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# Conservative algebras of 2-dimensional algebras, III

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**Abstract.** In the present paper we prove that every local and 2-local derivation on conservative algebras of 2-dimensional algebras are derivations. Also, we prove that every local and 2-local automorphism on conservative algebras of 2-dimensional algebras are automorphisms.

## 1 Introduction

The present paper is devoted to the study of conservative algebras. In 1972 Kantor [12] introduced conservative algebras as a generalization of Jordan algebras (also, see a good written survey about the study of conservative algebras [25]).

In 1990 Kantor [14] defined the multiplication $\cdot$ on the set of all algebras (i.e. all multiplications) on the $n$-dimensional vector space $V_n$ over a field $\mathbb{F}$ of characteristic zero as follows: $A \cdot B = [L_e^A, B]$, where $A$ and $B$ are multiplications and $e \in V_n$ is some fixed vector. If $n > 1$, then the algebra $W(n)$ does not belong to any well-known class of algebras (such as associative, Lie, Jordan, or Leibniz algebras). The algebra $W(n)$ is a conservative algebra [12].

In [12] Kantor classified all conservative 2-dimensional algebras and defined the class of terminal algebras as algebras satisfying some certain identity. He proved that every terminal algebra is a conservative algebra and classified all simple finite-dimensional terminal algebras with left quasi-unit over an algebraically closed field of characteristic zero [13]. Terminal algebras were also studied in [18], [19].

In 2017 Kaygorodov and Volkov [16] described automorphisms, one-sided ideals, and idempotents of $W(2)$. Also a similar problem is solved for the algebra $W_2$ of all commutative algebras on the 2-dimensional vector space and for the algebra

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$S_2$ of all commutative algebras with zero multiplication trace on the 2-dimensional vector space. The papers [15], [17] are also devoted to the study of conservative algebras and superalgebras.

Let $\mathcal{A}$ be an algebra. A linear operator $\nabla$ on $\mathcal{A}$ is called a local derivation if for every $x \in \mathcal{A}$ there exists a derivation $\phi_x$ of $\mathcal{A}$, depending on $x$, such that $\nabla(x) = \phi_x(x)$. The history of local derivations had begun from the paper of Kadison [11]. Kadison introduced the concept of local derivation and proved that each continuous local derivation from a von Neumann algebra into its dual Banach bimodule is a derivation.

A similar notion, which characterizes nonlinear generalizations of derivations, was introduced by Šemrl as 2-local derivations. In his paper [26] was proved that a 2-local derivation of the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space $H$ is a derivation. After his works, appear numerous new results related to the description of local and 2-local derivations of associative algebras (see, for example, [1], [3], [4], [20], [21], [23]).

The study of local and 2-local derivations of non-associative algebras was initiated in some papers of Ayupov and Kudaybergenov (for the case of Lie algebras, see [5], [6]). In particular, they proved that there are no non-trivial local and 2-local derivations on semisimple finite-dimensional Lie algebras. In [8] examples of 2-local derivations on nilpotent Lie algebras which are not derivations, were also given. Later, the study of local and 2-local derivations was continued for Leibniz algebras [7], Malcev algebras and Jordan algebras [2]. Local automorphisms and 2-local automorphisms, also were studied in many cases, for example, they were studied on Lie algebras [5], [10].

Now, a linear operator $\nabla$ on $\mathcal{A}$ is called a local automorphism if for every $x \in \mathcal{A}$ there exists an automorphism $\phi_x$ of $\mathcal{A}$, depending on $x$, such that $\nabla(x) = \phi_x(x)$. The concept of local automorphism was introduced by Larson and Sourour [22] in 1990. They proved that, invertible local automorphisms of the algebra of all bounded linear operators on an infinite-dimensional Banach space $X$ are automorphisms.

A similar notion, which characterizes non-linear generalizations of automorphisms, was introduced by Šemrl in [26] as 2-local automorphisms. Namely, a map $\Delta\colon \mathcal{A} \to \mathcal{A}$ (not necessarily linear) is called a 2-local automorphism, if for every $x, y \in \mathcal{A}$ there exists an automorphism $\phi_{x,y}\colon \mathcal{A} \to \mathcal{A}$ such that $\Delta(x) = \phi_{x,y}(x)$ and $\Delta(y) = \phi_{x,y}(y)$. After the work of Šemrl, it appeared numerous new results related to the description of local and 2-local automorphisms of algebras (see, for example, [5], [7], [9], [10], [21]).

In the present paper, we continue the study of derivations, local and 2-local derivations of conservative algebras of 2-dimensional algebras. We prove that every local and 2-local derivation of the conservative algebras of 2-dimensional algebras are derivations. In the present paper, we continue the study of automorphisms, local and 2-local automorphisms in the case of conservative algebras of 2-dimensional algebras. We prove that every local and 2-local automorphism of the conservative algebras of 2-dimensional algebras are automorphisms.

## 2 Preliminaries

Throughout this paper $\mathbb{F}$ is some fixed field of characteristic zero. A multiplication on 2-dimensional vector space is defined by a $2 \times 2 \times 2$ matrix. Their classification was given in many papers (see, for example, [24]). Let consider the space $W(2)$ of all multiplications on the 2-dimensional space $V_2$ with a basis $v_1$, $v_2$. The definition of the multiplication $\cdot$ on the algebra $W(2)$ is defined as follows: we fix the vector $v_1 \in V_2$ and define

$$(A \cdot B)(x, y) = A(v_1, B(x, y)) - B(A(v_1, x), y) - B(x, A(v_1, y))$$

for $x$, $y \in V_2$ and $A$, $B \in W(2)$. The algebra $W(2)$ is conservative [14]. Let consider the multiplications $\alpha_{i,j}^k$ $(i,\ j,\ k = 1, 2)$ on $V_2$ defined by the formula $\alpha_{i,j}^k(v_t, v_l) = \delta_{it}\delta_{jl}v_k$ for all $t$, $l \in \{1, 2\}$. It is easy to see that $\{\alpha_{i,j}^k | i, j, k = 1, 2\}$ is a basis of the algebra $W(2)$. The multiplication table of $W(2)$ in this basis is given in [15]. In this work we use another basis for the algebra $W(2)$ (from [16]). Let introduce the notation

$$e_1 = \alpha_{11}^1 - \alpha_{12}^2 - \alpha_{21}^2,\ e_2 = \alpha_{11}^2,\ e_3 = \alpha_{22}^2 - \alpha_{12}^1 - \alpha_{21}^1,\ e_4 = \alpha_{22}^1,\ e_5 = 2\alpha_{11}^1 + \alpha_{12}^2 + \alpha_{21}^2,$$

$$e_6 = 2a_{22}^2 + \alpha_{12}^1 + \alpha_{21}^1,\ e_7 = \alpha_{12}^1 - \alpha_{21}^1,\ e_8 = \alpha_{12}^2 - \alpha_{21}^2.$$

It is easy to see that the multiplication table of $W(2)$ in the basis $e_1, \ldots, e_8$ is the following.

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $-e_1$ | $-3e_2$ | $e_3$ | $3e_4$ | $-e_5$ | $e_6$ | $e_7$ | $-e_8$ |
| $e_2$ | $3e_2$ | $0$ | $2e_1$ | $e_3$ | $0$ | $-e_5$ | $e_8$ | $0$ |
| $e_3$ | $-2e_3$ | $-e_1$ | $-3e_4$ | $0$ | $e_6$ | $0$ | $0$ | $-e_7$ |
| $e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_5$ | $-2e_1$ | $-3e_2$ | $-e_3$ | $0$ | $-2e_5$ | $-e_6$ | $-e_7$ | $-2e_8$ |
| $e_6$ | $2e_3$ | $e_1$ | $3e_4$ | $0$ | $-e_6$ | $0$ | $0$ | $e_7$ |
| $e_7$ | $2e_3$ | $e_1$ | $3e_4$ | $0$ | $-e_6$ | $0$ | $0$ | $e_7$ |
| $e_8$ | $0$ | $e_2$ | $-e_3$ | $-2e_4$ | $0$ | $-e_6$ | $-e_7$ | $0$ |

The subalgebra generated by the elements $e_1, \ldots, e_6$ is the conservative (and, moreover, terminal) algebra $W_2$ of commutative 2-dimensional algebras. The subalgebra generated by the elements $e_1, \ldots, e_4$ is the conservative (and, moreover, terminal) algebra $S_2$ of all commutative 2-dimensional algebras with zero multiplication trace [15].

Let $\mathcal{A}$ be an algebra. A linear map $D \colon \mathcal{A} \to \mathcal{A}$ is called a derivation, if $D(xy) = D(x)y + xD(y)$ for any two elements $x$, $y \in \mathcal{A}$.

Our main tool for study of local and 2-local derivations of the algebras $S_2$, $W_2$ and $W(2)$ is the following lemma [15, Theorem 6], where the matrix of a derivation is calculated in the new basis $e_1, \ldots, e_8$.

**Lemma 1.** *A linear map $D\colon W(2) \to W(2)$ is a derivation if and only if the matrix of $D$ has the following matrix form:*

$$\begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\alpha & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\alpha & 2\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1}$$

*where $\alpha$, $\beta$ are elements in $\mathbb{F}$.*

Now, we give a characterization of automorphisms on conservative algebras of 2-dimensional algebras.

Let $\mathcal{A}$ be an algebra. A bijective linear map $\phi\colon \mathcal{A} \to \mathcal{A}$ is called an automorphism, if $\phi(xy) = \phi(x)\phi(y)$ for any elements $x$, $y \in \mathcal{A}$.

Our principal tool for study of local and 2-local automorphisms of the algebras $S_2$, $W_2$ and $W(2)$ is the following lemma, which was proved in [16, Theorem 11].

**Lemma 2.** *A linear map $\phi\colon W(2) \to W(2)$ is an automorphism if and only if the matrix of $\phi$ has the following matrix form:*

$$\begin{pmatrix} 1 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2ab & a^2b & b & 0 & 0 & 0 & 0 & 0 \\ 3a^2b^2 & a^3b^2 & 3ab^2 & b^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -ab & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & ab \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

*where $a$, $b$ are elements in $\mathbb{F}$ and $b \neq 0$.*

## 3 Local derivations of conservative algebras of 2-dimensional algebras

In this section we give a characterization of derivations on conservative algebras of 2-dimensional algebras.

Let $\mathcal{A}$ be an algebra. A linear map $\nabla\colon \mathcal{A} \to \mathcal{A}$ is called a local derivation, if for any element $x \in \mathcal{A}$ there exists a derivation $D_x\colon \mathcal{A} \to \mathcal{A}$ such that $\nabla(x) = D_x(x)$.

**Theorem 1.** *Every local derivation of the algebra $W(2)$ is a derivation.*

*Proof.* Let $\nabla$ be an arbitrary local derivation of $W(2)$ and write

$$\nabla(x) = B\bar{x}, x \in W(2),$$

where $B=(b_{i,j})_{i,j=1}^{8}$, $\bar{x}=(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)$ is the vector corresponding to $x$. Then for every $x\in W(2)$ there exist elements $a_x$, $b_x$ in $\mathbb{F}$ such that

$$B\bar{x}=\begin{pmatrix} 0 & a_x & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -b_x & 0 & 0 & 0 & 0 & 0 & 0\\ 2a_x & 0 & b_x & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 3a_x & 2b_x & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -a_x & b_x & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & b_x & a_x\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7\\ x_8\end{pmatrix}.$$

In other words

$$\begin{cases} b_{1,1}x_1+b_{1,2}x_2+b_{1,3}x_3+b_{1,4}x_4+b_{1,5}x_5+b_{1,6}x_6+b_{1,7}x_7+b_{1,8}x_8=a_xx_2;\\ b_{2,1}x_1+b_{2,2}x_2+b_{2,3}x_3+b_{2,4}x_4+b_{2,5}x_5+b_{2,6}x_6+b_{2,7}x_7+b_{2,8}x_8=-b_xx_2;\\ b_{3,1}x_1+b_{3,2}x_2+b_{3,3}x_3+b_{3,4}x_4+b_{3,5}x_5+b_{3,6}x_6+b_{3,7}x_7+b_{3,8}x_8=2a_xx_1+b_xx_3;\\ b_{4,1}x_1+b_{4,2}x_2+b_{4,3}x_3+b_{4,4}x_4+b_{4,5}x_5+b_{4,6}x_6+b_{4,7}x_7+b_{4,8}x_8=3a_xx_3+2b_xx_4;\\ b_{5,1}x_1+b_{5,2}x_2+b_{5,3}x_3+b_{5,4}x_4+b_{5,5}x_5+b_{5,6}x_6+b_{5,7}x_7+b_{5,8}x_8=0;\\ b_{6,1}x_1+b_{6,2}x_2+b_{6,3}x_3+b_{6,4}x_4+b_{6,5}x_5+b_{6,6}x_6+b_{6,7}x_7+b_{6,8}x_8=-a_xx_5+b_xx_6;\\ b_{7,1}x_1+b_{7,2}x_2+b_{7,3}x_3+b_{7,4}x_4+b_{7,5}x_5+b_{7,6}x_6+b_{7,7}x_7+b_{7,8}x_8=b_xx_7+a_xx_8;\\ b_{8,1}x_1+b_{8,2}x_2+b_{8,3}x_3+b_{8,4}x_4+b_{8,5}x_5+b_{8,6}x_6+b_{8,7}x_7+b_{8,8}x_8=0. \end{cases}$$

Taking $x=(1,0,0,0,0,0,0,0)$, $x=(0,0,1,0,0,0,0,0)$, $x=(0,0,0,1,0,0,0,0)$, etc, from this it follows that

$$\begin{gathered} b_{1,1}=b_{1,3}=b_{1,4}=b_{1,5}=b_{1,6}=b_{1,7}=b_{1,8}=\\ =b_{2,1}=b_{2,3}=b_{2,4}=b_{2,5}=b_{2,6}=b_{2,7}=b_{2,8}\\ =b_{3,2}=b_{3,4}=b_{3,5}=b_{3,6}=b_{3,7}=b_{3,8}\\ =b_{4,1}=b_{4,2}=b_{4,5}=b_{4,6}=b_{4,7}=b_{4,8}\\ =b_{5,1}=b_{5,2}=b_{5,3}=b_{5,4}=b_{5,5}=b_{5,6}=b_{5,7}=b_{5,8}\\ =b_{6,1}=b_{6,2}=b_{6,3}=b_{6,4}=b_{6,7}=b_{6,8}\\ =b_{7,1}=b_{7,2}=b_{7,3}=b_{7,4}=b_{7,5}=b_{7,6}\\ =b_{8,1}=b_{8,2}=b_{8,3}=b_{8,4}=b_{8,5}=b_{8,6}=b_{8,7}=b_{8,8}=0. \end{gathered}$$

Then for every $x\in W(2)$ there exist elements $a_x$, $b_x$ in $\mathbb{F}$ such that

$$\begin{cases} b_{1,2}x_2=a_xx_2;\\ b_{2,2}x_2=-b_xx_2;\\ b_{3,1}x_1+b_{3,3}x_3=2a_xx_1+b_xx_3;\\ b_{4,3}x_3+b_{4,4}x_4=3a_xx_3+2b_xx_4;\\ b_{6,5}x_5+b_{6,6}x_6=-a_xx_5+b_xx_6;\\ b_{7,7}x_7+b_{7,8}x_8=b_xx_7+a_xx_8. \end{cases} \tag{3}$$

Using 1-th and 3-th equalities of system (3) we get

$$\begin{cases} 2b_{1,2}x_1x_2 = 2a_xx_1x_2; \\ b_{3,1}x_1x_2 + b_{3,3}x_2x_3 = 2a_xx_1x_2 + b_xx_2x_3. \end{cases}$$

and

$$(b_{3,1} - 2b_{1,2})x_1x_2 + b_{3,3}x_2x_3 = b_xx_2x_3.$$

Hence, $b_{3,1} = 2b_{1,2}$. Similarly, using equalities of (3) we get

$$b_{4,3} = 3b_{1,2}, b_{2,2} = -b_{3,3}, b_{4,4} = -2b_{2,2}.$$

Using 1-th and 5-th equalities of system (3) we get

$$\begin{cases} b_{1,2}x_2x_5 = a_xx_2x_5; \\ b_{6,5}x_5x_2 + b_{6,6}x_6x_2 = -a_xx_5x_2 + b_xx_6x_2. \end{cases}$$

and

$$(b_{6,5} + b_{1,2})x_2x_5 + b_{6,6}x_6x_2 = b_xx_6x_2.$$

Hence, $b_{6,5} = -b_{1,2}$.

Using 2-th and 5-th equalities of system (3) we get

$$\begin{cases} b_{2,2}x_2x_6 = -b_xx_2x_6; \\ b_{6,5}x_5x_2 + b_{6,6}x_6x_2 = -a_xx_5x_2 + b_xx_6x_2. \end{cases}$$

and

$$b_{6,5}x_5x_2 + (b_{6,6} + b_{2,2})x_6x_2 = -a_xx_5x_2.$$

Hence, $b_{6,6} = -b_{2,2}$.

Using 1-th and 6-th equalities of system (3) we get

$$\begin{cases} b_{1,2}x_2x_8 = a_xx_2x_8; \\ b_{7,7}x_7x_2 + b_{7,8}x_8x_2 = b_xx_7x_2 + a_xx_8x_2. \end{cases}$$

and

$$b_{7,7}x_7x_2 + (b_{7,8} - b_{1,2})x_8x_2 = b_xx_7x_2.$$

Hence, $b_{7,8} = b_{1,2}$.

Using 2-th and 6-th equalities of system (3) we get

$$\begin{cases} b_{2,2}x_2x_7 = -b_xx_2x_7; \\ b_{7,7}x_7x_2 + b_{7,8}x_8x_2 = b_xx_7x_2 + a_xx_8x_2. \end{cases}$$

and

$$(b_{7,7} + b_{2,2})x_7x_2 + b_{7,8}x_8x_2 = a_xx_8x_2.$$

Hence, $b_{7,7} = -b_{2,2}$.

These equalities show that the matrix of the linear map $\nabla$ is of the form (1). Therefore, by lemma 1 $\nabla$ is a derivation. This completes the proof. □

Since a derivation on $W(2)$ is invariant on the subalgebras $S_2$ and $W_2$, we have the following corollary.

**Corollary 1.** *Every local derivation of the algebras $S_2$ and $W_2$ is a derivation.*

## 4 2-Local derivations of conservative algebras of 2-dimensional algebras

In this section we give another characterization of derivations on conservative algebras of 2-dimensional algebras.

A (not necessary linear) map $\Delta\colon \mathcal{A} \to \mathcal{A}$ is called a 2-local derivation, if for any elements $x, y \in \mathcal{A}$ there exists a derivation $D_{x,y}\colon \mathcal{A} \to \mathcal{A}$ such that $\Delta(x) = D_{x,y}(x)$, $\Delta(y) = D_{x,y}(y)$.

**Theorem 2.** *Every 2-local derivation of the algebras $S_2$, $W_2$ and $W(2)$ is a derivation.*

*Proof.* We will prove that every 2-local derivation of $W(2)$ is a derivation.

Let $\Delta$ be an arbitrary 2-local derivation of $W(2)$. T hen, by the definition, for every element $a \in W(2)$, there exists a derivation $D_{a,e_2}$ of $W(2)$ such that

$$\Delta(a) = D_{a,e_2}(a), \ \ \Delta(e_2) = D_{a,e_2}(e_2).$$

By lemma 1, the matrix $A^{a,e_2}$ of the derivation $D_{a,e_2}$ has the following matrix form:

$$A^{a,e_2} = \begin{pmatrix} 0 & \alpha_{a,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_{a,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\alpha_{a,e_2} & 0 & \beta_{a,e_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\alpha_{a,e_2} & 2\beta_{a,e_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_{a,e_2} & \beta_{a,e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_{a,e_2} & \alpha_{a,e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $v$ be an arbitrary element in $W(2)$. Then there exists a derivation $D_{v,e_2}$ of $W(2)$ such that

$$\Delta(v) = D_{v,e_2}(v), \ \ \Delta(e_2) = D_{v,e_2}(e_2).$$

By lemma 1, the matrix $A^{v,e_2}$ of the derivation $D_{v,e_2}$ has the following matrix form:

$$A^{v,e_2} = \begin{pmatrix} 0 & \alpha_{v,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta_{v,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\alpha_{v,e_2} & 0 & \beta_{v,e_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\alpha_{v,e_2} & 2\beta_{v,e_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\alpha_{v,e_2} & \beta_{v,e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_{v,e_2} & \alpha_{v,e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $\Delta(e_2) = D_{a,e_2}(e_2) = D_{v,e_2}(e_2)$, we have

$$\alpha_{a,e_2} = \alpha_{v,e_2}, \beta_{a,e_2} = \beta_{v,e_2},$$

that it

$$D_{v,e_2} = D_{a,e_2}.$$

Therefore, for any element $a$ of the algebra $W(2)$

$$\Delta(a) = D_{v,e_2}(a),$$

that it $D_{v,e_2}$ does not depend on $a$. Hence, $\Delta$ is a derivation by lemma 1.

The cases of the algebras $S_2$ and $W_2$ are also similarly proved. This ends the proof. □

## 5 2-Local automorphisms of conservative algebras of 2-dimensional algebras

A (not necessary linear) map $\Delta\colon \mathcal{A} \to \mathcal{A}$ is called a 2-local automorphism, if for any elements $x$, $y \in \mathcal{A}$ there exists an automorphism $\phi_{x,y}\colon \mathcal{A} \to \mathcal{A}$ such that $\Delta(x) = \phi_{x,y}(x)$, $\Delta(y) = \phi_{x,y}(y)$.

**Theorem 3.** *Every 2-local automorphism of the algebras $S_2$, $W_2$ and $W(2)$ is an automorphism.*

*Proof.* We prove that every 2-local automorphism of $W(2)$ is an automorphism.

Let $\Delta$ be an arbitrary 2-local automorphism of $W(2)$. Then, by the definition, for every element $x \in W(2)$,

$$x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 + x_8e_8,$$

there exist elements $a_{x,e_2}$, $b_{x,e_2}$ such that

$$A_{x,e_2} = \begin{pmatrix} 1 & a_{x,e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_{x,e_2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_{x,e_2}b_{x,e_2} & a^2_{x,e_2}b_{x,e_2} & b_{x,e_2} & 0 & 0 & 0 & 0 & 0 \\ 3a^2_{x,e_2}b^2_{x,e_2} & a^3_{x,e_2}b^2_{x,e_2} & 3a_{x,e_2}b^2_{x,e_2} & b^2_{x,e_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{x,e_2}b_{x,e_2} & b_{x,e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{x,e_2} & a_{x,e_2}b_{x,e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$\Delta(x) = A_{x,e_2}\bar{x}$, where $\bar{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ is the vector corresponding to $x$, and

$$\Delta(e_2) = A_{x,e_2}e_2 = (a_{x,e_2}, \frac{1}{b_{x,e_2}}, a^2_{x,e_2}b_{x,e_2}, a^3_{x,e_2}b^2_{x,e_2}, 0, 0, 0, 0).$$

Since the element $x$ was chosen arbitrarily, we have

$$\Delta(e_2) = (a_{x,e_2}, \frac{1}{b_{x,e_2}}, a^2_{x,e_2}b_{x,e_2}, a^3_{x,e_2}b^2_{x,e_2}, 0, 0, 0, 0)$$

$$= (a_{y,e_2}, \frac{1}{b_{y,e_2}}, a_{y,e_2}^2 b_{y,e_2}, a_{y,e_2}^3 b_{y,e_2}^2, 0, 0, 0, 0),$$

for each pair $x$, $y$ of elements in $W(2)$. Hence, $a_{x,e_2} = a_{y,e_2}$, $b_{x,e_2} = b_{y,e_2}$. Therefore

$$\Delta(x) = A_{y,e_2} x$$

for any $x \in W(2)$ and the matrix $A_{y,e_2}$ does not depend on $x$. Thus, by lemma 2 $\Delta$ is an automorphism.

The cases of the algebras $S_2$ and $W_2$ are also similarly proved. The proof is complete. □

## 6 Local automorphisms of conservative algebras of 2-dimensional algebras

Let $\mathcal{A}$ be an algebra. A linear map $\nabla\colon \mathcal{A} \to \mathcal{A}$ is called a local automorphism, if for any element $x \in \mathcal{A}$ there exists an automorphism $\phi_x\colon \mathcal{A} \to \mathcal{A}$ such that $\nabla(x) = \phi_x(x)$.

**Theorem 4.** *Every local automorphism of the algebras $S_2$, $W_2$ and $W(2)$ is an automorphism.*

*Proof.* We prove that every local automorphism of $W(2)$ is an automorphism.

Let $\nabla$ be an arbitrary local automorphism of $W(2)$ and $B$ be its matrix, i.e.,

$$\nabla(x) = B\bar{x}, x \in W(2),$$

where $\bar{x}$ is the vector corresponding to $x$. Then, by the definition, for every element $x \in W(2)$,

$$x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 + x_8e_8,$$

there exist elements $a_x$, $b_x$ such that

$$A_x = \begin{pmatrix} 1 & a_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_xb_x & a_x^2b_x & b_x & 0 & 0 & 0 & 0 & 0 \\ 3a_x^2b_x^2 & a_x^3b_x^2 & 3a_xb_x^2 & b_x^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_xb_x & b_x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_x & a_xb_x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\nabla(x) = B\bar{x} = A_x\bar{x}.$$

Using these equalities and by choosing subsequently $x = e_1$, $x = e_2$, $\ldots, x = e_8$ we get

$$B = \begin{pmatrix} 1 & a_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_{e_2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_{e_1}b_{e_1} & a_{e_2}^2 b_{e_2} & b_{e_3} & 0 & 0 & 0 & 0 & 0 \\ 3a_{e_1}^2 b_{e_1}^2 & a_{e_2}^3 b_{e_2}^2 & 3a_{e_3}b_{e_3}^2 & b_{e_4}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{e_5}b_{e_5} & b_{e_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{e_7} & a_{e_8}b_{e_8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $\nabla(e_6 + e_7) = \nabla(e_6) + \nabla(e_7)$, we have

$$b_{e_6+e_7} = b_{e_6}, b_{e_6+e_7} = b_{e_7}.$$

Hence,

$$b_{e_6} = b_{e_7}.$$

Similarly to this equality we get $b_{e_3} = b_{e_6}$ and $b_{e_6} = b_{e_2} \neq 0$. Hence,

$$b_{e_2} = b_{e_3} = b_{e_6} = b_{e_7}. \tag{4}$$

Since $\nabla(e_5 + e_8) = \nabla(e_5) + \nabla(e_8)$, we have

$$a_{e_5+e_8}b_{e_5+e_8} = a_{e_5}b_{e_5},\ a_{e_5+e_8}b_{e_5+e_8} = a_{e_8}b_{e_8}.$$

From this it follows that

$$a_{e_5}b_{e_5} = a_{e_8}b_{e_8}.$$

Similarly to this equality we get $a_{e_1}b_{e_1} = a_{e_8}b_{e_8}$. Hence,

$$a_{e_1}b_{e_1} = a_{e_5}b_{e_5} = a_{e_8}b_{e_8}. \tag{5}$$

Since $\nabla(e_4 + e_6) = \nabla(e_4) + \nabla(e_6)$, we have

$$b_{e_4+e_6}^2 = b_{e_4}^2,\ b_{e_4+e_6}^2 = b_{e_6}^2.$$

From this it follows that

$$b_{e_4}^2 = b_{e_6}^2.$$

Hence, by (4), we get

$$b_{e_4}^2 = b_{e_2}^2. \tag{6}$$

Since $\nabla(e_2 + e_8) = \nabla(e_2) + \nabla(e_8)$, we have

$$a_{e_2} = a_{e_2+e_8},\ a_{e_2+e_8}^2 b_{e_2+e_8} = a_{e_2}^2 b_{e_2},\ a_{e_2+e_8}b_{e_2+e_8} = a_{e_8}b_{e_8}.$$

Hence,

$$b_{e_2+e_8} = b_{e_2},\ a_{e_2+e_8}b_{e_2+e_8} = a_{e_2}b_{e_2}$$

and, therefore,

$$a_{e_2}b_{e_2} = a_{e_8}b_{e_8}. \tag{7}$$

Similarly, since $\nabla(e_2+e_3) = \nabla(e_2)+\nabla(e_3)$, we have

$$a_{e_2} = a_{e_2+e_3},\;\; b_{e_2}^{-1} = b_{e_2+e_3}^{-1},\;\; a_{e_2+e_3}^3 b_{e_2+e_3}^2 + 3a_{e_2+e_3}b_{e_2+e_3}^2 = a_{e_2}^3 b_{e_2}^2 + 3a_{e_3}b_{e_3}^2.$$

Hence,

$$b_{e_2} = b_{e_2+e_3}$$

and by (4) and $a_{e_2} = a_{e_2+e_3}$ we get

$$a_{e_2}^3 + 3a_{e_2} = a_{e_2}^3 + 3a_{e_3}.$$

Therefore, $a_{e_2} = a_{e_3}$ and

$$a_{e_2}b_{e_2}^2 = a_{e_3}b_{e_3}^2. \tag{8}$$

Finally, since $\nabla(e_1+e_8) = \nabla(e_1)+\nabla(e_8)$, we have

$$a_{e_1+e_8}b_{e_1+e_8} = a_{e_1}b_{e_1},\;\; a_{e_1+e_8}b_{e_1+e_8} = a_{e_8}b_{e_8}.$$

Hence,

$$a_{e_1}b_{e_1} = a_{e_8}b_{e_8}.$$

By (7), from the last equalities it follows that

$$a_{e_1}b_{e_1} = a_{e_2}b_{e_2},\;\; a_{e_1}^2 b_{e_1}^2 = (a_{e_1}b_{e_1})^2 = (a_{e_2}b_{e_2})^2 = a_{e_2}^2 b_{e_2}^2. \tag{9}$$

By (4), (5), (6), (7), (8), (9) the matrix $B$ has the following matrix form

$$B = \begin{pmatrix} 1 & a_{e_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{b_{e_2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2a_{e_2}b_{e_2} & a_{e_2}^2 b_{e_2} & b_{e_2} & 0 & 0 & 0 & 0 & 0 \\ 3a_{e_2}^2 b_{e_2}^2 & a_{e_2}^3 b_{e_2}^2 & 3a_{e_2}b_{e_2}^2 & b_{e_2}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_{e_2}b_{e_2} & b_{e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{e_2} & a_{e_2}b_{e_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence, by lemma 2, the local automorphism $\nabla$ is an automorphism.

The cases of the algebras $S_2$ and $W_2$ are also similarly proved. This ends the proof. □

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