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NOTE ON "CONSTRUCTION OF UNINORMS ON BOUNDED LATTICES"

Xiu-Juan Hua, Hua-Peng Zhang and Yao Ouyang

In this note, we point out that Theorem 3.1 as well as Theorem 3.5 in G. D. Çaylı and F. Karaçal (Kybernetika *53* (2017), 394–417) contains a superfluous condition. We have also generalized them by using closure (interior, resp.) operators.

Keywords: bounded lattices, uninorms, closure operators, interior operators

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1. INTRODUCTION

Triangular norms (t-norms for short) and triangular conorms (t-conorms for short) on the unit interval $[0,1]$ introduced by Schweizer and Sklar in [12] are indispensable tools in fuzzy community. Uninorms on the unit interval introduced by Yager and Rybalov [14] are important generalizations of t-norms and t-conorms, which allow the neutral element e to locate anywhere of $[0,1]$. Notice that a uninorm degenerates to a t-norm (t-conorm, resp.) whenever $e = 1$ ($e = 0$, resp.). A uninorm with neutral element $e \in (0,1)$ is usually called proper.

Uninorms on bounded lattices initialed by Karaçal and Mesiar[10] have drawn many attention, see [2, 4, 9, 11, 13] and references therein. Various constructions, including the ones by using closure (interior, resp.) operators and by using t-subnorms (t-superconorms, resp.) are introduced [9, 11].

The aim of this paper is to point out that the main results (Theorems 3.1 and 3.5) of [2] contain superfluous conditions. We will give an example to support our claim and further provide an improvement of these two theorems. The remainder of this paper is organized as follows. In section 2, we recall some necessary knowledge concerning lattices and aggregation functions on bounded lattices. In section 3, we give an example to illustrate that Theorems 3.1 and 3.5 of [2] contain superfluous conditions, while in section 4 we improve these two results.

2. PRELIMINARIES

This section includes some necessary knowledge.

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2.1. Bounded lattices and Closure operators on a lattice

A *lattice* is a nonempty set L equipped with a partial order $\leq$ such that any two elements x and y have a greatest lower bound (called meet or infimum), denoted by $x \wedge y$, as well as a smallest upper bound (called join or supremum), denoted by $x \vee y$. For $a, b \in L$, we also write $b \geq a$ if $a \leq b$ holds. The symbol $a < b$ means that $a \leq b$ and $a \neq b$. If neither $a \leq b$ nor $a \geq b$, then we say that a is incomparable with b and write $a \parallel b$. Let $a \in L$ be fixed, the set of all $b \in L$ with $a \parallel b$ will be denoted by I_a.

A lattice $(L, \leq, \wedge, \vee)$ is called *bounded* if it has a top element 1 and a bottom element 0, i.e., for any $x \in L$ we have $0 \leq x \leq 1$. Let $(L, \leq, \wedge, \vee)$ be a lattice and $a, b \in L$ with $a \leq b$. The subinterval $[a, b]$ is a sublattice of L defined as

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Other subintervals such as $[a, b)$ and (a, b) can be defined similarly.

For more information about lattices, we refer to [1].

Definition 2.1. (Everett [8]) Let $(L, \leq, \wedge, \vee)$ be a lattice. A mapping $cl\colon L \to L$ is said to be *a closure operator* if, for any $x, y \in L$, it satisfies the following three conditions:

(i) $x \leq cl(x)$ (expansion);

(ii) $cl(x \vee y) = cl(x) \vee cl(y)$ (preservation of join);

(iii) $cl(cl(x)) = cl(x)$ (idempotence).

By (i), the condition (iii) is equivalent to $cl(cl(x)) \leq cl(x)$. In addition, (ii) implies (ii)' $x \leq y \Longrightarrow cl(x) \leq cl(y)$. Note that Birkhoff [1] defines a closure operator by (i), (ii)' and (iii).

Any lattice can naturally induce a family of closure operators on itself.

Example 2.2. (Drossos and Navara [7]) Let $(L, \leq, \wedge, \vee)$ be a lattice and $a \in L$ be given. Then the mapping $cl_a\colon L \to L$ defined as

$$cl_a(x) = x \vee a \ (\forall\ x \in L)$$

is a closure operator.

Dually, we can define interior operators on a lattice.

Definition 2.3. Let $(L, \leq, \wedge, \vee)$ be a lattice. A mapping $int\colon L \to L$ is said to be *an interior operator* if, for any $x, y \in L$, it satisfies the following three conditions:

(i) $int(x) \leq x$ (contraction);

(ii) $int(x \wedge y) = int(x) \wedge int(y)$ (preservation of meet);

(iii) $int(int(x)) = int(x)$ (idempotence).

Similar to the closure operators, $int_a(x) = x \wedge a, x \in L$ is an interior operator on L, where $a \in L$ is an arbitrary but fixed element.

2.2. T-norms and Uninorms on a bounded lattice

Aggregation functions such as t-norms, t-conorms and uninorms can be defined on a bounded lattice.

Let $[a,b]$ be a subinterval of a bounded lattice L. A binary operation $T\colon [a,b]\times[a,b]\to[a,b]$ is said to be a *t-norm* on $[a,b]$ [5, 6] if it is commutative, increasing (in each variable), associative and has a neutral element b, i. e., $T(x,b)=x$ for any $x\in[a,b]$. If $[a,b]=L$, then we define t-norms on the lattice L. The strongest t-norm on L is $T_\wedge$ defined by $T_\wedge(x,y)=x\wedge y$ for any $x,y\in L$, while the weakest t-norm on L is the drastic product T_D which takes value $x\wedge y$ if $1\in\{x,y\}$ and 0 otherwise. That is to say, for any t-norm T on L, we have $T_D\le T\le T_\wedge$. If we replace the boundary condition $T(b,x)=x$ by $T(b,x)\le x$ then we define a *t-subnorm* on $[a,b]$ [9]. Obviously, each t-norm on $[a,b]$ is a t-subnorm on $[a,b]$, but not vice versa.

A *t-conorm* on $[a,b]$ is a binary operation $S\colon [a,b]\times[a,b]\to[a,b]$, which is commutative, increasing (in each variable), associative and has a neutral element a, i. e., $S(x,a)=x$ for any $x\in[a,b]$. The weakest t-conorm on L is $S_\vee$ defined by $S_\vee(x,y)=x\vee y$ for any $x,y\in L$, while the strongest t-conorm on L is the drastic sum S_D which takes value $x\vee y$ if $0\in\{x,y\}$ and 1 otherwise. If we replace the boundary condition $S(a,x)=x$ for a t-conorm S on $[a,b]$ by $S(a,x)\ge x$ then we define a *t-superconorm* on $[a,b]$. Obviously, each t-conorm on $[a,b]$ is a t-superconorm on $[a,b]$, but not vice versa.

Definition 2.4. (Karaçal and Mesiar [10]) Let $(L,\le,\wedge,\vee,0,1)$ be a bounded lattice. A binary operation $U\colon L\times L\to L$ is called a *uninorm* on L if, for any $x,y,z\in L$, the following conditions are fulfilled:

(i) $U(x,y)=U(y,x)$ (commutativity);

(ii) If $x\le y$, then $U(x,z)\le U(y,z)$ (increasingness);

(iii) $U(U(x,y),z)=U(x,U(y,z))$ (associativity);

(iv) There is an element $e\in L$ such that $U(x,e)=x$ (neutrality).

Ouyang and Zhang [11] proposed a rather effective method to construct uninorms on L with a given t-norm T (t-conorm S, resp.) on the subinterval $[0,e]$ ($[e,1]$, resp.) of L.

Theorem 2.5. (Ouyang and Zhang [11]) Let $(L,\le,0,1)$ be a bounded lattice with $e\in L\setminus\{0,1\}$. Give a t-norm T_e on $[0,e]^2$ and t-conorm S_e on $[e,1]^2$.

(1) If $cl: L\to L$ is a closure operator, then the function $U_{cl}: L\times L\to L$ is a uninorm on L with the neutral element e, where

$$U_{cl}(x,y)=\begin{cases} T_e(x,y) & \text{if } x,y\in[0,e]^2,\\ y & \text{if } x\in[0,e] \text{ and } y\in L\setminus[0,e],\\ x & \text{if } y\in[0,e] \text{ and } x\in L\setminus[0,e],\\ cl(x)\vee cl(y) & \text{otherwise.}\end{cases}$$

(2) If $int : L \to L$ is an interior operator, then the function $U_{int} : L \times L \to L$ is a uninorm on L with the neutral element e, where

$$U_{int}(x,y) = \begin{cases} S_e(x,y) & \text{if } (x,y) \in [e,1]^2, \\ y & \text{if } x \in [e,1] \text{ and } y \in L \setminus [e,1], \\ x & \text{if } y \in [e,1] \text{ and } x \in L \setminus [e,1], \\ int(x) \wedge int(y) & \text{otherwise.} \end{cases}$$

Many constructions of uninorms can be seen as a special case of Theorem 2.5. For example, if we put $cl(x) = x \vee 1 = 1$ ($int(x) = x \wedge 0 = 0$, resp.) in Theorem 2.5, then we retrieve the corresponding uninorms constructed by Karaçal and Mesiar (see Theorem 1 of [10]).

$$U_{t_1}(x,y) = \begin{cases} T_e(x,y) & \text{if } (x,y) \in [0,e]^2, \\ x & \text{if } y \in [0,e] \text{ and } x \in L \setminus [0,e], \\ y & \text{if } x \in [0,e] \text{ and } y \in L \setminus [0,e], \\ 1 & \text{otherwise,} \end{cases}$$

$$U_{s_1}(x,y) = \begin{cases} S_e(x,y) & \text{if } (x,y) \in [e,1]^2, \\ x & \text{if } y \in [e,1] \text{ and } x \in L \setminus [e,1], \\ y & \text{if } x \in [e,1] \text{ and } y \in L \setminus [e,1], \\ 0 & \text{otherwise.} \end{cases}$$

If we put $cl(x) = x \vee 0 = x$ ($int(x) = x \wedge 1 = x$, resp.) in Theorem 2.5, then we retrieve the corresponding uninorms constructed by Çaylı, Karaçal and Mesiar (see Theorem 1 of [3]).

$$U_{t_2}(x,y) = \begin{cases} T_e(x,y) & \text{if } (x,y) \in [0,e]^2, \\ y & \text{if } (x,y) \in [0,e] \times I_e, \\ x & \text{if } (x,y) \in I_e \times [0,e], \\ x \vee y & \text{otherwise,} \end{cases}$$

$$U_{s_2}(x,y) = \begin{cases} S_e(x,y) & \text{if } (x,y) \in [e,1]^2, \\ y & \text{if } (x,y) \in [e,1] \times I_e \\ x & \text{if } (x,y) \in I_e \times [e,1], \\ x \wedge y & \text{otherwise.} \end{cases}$$

Note that if we take $T_e = T_\wedge$ ($S_e = S_\vee$, resp.) in U_{t_2} (U_{s_2}, resp.) then we construct an idempotent uninorm U, i. e., $U(x,x) = x$ for all $x \in L$.

It should be stressed that [9] even provided a more effective method than [11]. There exist some other constructions of uninorms in the literature, however, cannot be derived from [9, 11]. For example, the following theorem is such a case.

Theorem 2.6. (cCaylı and Karaçal [2]) Let $(L, \leq, 0, 1)$ be a bounded lattice with $e \in L \setminus \{0, 1\}$, T_e a t-norm on $[0, e]^2$ and S_e a t-conorm on $[e, 1]^2$.

(1) Suppose that either $x \vee y > e$ for all $x, y \in I_e$ or $x \vee y \in I_e$ for all $x, y \in I_e$, then the function $U_{t_3} : L \times L \to L$ is a uninorm on L with the neutral element e, where

$$U_{t_3}(x,y) = \begin{cases} T_e(x,y) & \text{if } x, y \in [0,e]^2, \\ x \vee y & \text{if } (x,y) \in [0,e] \times (e,1] \cup (e,1] \times [0,e] \cup I_e \times I_e, \\ x & \text{if } (x,y) \in I_e \times [0,e], \\ y & \text{if } (x,y) \in [0,e] \times I_e, \\ 1 & \text{otherwise.} \end{cases}$$

(2) Suppose that either $x \wedge y < e$ for all $x, y \in I_e$ or $x \wedge y \in I_e$ for all $x, y \in I_e$,, then the function $U_{s_3} : L \times L \to L$ is a uninorm on L with the neutral element e, where

$$U_{s_3}(x,y) = \begin{cases} S_e(x,y) & \text{if } (x,y) \in [e,1]^2, \\ x \wedge y & \text{if } (x,y) \in [0,e) \times [e,1] \cup [e,1] \times [0,e) \cup I_e \times I_e, \\ x & \text{if } (x,y) \in I_e \times [e,1], \\ y & \text{if } (x,y) \in [e,1] \times I_e, \\ 0 & \text{otherwise.} \end{cases}$$

In next section, we will show that the condition "$x \vee y > e$ ($x \wedge y < e$, resp.) for all $x, y \in I_e$" in Theorem 2.6 is superfluous and then improve Theorem 2.6 in Section 4.

3. ANALYSIS ON THEOREM 3.1 OF [2]

Let us focus on the condition "$x \vee y > e$ ($x \wedge y < e$, resp.) for all $x, y \in I_e$" in Theorem 2.6 (see also Theorems 3.1 and 3.5 of [2]). Since for any $x \in I_e$ it always holds $x \vee x = x \in I_e$. We can understand this condition from two different points of view. One is that $x \vee y > e$ ($x \wedge y < e$, resp.) for all $x, y \in I_e$ with $x \neq y$ and the other is that $I_e = \emptyset$.

If the condition reads as the former way then Theorem 2.6 is not correct. This can be seen from the following example.

Example 3.1. Let L be the lattice given by Figure 1. It is easy to see that $I_e = \{b, c\}$ and, $b \vee c = d > e$ and $b \wedge c = a < e$. Thus all conditions in Theorem 2.6 posed on the lattice L are satisfied. Let $T_e \colon [0, e]^2 \to [0, e]$ be $T_\wedge$ and $S_e \colon [e, 1]^2 \to [e, 1]$ be $S_\vee$. Then the function $U_{t_3} \colon L \times L \to L$ ($U_{s_3} \colon L \times L \to L$, resp.) is given by Table 1 (Table 2, resp.). Clearly, U_{t_3} is commutative, increasing in each place and satisfies $U_{t_3}(x, e) = x$ for all $x \in L$. But U_{t_3} is not associative. In fact, $U_{t_3}(U_{t_3}(b, c), c) = U_{t_3}(d, c) = 1$, but $U_{t_3}(b, U_{t_3}(c, c)) = U_{t_3}(b, c) = d$. Thus

$$U_{t_3}(U_{t_3}(b,c),c) \neq U_{t_3}(b, U_{t_3}(c,c)).$$

Similarly, U_{s_3} is commutative, increasing in each place and satisfies $U_{s_3}(x, e) = x$ for all $x \in L$. But U_{s_3} is also not associative. In fact, $U_{s_3}(U_{s_3}(b, c), c) = U_{s_3}(a, c) = 0$, but $U_{s_3}(b, U_{s_3}(c, c)) = U_{s_3}(b, c) = a$. Thus

$$U_{s_3}(U_{s_3}(b,c),c) \neq U_{s_3}(b, U_{s_3}(c,c)).$$

So, the function U_{t_3} (U_{s_3}, resp.) constructed via Theorem 2.6 is not a uninorm.

U_{t_3}	0	a	e	d	1	b	c
0	0	0	0	d	1	b	c
a	0	a	a	d	1	b	c
e	0	a	e	d	1	b	c
d	d	d	d	1	1	1	1
1	1	1	1	1	1	1	1
b	b	b	b	1	1	b	d
c	c	c	c	1	1	d	c

Tab. 1. The function U_{t_3} constructed via Theorem 2.6.

U_{s_3}	0	a	e	d	1	b	c
0	0	0	0	0	0	0	0
a	0	0	a	a	a	0	0
e	0	a	e	d	1	b	c
d	0	a	d	d	1	b	c
1	0	a	1	1	1	b	c
b	0	0	b	b	b	b	a
c	0	0	c	c	c	a	c

Tab. 2. The function U_{s_3} constructed via Theorem 2.6.

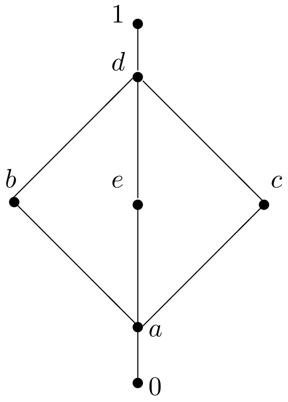

Fig. 1. Hasse diagram of the lattice L in Example 3.1.

So, to ensure the validity of Theorem 2.6, then condition "$x \vee y > e$ ($x \wedge y < e$, resp.) for all $x, y \in I_e$" can only be treated as the latter way, i. e., $I_e = \emptyset$. Note also that the condition "$x \vee y \in I_e$ ($x \wedge y \in I_e$, resp.) for all $x, y \in I_e$ is equivalent to that $(I_e, \vee)$ ($(I_e, \wedge)$, resp.) is closed. Moreover, if $I_e = \emptyset$ then $(I_e, \vee)$ as well as $(I_e, \wedge)$ is obviously closed. Thus, the condition "$x \vee y > e$ ($x \wedge y < e$, resp.) for all $x, y \in I_e$" in Theorem 2.6 is superfluous.

4. IMPROVEMENT OF THEOREM 2.6

The following theorem gives a generalized form of Theorem 2.6.

Theorem 4.1. Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$ and T_e be a t-norm on $[0, e]$. Then the function $U_{cl}^e : L \times L \to L$ defined by

$$U_{cl}^e(x,y) = \begin{cases} T_e(x,y) & \text{if } (x,y) \in [0,e]^2, \\ x \vee y & \text{if } (x,y) \in I_e \times I_e, \\ x & \text{if } \; y \in [0,e] \; \text{ and } \; x \in L \setminus [0,e], \\ y & \text{if } \; x \in [0,e] \; \text{ and } \; y \in L \setminus [0,e], \\ cl(x) \vee cl(y) & \text{otherwise,} \end{cases} \tag{1}$$

is a uninorm on L with the neutral element e for every closure operator $cl \colon L \to L$ if and only if for all $x, y \in I_e$, either $x \vee y = 1$ or $x \vee y \in I_e$ holds.

The following observation is useful to simplify our proof.

Observation. $U_{cl}^e(x, y)$ in Theorem 4.1 is the same as $U_{cl}(x, y)$ in Theorem 2.5 in the region $L^2 \setminus I_e^2$.

Proof. *Necessity.* Notice that for any $x, y \in I_e$ either $x \vee y \in I_e$ or $x \vee y > e$ holds. If there are $x, y \in I_e$ such that $x \vee y \in (e, 1)$ then for the closure operator defined by $cl(x) = 1, \forall\, x \in L$, we have

$$U_{cl}^e(U_{cl}^e(x,y), y) = U_{cl}^e(x \vee y, y) = cl(x \vee y) \vee cl(y) = cl(x) \vee cl(y) = 1$$

but

$$U_{cl}^e(x, U_{cl}^e(y,y)) = U_{cl}^e(x,y) = x \vee y \in (e, 1).$$

Hence U_{cl}^e is not associative and thus is not a uninorm. To ensure the associativity of U_{cl}^e for all closure operators, it must hold that $\forall\, x, y \in I_e$, either $x \vee y = 1$ or $x \vee y \in I_e$.

Sufficiency. It is obvious that U_{cl}^e is commutative and $U_{cl}^e(e, x) = x$ for all $x \in L$. We need only to verify the increasingness and associativity of U_{cl}^e.

Increasingness. Let $x, y, z \in L$ with $y < z$. We need to verify the inequality $U_{cl}^e(x, y) \leq U_{cl}^e(x, z)$. By Theorem 2.5 and the above observation, it is sufficient to consider the case that $x \in I_e$ and at least one of y, z is in I_e. We split the proof into two possible cases.

Case 1. $y \in [0, e]$ and $z \in I_e$.

$$U^e_{cl}(x, y) = x \leq x \vee z = U^e_{cl}(x, z).$$

Case 2. $y \in I_e$, then we have either $z \in I_e$ or $z \in (e, 1]$.

If $z \in I_e$ then $U^e_{cl}(x, y) = x \vee y \leq x \vee z = U^e_{cl}(x, z)$.

If $z \in (e, 1]$ then $U^e_{cl}(x, y) = x \vee y \leq x \vee z \leq cl(x) \vee cl(z) = U^e_{cl}(x, z)$.

Associativity. Let $x, y, z \in L$ be given. We need to verify the equality $U^e_{cl}(U^e_{cl}(x, y), z) = U^e_{cl}(x, U^e_{cl}(y, z))$. Note that $U^e_{cl}(x, y) \notin I_e$ whenever $x, y \notin I_e$. So, by Theorem 2.5 and the above observation, we need only to consider the case that there are at least two elements of x, y, z being in I_e. The proof is again split into four possible cases.

Case 1. $x, y \in I_e$ but $z \notin I_e$.

For $z \in [0, e]$ we have

$$U^e_{cl}(U^e_{cl}(x, y), z) = U^e_{cl}(x, y) = U^e_{cl}(x, U^e_{cl}(y, z)).$$

For $z \in (e, 1]$, if $x \vee y = 1$ then

$$\begin{aligned} U^e_{cl}(U^e_{cl}(x, y), z) &= U^e_{cl}(x \vee y, z) = U^e_{cl}(1, z) = 1 = cl(x) \vee cl(y) \\ &= cl(x) \vee cl(y) \vee cl(z) = U^e_{cl}(x, U^e_{cl}(y, z)). \end{aligned}$$

Otherwise, we have $x \vee y \in I_e$, thus

$$\begin{aligned} U^e_{cl}(U^e_{cl}(x, y), z) &= U^e_{cl}(x \vee y, z) = cl(x \vee y) \vee cl(z) \\ &= cl(x) \vee cl(y) \vee cl(z) = cl(x) \vee cl(cl(y) \vee cl(z)) \\ &= U^e_{cl}(x, cl(y) \vee cl(z)) = U^e_{cl}(x, U^e_{cl}(y, z)). \end{aligned}$$

Case 2. $y, z \in I_e$ but $x \notin I_e$. By the commutativity of U^e_{cl}, it is the same as Case 1.

Case 3. $x, z \in I_e$ but $y \notin I_e$.

If $y \in [0, e]$ then

$$U^e_{cl}(U^e_{cl}(x, y), z) = U^e_{cl}(x, z) = U^e_{cl}(x, U^e_{cl}(y, z)).$$

If $y \in (e, 1]$ then

$$\begin{aligned} U^e_{cl}(U^e_{cl}(x, y), z) &= U^e_{cl}(cl(x) \vee cl(y), z) = cl(cl(x) \vee cl(y)) \vee cl(z) \\ &= cl(x) \vee cl(y) \vee cl(z) = cl(x) \vee cl(cl(y) \vee cl(z)) \\ &= U^e_{cl}(x, cl(y) \vee cl(z)) = U^e_{cl}(x, U^e_{cl}(y, z)). \end{aligned}$$

Case 4. $x, y, z \in I_e$.

If $x \vee y = 1$ then, together with the monotonicity of U^e_{cl}, we have

$$\begin{aligned} U^e_{cl}(U^e_{cl}(x,y),z) &= U^e_{cl}(x \vee y, z) = U^e_{cl}(1,z) = 1 = U^e_{cl}(x,y) \\ &\leq U^e_{cl}(x, y \vee z) = U^e_{cl}(x, U^e_{cl}(y,z)), \end{aligned}$$

i. e., $U^e_{cl}(U^e_{cl}(x,y),z) = U^e_{cl}(x, U^e_{cl}(y,z)) = 1$.

If $y \vee z = 1$, we also have $U^e_{cl}(U^e_{cl}(x,y),z) = U^e_{cl}(x, U^e_{cl}(y,z)) = 1$.

Otherwise, we conclude that both $x \vee y \in I_e$ and $y \vee z \in I_e$ hold. Thus

$$U^e_{cl}(U^e_{cl}(x,y),z) = U^e_{cl}(x \vee y, z) = x \vee y \vee z = U^e_{cl}(x, y \vee z) = U^e_{cl}(x, U^e_{cl}(y,z)).$$

Now we have proved that U^e_{cl} is a commutative, increasing (in each place) and associative binary operation on L with neutral element e, i. e., U^e_{cl} is a uninorm on L. □

Note 4.2. We point out that the sufficiency can also be proved by using Corollary 4.6 in [9]. To avoid introducing the notations and terminology of [9], we give a direct proof here.

If we take the closure operator cl as $cl(x) = x \vee 1 = 1$ for all $x \in L$ then we get the following corollary, which is an improvement of Theorem 2.6(1).

Corollary 4.3. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0,1\}$ and T_e a t-norm on $[0,e]^2$. If for all $x, y \in I_e$, either $x \vee y = 1$ or $x \vee y \in I_e$ holds then the function $U^e_{cl} : L \times L \to L$ defined by*

$$U^e_{cl}(x,y) = \begin{cases} T_e(x,y) & \text{if } (x,y) \in [0,e]^2, \\ x \vee y & \text{if } (x,y) \in I_e \times I_e, \\ x & \text{if } \ y \in [0,e] \ \text{ and } \ x \in L \setminus [0,e], \\ y & \text{if } \ x \in [0,e] \ \text{ and } \ y \in L \setminus [0,e], \\ 1 & \text{otherwise,} \end{cases} \tag{2}$$

is a uninorm on L with the neutral element e.

Notice that the condition *for all $x, y \in I_e$, either $x \vee y = 1$ or $x \vee y \in I_e$ holds* ensures $U^e_{cl}(x,y)$ is a uninorm for all closure operators cl and for all bounded lattices L. For some special closure operators, $cl(x) = x, x \in L$ for example, or for some special lattices, $x \vee y = 1, \forall\ x, y \in I_e$ for example, this condition is surplus. Note also that if $cl(x) = x$ for all $x \in L$ then U^e_{cl} is exactly U_{t_2}.

Theorem 4.4. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0,1\}$ and S_e a t-conorm on $[e,1]^2$. Then the function $U^e_{int} : L \times L \to L$ defined by*

$$U^e_{int}(x,y) = \begin{cases} S_e(x,y) & \text{if } (x,y) \in [e,1]^2, \\ x \wedge y & \text{if } (x,y) \in I_e \times I_e, \\ x & \text{if } \ y \in [e,1] \ \textit{ and } \ x \in L \setminus [e,1], \\ y & \text{if } \ x \in [e,1] \ \textit{ and } \ y \in L \setminus [e,1], \\ int(x) \wedge int(y) & \text{otherwise,} \end{cases} \tag{3}$$

is a uninorm on L with the neutral element e for every interior operator $int\colon L \to L$ if and only if for all $x, y \in I_e$, either $x \wedge y = 0$ or $x \wedge y \in I_e$ holds.

Proof. It is similar to that of Theorem 4.1. □

If we take the interior operator int as $int(x) = x \wedge 0 = 0$ for all $x \in L$ then we get the following corollary, which is an improvement of Theorem 2.6(2).

Corollary 4.5. Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \setminus \{0, 1\}$ and S_e a t-conorm on $[e, 1]$. If for all $x, y \in I_e$, either $x \wedge y = 0$ or $x \wedge y \in I_e$ holds then the function $U^e_{int} : L \times L \to L$ defined by

$$U^e_{int}(x, y) = \begin{cases} S_e(x, y) & \text{if } (x, y) \in [e, 1]^2, \\ x \wedge y & \text{if } (x, y) \in I_e \times I_e, \\ x & \text{if } \ y \in [e, 1] \ \ \textit{and} \ \ x \in L \setminus [e, 1], \\ y & \text{if } \ x \in [e, 1] \ \ \textit{and} \ \ y \in L \setminus [e, 1], \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

is a uninorm on L with the neutral element e.

Again the condition *for all* $x, y \in I_e$, *either* $x \wedge y = 0$ *or* $x \wedge y \in I_e$ *holds* ensures $U^e_{int}(x, y)$ is a uninorm for all interior operators int and for all bounded lattices L. For some special interior operators and some special lattices, this condition can be omitted. For the interior operator int defined by $int(x) = x, x \in L$, U^e_{int} is exactly U_{s_2}.

Note that if the set I_e has a maximal element and a minimal element, then for all $x, y \in I_e$ we have $x \vee y \in I_e$ and $x \wedge y \in I_e$. Thus, for such lattices L, U^e_{cl} and U^e_{int} can be introduced.

Finally, we stress that Theorem 4.1 as well as Theorem 4.4 cannot be derived from Theorem 2.5 in general.

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