(I_k)_{k\in A}$ be a family of fuzzy implications. Then $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ given by (9) in Definition 2.5 is a fuzzy implication, if and only if I_k satisfies (CB), whenever $k \in A$ and $b_k < 1$.

From now on, we call $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ an ordinal sum implication if it is a fuzzy implication.

Proposition 2.7. (Su et al. [37]) Let $(I_k)_{k\in A}$ be a family of fuzzy implications satisfying (CB) and I given by (9) in Definition 2.5. Then:

(1) I satisfies (CB).

(2) If $b_k < 1$ for all $k \in A$, then I satisfies (NP).

(3) If there exists $k_0 \in A$, such that $b_{k_0} = 1$, then I satisfies (NP) if and only if I_{k_0} satisfies (NP).

(4) If there exists $k_0 \in A$, such that $b_{k_0} < 1$, then I satisfies neither (IP) nor (OP).

(5) If there exists $I = (\langle a, 1, I_0\rangle)$ with $a > 0$, then I satisfies (OP) if and only if I_0 satisfies (OP).

Definition 2.8. (Bustince et al. [7, 8]) A binary function $O : [0,1]^2 \to [0,1]$ is said to be an overlap function if it satisfies the following conditions:

(O1) O is commutative;

(O2) $O(x,y) = 0$ if and only if $xy = 0$;

(O3) $O(x,y) = 1$ if and only if $xy = 1$;

(O4) O is increasing;

(O5) O is continuous.

Note that if an overlap function O has a neutral element e, then by (O3), this element e is necessarily equal to 1. Moreover, O is said to satisfy the property 1-section deflation [13] if

(O6) $O(1, y) \leq y$ for all $y \in [0, 1]$

and the property 1-section inflation [13] if

(O7) $O(1, y) \geq y$ for all $y \in [0, 1]$.

An overlap function O is called idempotent if $O(x, x) = x$ for all $x \in [0, 1]$ [14], we write $\mathbb{O}_{id}$ as the family of idempotent overlap functions.

In the following example, we list some commonly-used overlap functions [4, 15].

Example 2.9. (1) Any continuous t-norm without non-trivial zero divisors is an overlap function.

(2) For any $p > 0$, the function $O_p : [0, 1]^2 \to [0, 1]$ given by

$$O_p(x, y) = x^p y^p$$

is an overlap function.

(3) The function $O_{DB} : [0, 1]^2 \to [0, 1]$ given by

$$O_{DB}(x, y) = \begin{cases} \frac{2xy}{x+y}, & \text{if } x + y \neq 0, \\ 0, & \text{if } x + y = 0 \end{cases}$$

is an overlap function.

(4) The function $O_{mM} : [0, 1]^2 \to [0, 1]$ given by

$$O_{mM}(x, y) = \min\{x, y\} \max\{x^2, y^2\}$$

is a non-associative overlap function with 1 as the neutral element.

Definition 2.10. (Bustince et al. [9]) A binary function $G : [0, 1]^2 \to [0, 1]$ is said to be a grouping function if it satisfies the following conditions:

(G1) G is commutative;

(G2) $G(x, y) = 0$ if and only if $x = y = 0$;

(G3) $G(x, y) = 1$ if and only if $x = 1$ or $y = 1$;

(G4) G is increasing;

(G5) G is continuous.

Note that if a grouping function G has a neutral element e', then by (G2), this element e' is necessarily equal to 0. Moreover, G is said to satisfy the property 0-section deflation [13] if

(G6) $G(0, y) \geq y$ for all $y \in [0, 1]$

and the property 0-section inflation [13] if

(G7) $G(0, y) \leq y$ for all $y \in [0, 1]$.

A grouping function G is called idempotent if $G(x, x) = x$ for all $x \in [0, 1]$ [4], we write $\mathbb{G}_{id}$ as the family of idempotent grouping functions.

In the following example, we list some commonly-used grouping functions [4, 15].

Example 2.11. (1) Any continuous t-conorm without non-trivial one divisors is a grouping function.

(2) For any $p > 0$, the function $G_p : [0, 1]^2 \to [0, 1]$ given by

$$G_p(x, y) = 1 - (1 - x)^p(1 - y)^p$$

is a grouping function.

(3) The function $G_{DB} : [0, 1]^2 \to [0, 1]$ given by

$$G_{DB}(x, y) = \begin{cases} \frac{x+y-2xy}{2-(x+y)}, & \text{if } x + y \neq 2, \\ 1, & \text{if } x + y = 2 \end{cases}$$

is a grouping function.

(4) The function $G_{mM} : [0, 1]^2 \to [0, 1]$ given by

$$G_{mM}(x, y) = 1 - \min\{1 - x, 1 - y\} \max\{(1 - x)^2, (1 - y)^2\}$$

is a non-associative grouping function with 0 as the neutral element.

Like the dual relationship between t-norm and t-conorm, the overlap function and the grouping function have similar dual relationship. Let N be a strong fuzzy negation ($N \circ N = id_{[0,1]}$) and G a grouping function, then, the expression $O(x, y) = N(G(N(x), N(y)))$ is called the dual overlap function of G. Analogously, the grouping function G given by $G(x, y) = N(O(N(x), N(y)))$ is called the dual grouping function of O [4].

Definition 2.12. (Dimuro and Bedregal [14]) Let A be an at most countable index set, $(O_k)_{k\in A}$ be a family of overlap functions, and $([a_k, b_k])_{k\in A}$ be a family of pairwise disjoint close subintervals of $[0, 1]$. The ordinal sum of $(O_k)_{k\in A}$ is the binary function $(\langle a_k, b_k, O_k\rangle)_{k\in A} : [0, 1]^2 \to [0, 1]$ defined for all $x, y \in [0, 1]$, by:

$$(\langle a_k, b_k, O_k\rangle)_{k\in A}(x, y) = \begin{cases} a_k + (b_k - a_k)O_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & x, y \in [a_k, b_k], \\ \min(f_A(x), f_A(y)), & \text{otherwise}, \end{cases}$$

where $f_A : [0, 1] \to [0, 1]$ is given by:

$$f_A(x) = \begin{cases} a_k + (b_k - a_k)O_k(\frac{x-a_k}{b_k-a_k}, 1), & \exists k \in A, \text{s.t. } x \in [a_k, b_k], \\ x, & \text{otherwise}. \end{cases}$$

Notice that $f_A(x) = x$ for all $x \in [0, 1]$ while O_k has 1 as a neutral element. O is an overlap function if and only if O is representable as an ordinal sum of overlap functions $(O_k)_{k\in A}$.

Theorem 2.13. (Qiao and Hu [34]) Let O be an overlap function with 1 as a neutral element. If there is $\alpha \in [0,1]$ such that

(i) $O(\alpha, x) = \alpha \wedge x$ for all $x \in [0,1]$,

(ii) $O(x,y) = \alpha$ implies $x = \alpha$ or $y = \alpha$ for all $x, y \in [\alpha, 1]$.

Then there exist overlap functions O_1 and O_2 such that $O = (\langle 0, \alpha, O_1\rangle, \langle \alpha, 1, O_2\rangle)$.

We can show the dual concept of ordinal sum of grouping functions and give the sufficient conditions that one can be written as ordinal sum of two grouping functions [42]. We omit them here.

3. DISTRIBUTIVITY OF THE ORDINAL SUM IMPLICATIONS OVER OVERLAP AND GROUPING FUNCTIONS

In this section, we want to study distributive equations (5) − (8) in four parts for the ordinal sum implications over overlap and grouping functions.

3.1. Solution to $I(G(x,y),z) = O(I(x,z), I(y,z))$

Proposition 3.1. (Qiao and Hu [33]) Let O be an overlap function, G a grouping function and I a fuzzy implication satisfying (NP) in Eq.(5). Then $O = T_M$.

Proposition 3.2. (Qiao and Hu [33]) Let O be an overlap function, G a grouping function and I a fuzzy implication satisfying (OP) in Eq.(5). Then $G = S_M$.

Proposition 3.3. (Qiao and Hu [33]) Let O be an overlap function, G a grouping function and I a fuzzy implication satisfying (NP) and (OP) in Eq.(5). Then the following are equivalent.

(i) O and G are the solution of Eq.(5).

(ii) $O = T_M$ and $G = S_M$.

At the beginning, we have the following corollary from Proposition 3.3.

Corollary 3.4. Let O be an overlap function, G a grouping function and $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ an ordinal sum implication given by (9) in Definition 2.5. If (O, G, I) satisfies (5), and I satisfies (NP) and (OP), then $O = T_M$ and $G = S_M$.

Proposition 3.5. Let O be an overlap function with the neutral element 1, G a grouping function and $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ an ordinal sum implication given by (9) in Definition 2.5 satisfying (5). If there exists $k_0 \in A$ such that $b_{k_0} = 1$, $O(x,y) = a_{k_0}$ implies $x = a_{k_0}$ or $y = a_{k_0}$ for all $x, y \in [a_{k_0}, 1]$ and I_{k_0} does not satisfy (NP), then

$$O(x,y) = \begin{cases} a_{k_0} + (b_{k_0} - a_{k_0})O_{k_0}\left(\frac{x-a_{k_0}}{b_{k_0}-a_{k_0}}, \frac{y-a_{k_0}}{b_{k_0}-a_{k_0}}\right), & x, y \in [a_{k_0}, b_{k_0}], \\ \min(x,y), & \text{otherwise}, \end{cases} \tag{10}$$

where O_{k_0} is an overlap function with 1 as a neutral element.

P r o o f. First let $x = y = 1$, $z = a_{k_0} > 0$, since (O, G, I) satisfies (5), we have that

$$I(G(1,1), a_{k_0}) = O(I(1, a_{k_0}), I(1, a_{k_0}))$$

$I(1, a_{k_0}) = a_{k_0}$ by the definition of I, then $a_{k_0} = O(a_{k_0}, a_{k_0})$. If $x \in [0, a_{k_0}]$, it follows from

$$I(G(1,1), x) = O(I(1,x), I(1,x))$$

so we have $x = O(x,x)$ and $x = O(x,x) \leq O(a_{k_0}, x) \leq O(1,x) \leq x$. Thus $O(a_{k_0}, x) = x$. If $x \in [a_{k_0}, 1]$, we have $a_{k_0} = O(a_{k_0}, a_{k_0}) \leq O(a_{k_0}, x) \leq O(a_{k_0}, 1) \leq a_{k_0}$. Thus $O(a_{k_0}, x) = a_{k_0}$. From the above, we conclude that $O(a_{k_0}, x) = \min(a_{k_0}, x)$ for all $x \in [0,1]$.

Next, for all $x, y \in [a_{k_0}, 1]$, $O(x,y) = a_{k_0}$, implies $x = a_{k_0}$ or $y = a_{k_0}$ by the given condition. It follows from the Theorem 2.13 that $O = (\langle 0, a_{k_0}, O_1\rangle, \langle a_{k_0}, 1, O_{k_0}\rangle)$ for some overlap functions O_1 and O_{k_0}, that is

$$O(x,y) = \begin{cases} a_{k_0} + (1-a_{k_0}) O_{k_0}(\frac{x-a_{k_0}}{1-a_{k_0}}, \frac{y-a_{k_0}}{1-a_{k_0}}), & x,y \in [a_{k_0}, 1], \\ a_{k_0} O_1(\frac{x}{a_{k_0}}, \frac{y}{a_{k_0}}), & x, y \in [0, a_{k_0}], \\ \min(f_A(x), f_A(y)), & \text{otherwise}, \end{cases}$$

where $f_A : [0,1] \to [0,1]$ is given by:

$$f_A(x) = \begin{cases} a_{k_0} + (1-a_{k_0}) O_{k_0}(\frac{x-a_{k_0}}{1-a_{k_0}}, 1), & x \in [a_{k_0}, 1], \\ a_{k_0} O_1(\frac{x}{a_{k_0}}, 1), & x \in [0, a_{k_0}]. \end{cases}$$

Let $x, y \in [0, a_{k_0}]$, without loss of generality, we assume that $x \leq y$, then $x = O(x,x) \leq O(x,y) \leq O(x,1) = x$, and thus $O(x,y) = x = \min(x,y)$. In particular, for all $x \in [0, a_{k_0}]$, $a_{k_0} O_1(\frac{x}{a_{k_0}}, 1) = O(a_{k_0}, x) = x$. On the other hand, for all $x \in [a_{k_0}, 1]$, since O has a neutral element, we have $a_{k_0} + (1-a_{k_0}) O_{k_0}(\frac{x-a_{k_0}}{1-a_{k_0}}, 1) = O(x,1) = x$. We conclude that $f_A(x) = x$ for all $x \in [0,1]$ and O_{k_0} also with 1 as a neutral element. Then O has the ordinal sum of the form of (10). □

Proposition 3.6. Let O be an overlap function, G a grouping function with the neutral element 0 and $I = (\langle a_k, b_k, I_k\rangle)_{k \in A}$ an ordinal sum implication given by (9) in Definition 2.5 satisfying (5). If there exists $k_0 \in A$ such that $b_{k_0} < 1$ or $I = (\langle a, 1, I_0\rangle)$ with $a > 0$ and I_0 does not satisfy (OP), and $G(x,y) = a_k$ implies $x = a_k$ or $y = a_k$ for all $x, y \in [0, a_k]$, $G(x,y) = b_k$ implies $x = b_k$ or $y = b_k$ for all $x, y \in [0, b_k]$, then

$$G(x,y) = \begin{cases} a_k + (b_k - a_k) G_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & x,y \in [a_k, b_k], \\ \max(x,y), & \text{otherwise}, \end{cases} \tag{11}$$

where $\{G_k | k \in A\}$ is a family of grouping functions with 0 as a neutral element.

P r o o f. Let's first prove that a_k, b_k are idempotent elements of G for all $k \in A$. If there exists $k_0 \in A$, such that $a_{k_0} = 0$ or $b_{k_0} = 1$, then the conclusion clearly holds. Now we consider for any $k \in A$, such that $b_k < 1$. For any strictly decreasing sequence $(x_n)_{n\in N} \notin [a_k, b_k]$, such that $(x_n)_{n\in N} \searrow b_k$, since (O,G,I) satisfies (5), then $I(G(b_k,b_k),x_n) = O(I(b_k,x_n),I(b_k,x_n)) = 1$, and hence $G(b_k,b_k) \le x_n$ for all $n \in N$. Let $n \to \infty$, we have $G(b_k,b_k) \le b_k$. On the other hand, $G(b_k,b_k) \ge G(b_k,0) = b_k$, thus $G(b_k,b_k) = b_k$ for all $k \in A$. And in the same way we can get $G(a_k,a_k) = a_k$ for all $k \in A$. For any strictly increasing sequence $(y_n)_{n\in N} \notin [a_k,b_k]$ such that $(y_n)_{n\in N} \nearrow a_k$, since (O,G,I) satisfies (5), then $I(G(y_n,y_n),y_n) = O(I(y_n,y_n),I(y_n,y_n)) = 1$, thus $G(y_n,y_n) \le y_n$ for all $n \in N$. By the continuity of G, we know that $G(a_k,a_k) \le a_k$. On the other hand, $G(a_k,a_k) \ge G(a_k,0) = a_k$, thus $G(a_k,a_k) = a_k$ for all $k \in A$.

By the duality between grouping and overlap functions, we know many properties of grouping functions can be obtained in parallel. By the virtue of the proof of Proposition 2.3 (ii) in [25], we have for all $k \in A$,

$$G(a_k,x) = \max(a_k,x), \quad G(b_k,x) = \max(b_k,x) \ \text{ for all } \ x \in [0,1].$$

Next, by mathematical induction and Theorem 2.6 in [42] and Proposition 3.48 in [25], we can get that G has the ordinal sum of the form $G = (\langle a_k,b_k,G_k\rangle)$ for some grouping functions G_k, that is

$$G(x,y) = \begin{cases} a_k + (b_k - a_k)G_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & x,y \in [a_k,b_k], \\ \max(f_B(x), f_B(y)), & \text{otherwise}, \end{cases}$$

where $f_B : [0,1] \to [0,1]$ is given by:

$$f_B(x) = \begin{cases} a_k + (b_k - a_k)G_k(\frac{x-a_k}{b_k-a_k}, 0), & x \in [a_k,b_k], \\ x, & \text{otherwise}. \end{cases}$$

In particular, for all $x \in [a_k,b_k]$, $a_k + (b_k - a_k)G_k(\frac{x-a_k}{b_k-a_k}, 0) = G(x,a_k) = x$. We conclude that $f_B(x) = x$ for all $x \in [0,1]$ and G_k have 0 as a neutral element. Then we know that G has the ordinal sum of the form of (11). □

Proposition 3.7. Let O be an overlap function given by (10) in Proposition 3.5, G a grouping function given by (11) in Proposition 3.6 and $I = (\langle a_k,b_k,I_k\rangle)_{k\in A}$ an ordinal sum implication given by (9) in Definition 2.5. Then (G,O,I) satisfies (5), if and only if

(1) (G_k,T_M,I_k) satisfies (5), when $k \ne k_0$,

(2) $(G_{k_0},O_{k_0},I_{k_0})$ satisfies (5).

P r o o f. $(\Rightarrow)$ (1) Let $k \ne k_0$. For any $x_1,y_1,z_1 \in [0,1]$, then there exists $x,y,z \in [a_k,b_k]$ such that $x_1 = \frac{x-a_k}{b_k-a_k}, y_1 = \frac{y-a_k}{b_k-a_k}, z_1 = \frac{z-a_k}{b_k-a_k}$. If $x_1 = 0$ or $y_1 = 0$, obviously, $I_k(G_k(x_1,y_1),z_1) = T_M(I_k(x_1,z_1),I_k(y_1,z_1))$; If $x_1 \ne 0$ and $y_1 \ne 0$, then $x \ne a_k$ and $y \ne b_k$. We have $I(x,z) = a_k + (b_k-a_k)I_k(\frac{x-a_k}{b_k-a_k}, \frac{z-a_k}{b_k-a_k}) = a_k + (b_k-a_k)I_k(x_1,z_1)$, and $I(y,z) = a_k + (b_k - a_k)I_k(\frac{y-a_k}{b_k-a_k}, \frac{z-a_k}{b_k-a_k}) = a_k + (b_k - a_k)I_k(y_1,z_1)$, we can also have $G(x,y) = a_k + (b_k - a_k)G_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}) = a_k + (b_k - a_k)G_k(x_1,y_1) \in [a_k,b_k]$, since (G,O,I) satisfies (5), then $I(G(x,y),z) = O(I(x,z),I(y,z))$, thus $a_k + (b_k - a_k)I_k(G_k(x_1,y_1),z_1) = T_M(a_k + (b_k - a_k)I_k(x_1,z_1), a_k + (b_k - a_k)I_k(y_1,z_1))$. In conclusion, we can get $I_k(G_k(x_1,y_1),z_1) = T_M(I_k(x_1,z_1),I_k(y_1,z_1))$. (2) For the case of

$k = k_0$, the proof is similar as above.

($\Leftarrow$) We consider the relation between z and $[a_k, b_k]$ by the following three cases:

Case 1, $z \notin [a_k, b_k]$ for any $k \in A$.

Case 1.1, $x \leq z$ and $y \leq z$. No matter whether x, y is a member of interval or not, we all have $G(x, y) \leq z$, then $I(G(x, y), z) = 1 = O(I(x, z), I(y, z))$.

Case 1.2, $x \leq z$ and $y > z$. Then $G(x, y) = \max(x, y) > z$, $I(x, z) = 1, I(y, z) = z$. So we conclude that $I(G(x, y), z) = z = O(I(x, z), I(y, z))$.

Case 1.3, $x > z$ and $y \leq z$. Similar to Case 1.2.

Case 1.4, $x > z$ and $y > z$. No matter whether x, y is a member of interval or not, we all have $G(x, y) > z$, then $I(x, z) = z, I(y, z) = z$. Thus $I(G(x, y), z) = z = O(I(x, z), I(y, z))$.

Case 2, There exists $k_0 \in A$, such that $b_{k_0} = 1$ and $z \in [a_{k_0}, b_{k_0}]$.

Case 2.1, $x, y \notin [a_{k_0}, b_{k_0}]$. No matter whether x, y is a member of interval or not, we all have $G(x, y) \leq z$, then $I(x, z) = 1, I(y, z) = 1$. Thus $I(G(x, y), z) = 1 = O(I(x, z), I(y, z))$.

Case 2.2, $x \in [a_{k_0}, b_{k_0}]$ and $y \notin [a_{k_0}, b_{k_0}]$. Then $G(x, y) = \max(x, y) = x$, $I(y, z) = 1$, we have that $I(G(x, y), z) = I(x, z) = O(I(x, z), I(y, z))$.

Case 2.3, $x \notin [a_{k_0}, b_{k_0}]$ and $y \in [a_{k_0}, b_{k_0}]$. Similar to Case 2.2.

Case 2.4, $x, y \in [a_{k_0}, b_{k_0}]$. It's a direct consequence from the fact that $(G_{k_0}, O_{k_0}, I_{k_0})$ satisfies (5).

Case 3, There exists $k \in A$, such that $z \in [a_k, b_k]$, and $k \neq k_0$.

Case 3.1, $x, y \notin [a_k, b_k]$ for any $k \in A$. Then $G(x, y) = \max(x, y)$. Without loss of generality, we assume $x \leq y$. Whenever $y < a_k$, then $I(x, z) = 1, I(y, z) = 1$ and $I(G(x, y), z) = 1 = O(I(x, z), I(y, z))$; Whenever $x < a_k$ and $y > b_k$, then $I(x, z) = 1, I(y, z) = z$ and $I(G(x, y), z) = z = O(I(x, z), I(y, z))$; Whenever $x > b_k$, then $I(x, z) = z, I(y, z) = z$ and $I(G(x, y), z) = z = O(I(x, z), I(y, z))$.

Case 3.2, $x \in [a_k, b_k]$ and $y \notin [a_k, b_k]$. Whenever $y < a_k$, then $G(x, y) = \max(x, y) = x$, and $I(y, z) = 1$, thus $I(G(x, y), z) = I(x, z)$, $O(I(x, z), I(y, z)) = \min(I(x, z), 1) = I(x, z)$; Whenever $y > b_k$, then $G(x, y) = \max(x, y) = y$, and $I(y, z) = z$, we have $I(G(x, y), z) = I(y, z) = z$, $O(I(x, z), I(y, z)) = \min(I(x, z), z)$, since $I(x, z) \geq z$ by I satisfies (CB), we get $I(G(x, y), z) = z = O(I(x, z), I(y, z))$.

Case 3.3, $x \notin [a_k, b_k]$ and $y \in [a_k, b_k]$. Similar to Case 3.2.

Case 3.4, $x, y \in [a_k, b_k]$. It's a direct consequence from the fact that (G_k, T_M, I_k) satisfies (5). □

Observe the Proposition 2.7, we can conclude that I satisfies (NP) and (OP), i. e., for index set A, $|A| = 1, I = (\langle a, 1, I_0 \rangle)$ with $a > 0$ and I_0 satisfies (NP) and (OP). By these conclusions and the propositions above whether I satisfies (NP) and (OP), we have the following theorems.

Theorem 3.8. Let O be an overlap function, G a grouping function and $I = (\langle a, 1, I_0\rangle)$ with $a > 0$ an ordinal sum implication given by (9) in Definition 2.5, respectively. If I_0 satisfies (NP) and (OP), then (G, O, I) satisfies (5), if and only if $O = T_M$, $G = S_M$.

Theorem 3.9. Let O be an overlap function with the neutral element 1, G a grouping function with the neutral element 0 and $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ an ordinal sum implication given by (9) in Definition 2.5, respectively. If I satisfies neither (NP) nor (OP), i. e., for index set A, whenever $|A| = 1$, $I = (\langle a, 1, I_0\rangle)$ with $a > 0$ and I_0 satisfies neither (NP) nor (OP) , or whenever $|A| > 1$, there exist $k_0 \in A$, such that $b_{k_0} = 1$ and I_{k_0} does not satisfies (NP). $O(x, y) = a_{k_0}$ implies $x = a_{k_0}$ or $y = a_{k_0}$ for all $x, y \in [a_{k_0}, 1]$, and $G(x, y) = a_k$ implies $x = a_k$ or $y = a_k$ for all $x, y \in [0, a_k]$, $G(x, y) = b_k$ implies $x = b_k$ or $y = b_k$ for all $x, y \in [0, b_k]$. Then (G, O, I) satisfies (5), if and only if O is given by (10), G is given by (11), and

(1) (G_k, T_M, I_k) satisfies (5), when $k \neq k_0$,

(2) $(G_{k_0}, O_{k_0}, I_{k_0})$ satisfies (5).

Example 3.10. (i) Let $I = (\langle \frac{2}{3}, 1, I_{RS}\rangle)$, $O = (\langle \frac{2}{3}, 1, O_1\rangle)$ and $G = S_M$. Then

$$I(x,y) = \begin{cases} 1 & x \leq y, \\ \frac{2}{3} & (x,y) \in [\frac{2}{3}, 1]^2 \text{ and } x > y, \\ y & (x,y) \in [0,1]^2 \setminus [\frac{2}{3}, 1]^2 \text{ and } x > y, \end{cases}$$

and

$$O(x,y) = \begin{cases} \frac{2}{3} + 3(x - \frac{2}{3})(y - \frac{2}{3}) & (x,y) \in [\frac{2}{3}, 1]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

I does not satisfies (NP), and we can easily verify that (S_M, O, I) satisfies (5).

(ii) Let $I = (\langle \frac{1}{4}, \frac{1}{2}, I_{WB}\rangle)$, $G = (\langle \frac{1}{4}, \frac{1}{2}, G_1\rangle)$ and $O = T_M$. Then

$$I(x,y) = \begin{cases} \frac{1}{2} & (x,y) \in [\frac{1}{4}, \frac{1}{2}]^2, \quad x < \frac{1}{2}, \\ 1 & (x,y) \in [0,1]^2 \setminus [\frac{1}{4}, \frac{1}{2}]^2, \quad x \leq y, \\ y & (x,y) \in [0,1]^2 \setminus [\frac{1}{4}, \frac{1}{2}) \times [\frac{1}{4}, \frac{1}{2}], \quad x > y. \end{cases}$$

and

$$G(x,y) = \begin{cases} x + y - \frac{1}{4} - 4(x - \frac{1}{4})(y - \frac{1}{4}) & (x,y) \in [\frac{1}{4}, \frac{1}{2}]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

I does not satisfies (OP), and we can easily verify that (G, T_M, I) satisfies (5).

(iii) Let $I = (\langle \frac{1}{3}, 1, I_{LK}\rangle)$, $G = S_M$ and $O = T_M$. Then (S_M, T_M, I_{LK}) satisfies (5), and

$$I(x,y) = \begin{cases} \min(1, 1 - x + y) & (x,y) \in [\frac{1}{3}, 1]^2, \\ 1 & (x,y) \in [0,1]^2 \setminus [\frac{1}{3}, 1]^2, \quad x \leq y, \\ y & (x,y) \in [0,1]^2 \setminus [\frac{1}{3}, 1]^2, \quad x > y. \end{cases}$$

I satisfies (NP) and (OP), it is easy to verify that (G, O, I) satisfies (5).

(iv) Let

$$I_0(x,y) = \begin{cases} 1 & y = 1, \\ 1 - x & y < 1. \end{cases}$$

Then I_0 is a fuzzy implication which does not satisfy (NP). It is easy to verify that (G_1, T_M, I_{WB}) and (G_{mM}, O_{mM}, I_0) satisfy (5). Now let $I = (\langle \frac{1}{4}, \frac{1}{2}, I_{WB}\rangle, \langle \frac{3}{4}, 1, I_0\rangle)$ satisfy neither (NP) nor (OP), $G = (\langle \frac{1}{4}, \frac{1}{2}, G_1\rangle, \langle \frac{3}{4}, 1, G_{mM}\rangle)$ and $O = (\langle \frac{1}{4}, \frac{1}{2}, T_M\rangle, \langle \frac{3}{4}, 1, O_{mM}\rangle)$. Then we can verify that (G, O, I) satisfies (5) by Theorem 3.9. In fact

$$I(x,y) = \begin{cases} \frac{1}{2} & (x,y) \in [\frac{1}{4}, \frac{1}{2}) \times [\frac{1}{4}, \frac{1}{2}], \\ \frac{7}{4} - x & (x,y) \in [\frac{3}{4}, 1] \times [\frac{3}{4}, 1), \\ 1 & (x,y) \in [0,1]^2 \setminus \{[\frac{1}{4}, \frac{1}{2}) \times [\frac{1}{4}, \frac{1}{2}] \cup [\frac{3}{4}, 1] \times [\frac{3}{4}, 1)\}, \quad x \le y, \\ y & (x,y) \in [0,1]^2 \setminus \{[\frac{1}{4}, \frac{1}{2}) \times [\frac{1}{4}, \frac{1}{2}] \cup [\frac{3}{4}, 1] \times [\frac{3}{4}, 1)\}, \quad x > y. \end{cases}$$

$$G(x,y) = \begin{cases} x + y - \frac{1}{4} - 4(x - \frac{1}{4})(y - \frac{1}{4}) & (x,y) \in [\frac{1}{4}, \frac{1}{2}]^2, \\ 1 - \min(1-x, 1-y)\max((4-4x)^2, (4-4y)^2) & (x,y) \in [\frac{3}{4}, 1]^2, \\ \max(x,y) & \text{otherwise}, \end{cases}$$

and

$$O(x,y) = \begin{cases} \frac{3}{4} + \frac{1}{4}\min(4x-3, 4y-3)\max((4x-3)^2, (4y-3)^2) & (x,y) \in [\frac{3}{4}, 1]^2, \\ \min(x,y) & \text{otherwise}. \end{cases}$$

In Example 3.10(iv), we use O_{mM} and G_{mM} to induce O and G, respectively, but O_{mM} and G_{mM} are not associative, then O and G in this case are not associative. Thus, the distributivity equation of ordinal sum implications over t-norms and t-conorms is generalized. The plots of fuzzy implication I, group function G and overlap function O in Example 3.10(iv) are given in Figure 1 (a), (b) and (c), respectively.

3.2. Solution to $I(O(x, y), z) = G(I(x, z), I(y, z))$

The analysis of this subsection is similar to that of Eq.(5). We will just list some results and present some differences.

Proposition 3.11. (Qiao and Hu [33]) Let O be an overlap function, G a grouping function and I a fuzzy implication satisfying (NP) in Eq.(6). Then $G \in \mathbb{G}_{id}$.

Proposition 3.12. (Qiao and Hu [33]) Let O be an overlap function, G a grouping function and I a fuzzy implication satisfying (OP) and (NP) in Eq.(6). Then $O = T_M$.

Proposition 3.13. (Qiao and Hu [33]) Let O be an overlap function, G a grouping function satisfying (G6) and I a fuzzy implication satisfying (NP) and (OP) in Eq.(6). Then the following are equivalent.

(i) O and G are the solution of Eq.(6).

(ii) $O = T_M$ and $G = S_M$.

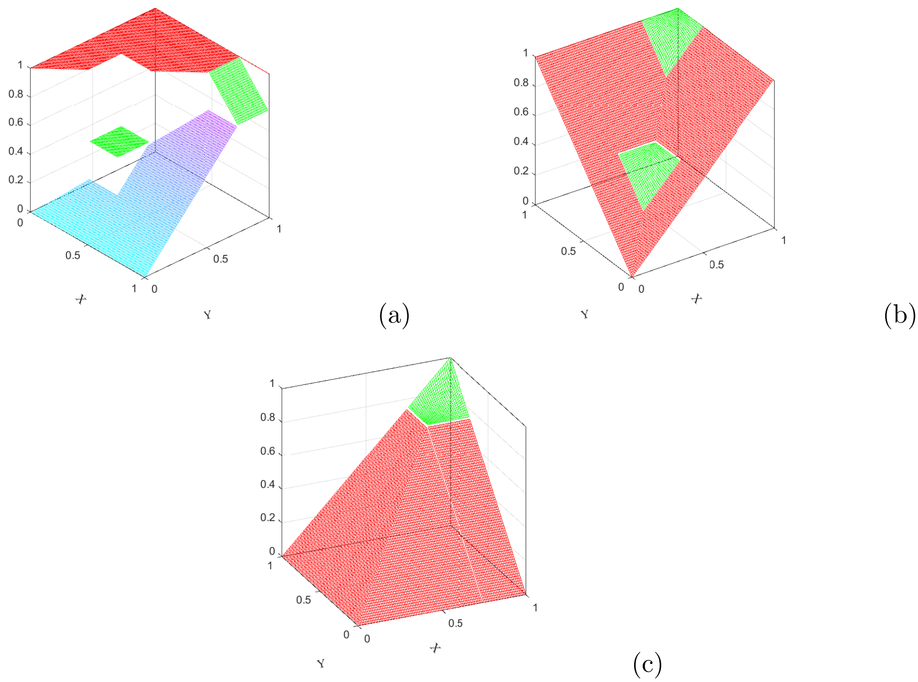


Fig. 1. Plots of Example 3.10(iv).

Corollary 3.14. Let O be an overlap function, G a grouping function with the neutral element 0 and I a fuzzy implication satisfying (NP) in Eq.(6). Then $G = S_M$.

Proof. It's straight from Proposition 3.11 and Proposition 3.2 in [33]. □

Corollary 3.15. Let O be an overlap function, G a grouping function satisfying (G6) and $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ an ordinal sum implication given by (9) in Definition 2.5. If (O, G, I) satisfies (6), and I satisfies (NP) and (OP), then $O = T_M$ and $G = S_M$.

Proposition 3.16. Let O be an overlap function, G a grouping function with the neutral element 0 and $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ an ordinal sum implication given by (9) in Definition 2.5 satisfying (6). If there exists $k_0 \in A$ such that $b_{k_0} = 1$, $G(x, y) = a_{k_0}$ implies $x = a_{k_0}$ or $y = a_{k_0}$ for all $x, y \in [0, a_{k_0}]$ and I_{k_0} does not satisfy (NP), then

$$G(x,y) = \begin{cases} a_{k_0} + (b_{k_0} - a_{k_0}) G_{k_0}\left(\frac{x-a_{k_0}}{b_{k_0}-a_{k_0}}, \frac{y-a_{k_0}}{b_{k_0}-a_{k_0}}\right), & x, y \in [a_{k_0}, b_{k_0}], \\ \max(x,y), & \text{otherwise}, \end{cases} \tag{12}$$

where G_{k_0} is a grouping function with 0 as a neutral element.

Proposition 3.17. Let O be an overlap function with the neutral element 1, G a grouping function satisfying (G6) and $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ an ordinal sum implication given by (9) in Definition 2.5 satisfying (6). If I does not satisfies (OP) and (NP), and $O(x,y) = a_k$ implies $x = a_k$ or $y = a_k$ for any $x, y \in [a_k, 1]$, $O(x,y) = b_k$ implies $x = b_k$ or $y = b_k$ for all $x, y \in [b_k, 1]$, then

$$O(x,y) = \begin{cases} a_k + (b_k - a_k)O_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & x, y \in [a_k, b_k], \\ \min(x,y), & \text{otherwise}, \end{cases} \tag{13}$$

where $\{O_k | k \in A\}$ is a family of grouping functions with 1 as a neutral element.

Proposition 3.18. Let O be an overlap function given by (13) in Proposition 3.17, G a grouping function given by (12) in Proposition 3.16 and $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ an ordinal sum implication given by (9) in Definition 2.5. Then (G, O, I) satisfies (6), if and only if

(1) (S_M, O_k, I_k) satisfies (6), when $k \neq k_0$,

(2) $(G_{k_0}, O_{k_0}, I_{k_0})$ satisfies (6).

By these conclusions and the propositions above whether I satisfies (NP) and (OP), we have the following theorems.

Theorem 3.19. Let O be an overlap function, G a grouping function satisfying (G6) and $I = (\langle a, 1, I_0\rangle)$ with $a > 0$ an ordinal sum implication given by (9) in Definition 2.5. If I_0 satisfies (NP) and (OP), then (G, O, I) satisfies (6), if and only if $O = T_M$, $G = S_M$.

Theorem 3.20. Let O be an overlap function with the neutral element 1, G a grouping function with the neutral element 0 and $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ an ordinal sum implication given by (9) in Definition 2.5. If I satisfies neither (NP) nor (OP), i.e., for index set A, whenever $|A| = 1$, $I = (\langle a, 1, I_0\rangle)$ with $a > 0$ and I_0 satisfies neither (NP) nor (OP), or whenever $|A| > 1$, there exist $k_0 \in A$, such that $b_{k_0} = 1$ and I_{k_0} does not satisfies (NP). $G(x,y) = a_{k_0}$ implies $x = a_{k_0}$ or $y = a_{k_0}$ for all $x, y \in [0, a_{k_0}]$, and $O(x,y) = a_k$ implies $x = a_k$ or $y = a_k$ for all $x, y \in [a_k, 1]$, $G(x,y) = b_k$ implies $x = b_k$ or $y = b_k$ for all $x, y \in [b_k, 1]$. Then (G, O, I) satisfies (6), if and only if O is given by (13), G is given by (12), and

(1) (S_M, O_k, I_k) satisfies (6), when $k \neq k_0$,

(2) $(G_{k_0}, O_{k_0}, I_{k_0})$ satisfies (6).

3.3. Solution to $I(x, O_1(y,z)) = O_2(I(x,y), I(x,z))$

Proposition 3.21. (Qiao and Hu [33]) Let O_1, O_2 be two overlap functions, and I be a fuzzy implication satisfying (NP) and (OP) in Eq.(7). Then the following are equivalent.

(i) O_1 and O_2 are the solution of Eq.(7).

(ii) $O_1 = O_2 = T_M$.

Corollary 3.22. Let O_1, O_2 be two overlap functions, and I an ordinal sum implication given by (9) in Definition 2.5. If (O_1, O_2, I) satisfies (7), and I satisfies (NP) and (OP), then $O_1 = O_2 = T_M$.

Proposition 3.23. Let O be an overlap function with the neutral element 1, and I an ordinal sum implication given by (9) in Definition 2.5. If O has an ordinal sum of the form

$$O(x,y) = \begin{cases} a_k + (b_k - a_k)O_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & x, y \in [a_k, b_k], \\ \min(x,y), & \text{otherwise}, \end{cases} \tag{14}$$

where O_k is a family of overlap functions with 1 as a neutral element, then (O, O, I) satisfies (7) if and only if (O_k, O_k, I_k) satisfies (7) for all $k \in A$.

Proof. ($\Rightarrow$) Let $k \in A$, for any $x_1, y_1, z_1 \in [0,1]$, then there exists $x, y, z \in [a_k, b_k]$ such that $x_1 = \frac{x-a_k}{b_k-a_k}, y_1 = \frac{y-a_k}{b_k-a_k}, z_1 = \frac{z-a_k}{b_k-a_k}$. Since (O, O, I) satisfies (7), then $I(x, O(y,z)) = O(I(x,y), I(x,z))$. Thus

$$\begin{aligned} I(x, O(y,z)) &= a_k + (b_k - a_k)I_k(\frac{x-a_k}{b_k-a_k}, O_k(\frac{y-a_k}{b_k-a_k}, \frac{z-a_k}{b_k-a_k})) \\ &= a_k + (b_k - a_k)I_k(x_1, O_k(y_1, z_1)). \end{aligned}$$

and

$$\begin{aligned} O(I(x,y), I(x,z)) &= a_k + (b_k - a_k)O_k(I_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), I_k(\frac{x-a_k}{b_k-a_k}, \frac{z-a_k}{b_k-a_k})) \\ &= a_k + (b_k - a_k)O_k(I_k(x_1, y_1), I_k(x_1, z_1)). \end{aligned}$$

We conclude that $I_k(x_1, O_k(y_1, z_1) = O_k(I_k(x_1, y_1), I_k(x_1, z_1))$.

($\Leftarrow$) If $x = 0$, then $I(x, O(y,z)) = 1 = O(I(x,y), I(x,z))$. If $x \neq 0$, we will consider the following cases:

Case 1, $x \notin [a_k, b_k]$ for any $k \in A$.

Case 1.1, $y, z \notin [a_k, b_k]$ for any $k \in A$. Then $O(y,z) = \min(y,z)$, without loss of generality, let's assume $y \leq z$. If $x \leq y$, then $O(I(x,y), I(x,z)) = O(1,1) = 1 = I(x, O(y,z))$; If $y \leq x \leq z$, then $O(I(x,y), I(x,z)) = O(y,1) = y = I(x, O(y,z))$; If $z \leq x$, then $O(I(x,y), I(x,z)) = O(y,z) = y = I(x, O(y,z))$.

Case 1.2, There exists $k_0 \in A$ such that $y, z \in [a_{k_0}, b_{k_0}]$. Then $O(y,z) = a_{k_0} + (b_{k_0} - a_{k_0})O_{k_0}(\frac{y-a_{k_0}}{b_{k_0}-a_{k_0}}, \frac{z-a_{k_0}}{b_{k_0}-a_{k_0}}) \in [a_{k_0}, b_{k_0}]$. If $x < a_{k_0}$, then $I(x,y) = I(x,z) = 1$, thus $O(I(x,y), I(x,z)) = O(1,1) = 1 = I(x, O(y,z))$. If $x > b_{k_0}$, then $I(x,y) = y$, and $I(x,z) = z$, thus $O(I(x,y), I(x,z)) = O(y,z) = I(x, O(y,z))$.

Case 1.3, There exists $k_0 \in A$ such that $y \in [a_{k_0}, b_{k_0}]$, and $z \notin [a_k, b_k]$ for any $k \in A$. If $x < a_{k_0}$, when $z < a_{k_0}$, then $I(x,y) = 1$ and $O(y,z) = z$, thus $O(I(x,y), I(x,z)) = O(1, I(x,z)) = I(x,z) = I(x, O(y,z))$; When $z > b_{k_0}$, then $O(I(x,y), I(x,z)) = O(1,1) = 1 = I(x, O(y,z))$. If $x > b_{k_0}$, when $z < a_{k_0}$, then $I(x,y) = y$ and $I(x,z) = z$, thus $O(I(x,y), I(x,z)) = O(y,z) = z = I(x,z) = I(x, O(y,z))$; When $z > b_{k_0}$, then $O(I(x,y), I(x,z)) = O(y,1) = y = I(x, O(y,z))$.

Case 1.4, There exists $k_0 \in A$ such that $z \in [a_{k_0}, b_{k_0}]$, and $y \notin [a_k, b_k]$ for any $k \in A$. Similar to Case 1.3.

Case 2, There exists $k_0 \in A$ such that $x \in [a_{k_0}, b_{k_0}]$. We need to consider the following four subcases.

Case 2.1, $y, z \notin [a_{k_0}, b_{k_0}]$. If $y, z \in [a_{k_1}, b_{k_1}]$ for $k_1 \neq k_0$, then $O(y,z) = a_{k_1} + (b_{k_1} - a_{k_1})O_{k_1}(\frac{y-a_{k_1}}{b_{k_1}-a_{k_1}}, \frac{z-a_{k_1}}{b_{k_1}-a_{k_1}}) \in [a_{k_1}, b_{k_1}]$. Whenever $b_{k_1} < a_{k_0}$, then $I(x,y) = y$ and $I(x,z) = z$, thus $O(I(x,y), I(x,z)) = O(y,z) = I(x, O(y,z))$; Whenever $a_{k_1} < b_{k_0}$, then $I(x,y) = I(x,z) = 1$, thus $O(I(x,y), I(x,z)) = O(1,1) = 1 = I(x, O(y,z))$. If $y, z \notin [a_k, b_k]$ for any $k \in A$, then $O(y,z) = \min(y,z)$. Without loss of generality, we assume that $y \leq z$. Whenever $y \leq z \leq x$, then $I(x,y) = y$ and $I(x,z) = z$, thus $O(I(x,y), I(x,z)) = O(y,z) = y = I(x, O(y,z))$; Whenever $y \leq x \leq z$, then $I(x,y) = y$ and $I(x,z) = 1$, thus $O(I(x,y), I(x,z)) = O(y,1) = y = I(x, O(y,z))$; Whenever $x \leq y \leq z$, then $I(x,y) = I(x,z) = 1$, thus $O(I(x,y), I(x,z)) = O(1,1) = 1 = I(x, O(y,z))$.

Case 2.2, $y \in [a_{k_0}, b_{k_0}]$, $z \notin [a_{k_0}, b_{k_0}]$. If $z < a_{k_0}$, no matter whether z is a member of interval or not, we all have $I(x,z) = z < I(x,y)$, then $O(I(x,y), I(x,z)) = O(I(x,y), z) = z$, thus $I(x, O(y,z)) = I(x,z) = z$; If $z > b_{k_0}$, no matter whether z is a member of interval or not, we all have $I(x,z) = 1 \geq I(x,y)$, then $O(I(x,y), I(x,z)) = O(I(x,y), 1) = I(x,y)$, thus $I(x, O(y,z)) = I(x,y)$.

Case 2.3, $y \notin [a_{k_0}, b_{k_0}]$, $z \in [a_{k_0}, b_{k_0}]$. Similar to Case 2.2.

Case 2.4, $y, z \in [a_{k_0}, b_{k_0}]$. It's a direct consequence from the fact that (O_k, O_k, I_k) satisfies (7). □

Proposition 3.24. Let O_1, O_2 be two overlap functions with 1 as a neutral element and $I = (\langle a_k, b_k, I_k \rangle)_{k \in A}$ an ordinal sum implication given by (9) in Definition 2.5 satisfying (7). If I satisfies neither (NP) nor (OP), $\mathcal{O}(x,y) = a_k$ implies $x = a_k$ or $y = a_k$ for all $x, y \in [a_k, 1]$, $\mathcal{O}(x,y) = b_k$ implies $x = b_k$ or $y = b_k$ for all $x, y \in [b_k, 1]$, $\mathcal{O} \in \{O_1, O_2\}$, then O_1 and O_2 have the ordinal sum of the forms as follows:

$$O_1(x,y) = \begin{cases} a_k + (b_k - a_k)O_{1k}(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & x, y \in [a_k, b_k], \\ \min(x,y), & \text{otherwise}, \end{cases} \tag{15}$$

$$O_2(x,y) = \begin{cases} a_k + (b_k - a_k)O_{2k}(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & x, y \in [a_k, b_k], \\ \min(x,y), & \text{otherwise}, \end{cases} \tag{16}$$

where $\{O_{1k} | k \in A\}$ and $\{O_{2k} | k \in A\}$ are two families of overlap functions with 1 as a neutral element.

P r o o f. We only prove the structure of O_1, let's first prove that a_k, b_k are idempotent elements of O_1 for all $k \in A$. Now we consider $k \in A$, for any strictly increasing sequence $(y_n)_{n \in N} \notin [a_k, b_k]$, such that $(y_n)_{n \in N} \nearrow a_k$. Since (O_1, O_2, I) satisfies (7), then $I(y_n, O_1(a_k, a_k)) = O_2(I(y_n, a_k), I(y_n, a_k)) = O_2(1,1) = 1$, and hence $O_1(a_k, a_k) \geq y_n$. Let $n \to \infty$, we have $O_1(a_k, a_k) \geq a_k$. On the other hand, $O_1(a_k, a_k) \leq a_k$, thus $O_1(a_k, a_k) = a_k$ for all $k \in A$. And in the same way we can get $O_1(b_k, b_k) = b_k$ for all

$k \in A$. Since O_1 is continuous and a_k, b_k are idempotent elements of O_1, then by virtue of Proposition 2.3 [25] we have for all $k \in A$,

$$O_1(a_k, x) = \min(a_k, x), \quad O_1(b_k, x) = \min(b_k, x) \text{ for all } \quad x \in [0,1].$$

Next, by mathematical induction and the Theorem 2.13 , we can get that O_1 has the ordinal sum of the form $O_1 = (\langle a_k, b_k, O_{1k} \rangle)$ for some overlap functions O_{1k}, that is

$$O_1(x,y) = \begin{cases} a_k + (b_k - a_k) O_{1k}(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & x, y \in [a_k, b_k], \\ \min(f_A(x), f_A(y)), & \text{otherwise}, \end{cases}$$

where $f_A : [0,1] \to [0,1]$ is given by:

$$f_A(x) = \begin{cases} a_k + (b_k - a_k) O_{1k}(\frac{x-a_k}{b_k-a_k}, 1), & x \in [a_k, b_k], \\ x, & \text{otherwise}. \end{cases}$$

In particular, for all $x \in [a_k, b_k]$, $a_k + (b_k - a_k) O_{1k}(\frac{x-a_k}{b_k-a_k}, 1) = O_1(x, b_k) = x$. We conclude that $f_A(x) = x$ for all $x \in [0,1]$ and O_{1k} has 1 as a neutral element, we can get that O_1 has the ordinal sum of the form of (15). □

By the propositions above and the conditions about whether I satisfying (NP) and (OP), we have the following theorems.

Theorem 3.25. Let O_1, O_2 be two overlap functions, and $I = (\langle a, 1, I_0 \rangle)$ with $a > 0$ an ordinal sum implication given by (9) in Definition 2.5, respectively. If (O_1, O_2, I) satisfies (7), and I_0 satisfies (NP) and (OP), if and only if $O_1 = O_2 = T_M$.

Theorem 3.26. Let O_1, O_2 be two overlap functions with 1 as a neutral element and $I = (\langle a_k, b_k, I_k \rangle)_{k \in A}$ an ordinal sum implication given by (9) in Definition 2.5. If I satisfies neither (NP) nor (OP), i. e., for index set A, whenever $|A| = 1$, $I = (\langle a, 1, I_0 \rangle)$ with $a > 0$ and I_0 satisfies neither (NP) nor (OP), or whenever $|A| > 1$, there exists $k_0 \in A$ such that $b_{k_0} = 1$ and I_{k_0} does not satisfy (NP). $\mathcal{O}(x,y) = a_k$ implies $x = a_k$ or $y = a_k$ for all $x, y \in [a_k, 1]$, $\mathcal{O}(x,y) = b_k$ implies $x = b_k$ or $y = b_k$ for all $x, y \in [b_k, 1]$, $\mathcal{O} \in \{O_1, O_2\}$. Then (O_1, O_2, I) satisfies (7), if and only if O_1 is given by (15), O_2 is given by (16), and (O_{1k}, O_{2k}, I_k) satisfies (7), for all $k \in A$.

Example 3.27. (i) Let $I = (\langle \frac{2}{3}, 1, I_{GG} \rangle)$, and $O_1 = O_2 = T_M$. Then (T_M, T_M, I_{GG}) satisfies (7), and

$$I(x,y) = \begin{cases} \frac{2}{3} + \frac{1}{3}(\frac{3y-2}{3x-2}) & (x,y) \in [\frac{2}{3}, 1]^2, \quad x > y, \\ 1 & x \le y, \\ y & (x,y) \in [0,1]^2 \setminus [\frac{2}{3}, 1]^2, \quad x > y. \end{cases}$$

I satisfies (NP) and (OP), it is easy to verify that (O_1, O_2, I) satisfies (7).

(ii) Let

$$I_0(x,y) = \begin{cases} 1 & y = 1, \\ 1 - x & y < 1. \end{cases}$$

Then I_0 is a fuzzy implication satisfies neither (NP) nor (OP), let $I = (\langle \frac{3}{5}, 1, I_0 \rangle)$, $O_1 = (\langle \frac{3}{5}, 1, O_{mM} \rangle)$ and $O_2 = T_M$. It is easy to verify that (O_{mM}, T_M, I_0) satisfy (7). In fact

$$I(x,y) = \begin{cases} \frac{8}{5} - x & (x,y) \in [\frac{3}{5}, 1] \times [\frac{3}{5}, 1), \\ 1 & (x,y) \in [0,1]^2 \setminus [\frac{3}{5}, 1] \times [\frac{3}{5}, 1) \text{ and } x \leq y, \\ y & (x,y) \in [0,1]^2 \setminus [\frac{3}{5}, 1] \times [\frac{3}{5}, 1) \text{ and } x > y. \end{cases}$$

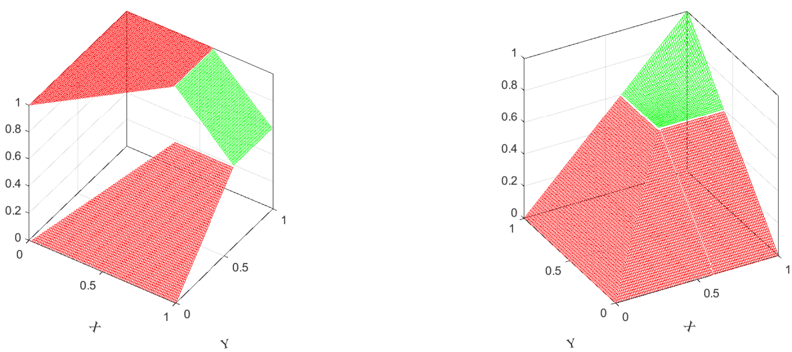
and

$$O_1(x,y) = \begin{cases} \frac{3}{5} + \frac{2}{5} \min(\frac{5x-3}{2}, \frac{5y-3}{2}) \max((\frac{5x-3}{2})^2, (\frac{5y-3}{2})^2) & (x,y) \in [\frac{3}{5}, 1]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

we can easily conclude that (O_1, O_2, I) satisfies (7) by Theorem 3.26.

In Example 3.27 (ii), we use O_{mM} to induce O, but O_{mM} is not associative, then O in this case is not associative. Thus, the distributivity equation of ordinal sum implications over t-norms and t-conorms is generalized.

The plots of fuzzy implication I and overlap function O_1 in Example 3.27(ii) are given in Figure 2 (a) and (b), respectively.



(a) (b)

Fig. 2. Plots of Example 3.27(ii).

3.4. Solution to $I(x, G_1(y,z)) = G_2(I(x,y), I(x,z))$

The analysis of this subsection is similar to that of Eq.(7). We will just list some results and present some differences.

Proposition 3.28. (Qiao and Hu [33]) Let G_1, G_2 be two grouping functions, and I be a fuzzy implication satisfying (NP) and (OP) in Eq.(8). Then the following are equivalent.

(i) G_1 and G_2 are the solution of Eq.(8).

(ii) $G_1 = G_2 = S_M$.

Corollary 3.29. Let G_1, G_2 be two grouping functions, and I an ordinal sum implication given by (9) in Definition 2.5. If (G_1, G_2, I) satisfies (8), and I satisfies (NP) and (OP), then $G_1 = G_2 = S_M$.

Proposition 3.30. Let G be a grouping function with the neutral element 0, and I an ordinal sum implication given by (9) in Definition 2.5. If G has an ordinal sum of the form

$$G(x,y) = \begin{cases} a_k + (b_k - a_k)G_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & x, y \in [a_k, b_k], \\ \max(x,y), & \text{otherwise}, \end{cases} \tag{17}$$

where G_k is a family of grouping functions with 0 as a neutral element, then (G, G, I) satisfies (8) if and only if (G_k, G_k, I_k) satisfies (8) for all $k \in A$.

Proposition 3.31. Let G_1, G_2 be two grouping functions with 0 as a neutral element and $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ an ordinal sum implication given by (9) in Definition 2.5 satisfying (8). If I satisfies neither (NP) nor (OP), $\mathcal{G}(x,y) = a_k$ implies $x = a_k$ or $y = a_k$ for all $x, y \in [0, a_k]$, $\mathcal{G}(x,y) = b_k$ implies $x = b_k$ or $y = b_k$ for all $x, y \in [0, b_k]$, $\mathcal{G} \in \{G_1, G_2\}$, then G_1 and G_2 have the ordinal sum of the forms as follows:

$$G_1(x,y) = \begin{cases} a_k + (b_k - a_k)G_{1k}(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & x, y \in [a_k, b_k], \\ \max(x,y), & \text{otherwise}, \end{cases} \tag{18}$$

$$G_2(x,y) = \begin{cases} a_k + (b_k - a_k)G_{2k}(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), & x, y \in [a_k, b_k], \\ \max(x,y), & \text{otherwise}, \end{cases} \tag{19}$$

where $\{G_{1k} | k \in A\}$ and $\{G_{2k} | k \in A\}$ are two families of grouping functions with 0 as a neutral element.

By these conclusions and the propositions above whether I satisfies (NP) and (OP), we have the following theorems.

Theorem 3.32. Let G_1, G_2 be two grouping functions, and $I = (\langle a, 1, I_0\rangle)$ with $a > 0$ an ordinal sum implication given by (9) in Definition 2.5, respectively. If (G_1, G_2, I) satisfies (8), and I_0 satisfies (NP) and (OP), if and only if $G_1 = G_2 = S_M$.

Theorem 3.33. Let G_1, G_2 be two grouping functions with 0 as a neutral element and $I = (\langle a_k, b_k, I_k\rangle)_{k\in A}$ an ordinal sum implication given by (9) in Definition 2.5. If I satisfies neither (NP) nor (OP), i. e., for index set A, whenever $|A| = 1$, $I = (\langle a, 1, I_0\rangle)$ with $a > 0$ and I_0 satisfies neither (NP) nor (OP), or whenever $|A| > 1$, there exists $k_0 \in A$ such that $b_{k_0} = 1$ and I_{k_0} does not satisfies (NP). $\mathcal{G}(x,y) = a_k$ implies $x = a_k$ or $y = a_k$ for all $x, y \in [0, a_k]$, $\mathcal{G}(x,y) = b_k$ implies $x = b_k$ or $y = b_k$ for all $x, y \in [0, b_k]$, $\mathcal{G} \in \{G_1, G_2\}$. Then (G_1, G_2, I) satisfies (8), if and only if G_1 is given by (18), G_2 is given by (19), and (G_{1k}, G_{2k}, I_k) satisfies (8), for all $k \in A$.

4. CONCLUDING REMARKS

In this paper, we have studied the four distributive equations of a new class of ordinal sum fuzzy implication proposed by Su et al. [37] with respect to two newly-born aggregate operators, viz., overlap and grouping functions. Sufficient and necessary conditions for ordinal sum implications satisfy (5) – (8) are given. This research will bring benefits to the related fields such as approximate reasoning and fuzzy control. In the past year, the research on the structure of ordinal sum of fuzzy implications and distributivity of fuzzy implication over t-norms and t-conorms have made a lot of achievements. See references [52, 53]. As a future work, we intend to focus on studying the distributivity and conditional distributivity of the general ordinal sum implications over overlap and grouping functions.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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