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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF SECOND-ORDER DIFFERENTIAL EQUATIONS WITH NONPOSITIVE NEUTRAL COEFFICIENTS

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Abstract. In this work, we present necessary and sufficient conditions for oscillation of all solutions of a second-order functional differential equation of type

$$(r(t)(z'(t))^{\gamma})' + \sum_{i=1}^{m} q_i(t)x^{\alpha_i}(\sigma_i(t)) = 0, \quad t \geqslant t_0,$$

where $z(t) = x(t) + p(t)x(\tau(t))$. Under the assumption $\int^{\infty}(r(\eta))^{-1/\gamma}\,\mathrm{d}\eta = \infty$, we consider two cases when $\gamma > \alpha_i$ and $\gamma < \alpha_i$. Our main tool is Lebesgue's dominated convergence theorem. Finally, we provide examples illustrating our results and state an open problem.

Keywords: oscillation; non-oscillation; neutral; delay; Lebesgue's dominated convergence theorem

MSC 2020: 34C10, 34K11

1. Introduction

In this article we consider the neutral differential equation

$$(1.1)\quad (r(t)(z'(t))^{\gamma})' + \sum_{i=1}^{m} q_i(t)x^{\alpha_i}(\sigma_i(t)) = 0, \quad z(t) = x(t) + p(t)x(\tau(t)), \quad t \geqslant t_0,$$

where γ and α_i are the quotients of odd positive integers, and the functions p, q_i, r, σ_i, τ are continuous such that

(A1) $\sigma_i \in C([0,\infty),\mathbb{R}_+)$, $\tau \in C^2([0,\infty),\mathbb{R}_+)$, $\sigma_i(t) < t$, $\tau(t) < t$, $\lim\limits_{t\to\infty}\sigma_i(t) = \infty$, $\lim\limits_{t\to\infty}\tau(t) = \infty$;

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(A2) $r \in C^1([0,\infty),\mathbb{R}_+)$, $q_i \in C([0,\infty),\mathbb{R}_+)$; $0 < r(t)$, $0 \leqslant q_i(t)$ for all $t \geqslant 0$ and $i = 1,2,\dots,m$; $\sum q_i(t)$ is not identically zero in any interval $[b,\infty)$;
(A3) $\int_0^\infty r^{-1/\gamma}(s)\,\mathrm{d}s = \infty$, $\Pi(t) = \int_0^t r^{-1/\gamma}(\eta)\,\mathrm{d}\eta$;
(A4) $-1 < -p_0 \leqslant p(t) \leqslant 0$ for $t \geqslant t_0$;
(A5) there exists a differentiable function $\sigma_0(t)$ satisfying the properties $0 < \sigma_0(t) = \min\{\sigma_i(t)\colon\ t \geqslant t^* > t_0\}$ and $\sigma_0'(t) \geqslant \alpha$ for $t \geqslant t^* > t_0$, $\alpha > 0$, $i = 1,2,\dots,m$.

In 1978, Brands has proved that for bounded delays, the solutions of

$$x''(t) + q(t)x(t-\sigma(t)) = 0$$

are oscillatory if and only if the solutions of $x''(t) + q(t)x(t) = 0$ are oscillatory (see [9]). In [10], [12] Chatzarakis et al. have considered a more general second-order half-linear differential equation of the form

$$(r(x')^\alpha)'(t) + q(t)x^\alpha(\sigma(t)) = 0, \tag{1.2}$$

and established new oscillation criteria for (1.2) when

$$\lim_{t\to\infty} \Pi(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} \Pi(t) < \infty.$$

Wong in [29] has obtained the necessary and sufficient conditions for oscillation of solutions of

$$(x(t) + px(t-\tau))'' + q(t)f(x(t-\sigma)) = 0, \quad -1 < p < 0,$$

in which the neutral coefficient and delays are constants. However, we have seen in [5], [13] that the authors Baculíková and Džurina have studied

$$(r(t)(z'(t))^\gamma)' + q(t)x^\alpha(\sigma(t)) = 0, \quad z(t) = x(t) + p(t)x(\tau(t)), \quad t \geqslant t_0, \tag{1.3}$$

and established sufficient conditions for oscillation of solutions of (1.3) using comparison techniques when $\gamma = \alpha = 1$, $0 \leqslant p(t) < \infty$ and $\lim\limits_{t\to\infty} \Pi(t) = \infty$. In the same technique, Baculíková and Džurina (see [6]) obtained sufficient conditions for oscillation of the solutions of (1.3) by considering the assumptions $0 \leqslant p(t) < \infty$ and $\lim\limits_{t\to\infty} \Pi(t) = \infty$. In [28], Tripathy et al. have studied (1.3) and established several sufficient conditions for oscillations of the solutions of (1.3) by considering the assumptions $\lim\limits_{t\to\infty} \Pi(t) = \infty$ and $\lim\limits_{t\to\infty} \Pi(t) < \infty$ for different ranges of the neutral coefficient p. In [8], Bohner et al. have obtained sufficient conditions for oscillation of solutions of (1.3) when $\gamma = \alpha$, $\lim\limits_{t\to\infty} \Pi(t) < \infty$ and $0 \leqslant p(t) < 1$. Grace et al. in [15] have established sufficient conditions for the oscillation of the solutions of (1.3) when $\gamma = \alpha$ and by considering the assumptions $\lim_{t\to\infty} \Pi(t) < \infty$, $\lim_{t\to\infty} \Pi(t) = \infty$ and $0 \leqslant p(t) < 1$. In [18], Li et al. have established sufficient conditions for oscillation of the solutions of (1.3), under the assumptions $\lim_{t\to\infty} \Pi(t) < \infty$ and $p(t) \geqslant 0$. Karpuz and Santra in [17] have obtained several sufficient conditions for the oscillatory and asymptotic behavior of the solutions of

$$(r(t)(x(t) + p(t)x(\tau(t)))')' + q(t)f(x(\sigma(t))) = 0,$$

by considering the assumptions $\lim_{t\to\infty} \Pi(t) < \infty$ and $\lim_{t\to\infty} \Pi(t) = \infty$, for different ranges of p.

For more information on oscillation of second order neutral differential equations, we refer the reader to [1]–[4], [7], [11], [14], [15], [19]–[27], [30] and the references cited therein. Note that most of the works have considered sufficient conditions, and merely a few works deals with the necessary and sufficient conditions. Hence, unlike the above methods, the main purpose of this article is to establish conditions that are both necessary and sufficient for oscillation of all solutions of (1.1).

Neutral differential equations have several applications in the natural sciences and engineering. For example, they often appear in the study of distributed networks containing lossless transmission lines (see, e.g., [16]). In this paper, we restrict our attention to studying oscillation and non-oscillation of (1.1), which includes the class of functional differential equations of neutral type.

By a solution to equation (1.1) we mean a function $x \in \mathrm{C}([T_x, \infty), \mathbb{R})$, where $T_x \geqslant t_0$, such that $rz' \in \mathrm{C}^1([T_x, \infty), \mathbb{R})$ and satisfies (1.1) on the interval $[T_x, \infty)$. A solution x of (1.1) is said to be *proper* if x is not identically zero eventually, i.e. $\sup\{|x(t)|\colon\ t \geqslant T\} > 0$ for all $T \geqslant T_x$. We assume that (1.1) possesses such solutions. A *solution* of (1.1) is called *oscillatory* if it has arbitrarily large zeros on $[T_x, \infty)$; otherwise, it is said to be *non-oscillatory. Equation* (1.1) itself is said to be *oscillatory* if all of its solutions are oscillatory.

R e m a r k 1.1. When the domain is not specified explicitly, all functional inequalities considered in this paper are assumed to hold eventually, i.e. they are satisfied for all t large enough.

2. Main results

Lemma 2.1. *Assume that* (A1)–(A4) *hold for* $t \geqslant t_0$. *If* x *is an eventually positive solution of* (1.1), *then* z *satisfies one of the following two cases:*

(i) $z(t) < 0$, $z'(t) > 0$, $(r(z')^\gamma)'(t) \leqslant 0$;

(ii) $z(t) > 0$, $z'(t) > 0$, $(r(z')^\gamma)'(t) \leqslant 0$

for $t \geqslant t_1$.

Proof. Let x be an eventually positive solution. Hence, there exists a $t_0 \geqslant 0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma_i(t)) > 0$ for all $t \geqslant t_0$ and $i = 1, 2, \ldots, m$. From (1.1) it follows that

$$(r(t)(z'(t))^\gamma)' = -\sum_{i=1}^{m} q_i(t) x^{\alpha_i}(\sigma_i(t)) \leqslant 0 \quad \text{for } t \geqslant t_0. \tag{2.1}$$

Therefore, $r(t)(z'(t))^\gamma$ is non-increasing for $t \geqslant t_0$. Assume that $r(t)(z'(t))^\gamma < 0$ for $t \geqslant t_1 > t_0$. Hence,

$$r(t)(z'(t))^\gamma \leqslant r(t_1)(z'(t_1))^\gamma < 0 \quad \text{for all } t \geqslant t_1,$$

that is,

$$z'(t) \leqslant \Big(\frac{r(t_1)}{r(t)}\Big)^{1/\gamma} z'(t_1) \quad \text{for } t \geqslant t_1.$$

Using integration from t_1 to t, we have

$$z(t) \leqslant z(t_1) + (r(t_1))^{1/\gamma} z'(t_1)(\Pi(t) - \Pi(t_1)) \to -\infty \tag{2.2}$$

as $t \to \infty$ due to (A3). Now, we consider the two possibilities, namely, x is bounded and x is unbounded.

If x is unbounded, then there exists a sequence $\{\eta_k\} \to \infty$ as $k \to \infty$ and $x(\eta_k) = \sup\{x(\eta)\colon \eta \leqslant \eta_k\}$. By $\tau(\eta_k) \leqslant \eta_k$, we have $x(\tau(\eta_k)) \leqslant x(\eta_k)$ and hence

$$z(\eta_k) = x(\eta_k) + p(\eta_k)x(\tau(\eta_k)) \geqslant (1 + p(\eta_k))x(\eta_k) \geqslant (1 - p_0)x(\eta_k) \geqslant 0$$

contradicts the fact that $\lim\limits_{k\to\infty} z(\eta_k) = -\infty$. Ultimately, x is bounded. Then z is also bounded, which is a contradiction.

Therefore $r(t)(z'(t))^\gamma > 0$ for all $t \geqslant t_1$. From $r(t)(z'(t))^\gamma > 0$ and $r(t) > 0$, it follows that $z'(t) > 0$. Then z satisfies only one of the two cases (i) and (ii) for all $t \geqslant t_1$. This completes the proof. $\square$

Lemma 2.2. *Assume that* (A1)–(A4) *hold. If x is an eventually positive solution of* (1.1), *then any one of the following two cases holds:*

(1) *if z satisfies* (i), *then* $\lim_{t\to\infty} x(t) = 0$*;*

(2) *if z satisfies* (ii), *then there exist $t_1 > t_0$ and $\delta > 0$ such that*

$$0 < z(t) \leqslant \delta\Pi(t), \tag{2.3}$$

$$(\Pi(t) - \Pi(t_1))\biggl(\int_t^\infty \sum_{i=1}^m q_i(\zeta)x^{\alpha_i}(\sigma_i(\zeta))\,\mathrm{d}\zeta\biggr)^{1/\gamma} \leqslant z(t) \leqslant x(t) \tag{2.4}$$

hold for all $t \geqslant t_1$.

Proof. Let x be an eventually positive solution of (1.1). Then there exists a $t_0 > 0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma_i(t)) > 0$ for all $t \geqslant t_0$ and $i = 1, 2, \ldots, m$. Applying Lemma 2.1 for $t \geqslant t_1 > t_0$ we have the following two cases:

Case 1: Let z satisfy (i) for all $t \geqslant t_1$. Note that $\lim_{t\to\infty} z(t)$ exists. As $0 > z(t) \geqslant x(t) - p_0 x(\tau(t))$, then

$$0 \geqslant \lim_{t\to\infty} z(t) \geqslant \lim_{t\to\infty} (x(t) - p_0 x(\tau(t))) \geqslant (1 - p_0) \limsup_{t\to\infty} x(t)$$

implies that $\limsup_{t\to\infty} x(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$.

Case 2: Let z satisfy (ii) for all $t \geqslant t_1$. In this case, $x(t) \geqslant z(t) > 0$ and z is increasing. From $r(t)(z'(t))^\gamma > 0$ and being non-increasing, we have

$$z'(t) \leqslant \Bigl(\frac{r(t_1)}{r(t)}\Bigr)^{1/\gamma} z'(t_1) \quad \text{for } t \geqslant t_1.$$

Integrating this inequality from t_1 to t,

$$z(t) \leqslant z(t_1) + (r(t_1))^{1/\gamma} z'(t_1)(\Pi(t) - \Pi(t_1)).$$

Since $\lim_{t\to\infty} \Pi(t) = \infty$, there exists a positive constant δ such that (2.3) holds. On the other hand, $\lim_{t\to\infty} r(t)(z'(t))^\gamma$ exists and integrating (1.1) from t to a, we obtain

$$r(a)(z'(a))^\gamma - r(t)(z'(t))^\gamma = -\int_t^a \sum_{i=1}^m q_i(\eta)x^{\alpha_i}(\sigma_i(\eta))\,\mathrm{d}\eta.$$

Taking limit as $a \to \infty$,

$$r(t)(z'(t))^\gamma \geqslant \int_t^\infty \sum_{i=1}^m q_i(\eta)x^{\alpha_i}(\sigma_i(\eta))\,\mathrm{d}\eta, \tag{2.5}$$

that is,

$$z'(t) \geqslant \left(\frac{1}{r(t)} \int_t^\infty \sum_{i=1}^m q_i(\eta) x^{\alpha_i}(\sigma_i(\eta))\,\mathrm{d}\eta\right)^{1/\gamma}.$$

Therefore

$$\begin{aligned} z(t) &\geqslant \int_{t_1}^t \left(\frac{1}{r(\eta)} \int_\eta^\infty \sum_{i=1}^m q_i(\zeta) x^{\alpha_i}(\sigma_i(\zeta))\,\mathrm{d}\zeta\right)^{1/\gamma} \mathrm{d}\eta \\ &\geqslant \int_{t_1}^t \left(\frac{1}{r(\eta)} \int_t^\infty \sum_{i=1}^m q_i(\zeta) x^{\alpha_i}(\sigma_i(\zeta))\,\mathrm{d}\zeta\right)^{1/\gamma} \mathrm{d}\eta \\ &= (\Pi(t) - \Pi(t_1))\left(\int_t^\infty \sum_{i=1}^m q_i(\zeta) x^{\alpha_i}(\sigma_i(\zeta))\,\mathrm{d}\zeta\right)^{1/\gamma}. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.3. *Assume that* (A1)–(A4) *hold. If x is an eventually positive unbounded solution of* (1.1), *then z satisfies* (ii) *only.*

Theorem 2.1. *Assume that there exists a constant β_1, the quotient of odd positive integers such that $0 < \alpha_i < \beta_1 < \gamma$. If* (A1)–(A4) *hold, then every solution of* (1.1) *either oscillates or converges to zero as $t \to \infty$ if and only if*

$$\int_0^\infty \sum_{i=1}^m q_i(\eta) \Pi^{\alpha_i}(\sigma_i(\eta))\,\mathrm{d}\eta = \infty. \tag{2.6}$$

Proof. We prove the sufficiency by contradiction. Initially, we assume that a solution x is eventually positive which means it does not converge to zero. So, Lemma 2.1 holds and z satisfies any one of the two cases (i) and (ii). In Lemma 2.2, Case 1 leads to $\lim\limits_{t\to\infty} x(t) = 0$, which is a contradiction.

For Case 2, we can find a $t_1 > 0$ such that

$$x(t) \geqslant z(t) \geqslant (\Pi(t) - \Pi(t_1)) w^{1/\gamma}(t) \geqslant 0 \quad \text{for } t \geqslant t_1,$$

where

$$w(t) = \int_t^\infty \sum_{i=1}^m q_i(\zeta) x^{\alpha_i}(\sigma_i(\zeta))\,\mathrm{d}\zeta \geqslant 0.$$

As $\lim\limits_{t\to\infty} \Pi(t) = \infty$, there exists a $t_2 > t_1$ such that $\Pi(t) - \Pi(t_1) \geqslant \frac{1}{2}\Pi(t)$ for $t \geqslant t_2$ and hence

$$z(t) \geqslant \frac{1}{2}\Pi(t) w^{1/\gamma}(t). \tag{2.7}$$

Using (2.3), $\alpha_i - \beta_1 < 0$ and (2.7), we have

$$\begin{aligned} x^{\alpha_i}(t) &\geqslant z^{\alpha_i-\beta_1}(t) z^{\beta_1}(t) \geqslant (\delta\Pi(t))^{\alpha_i-\beta_1} z^{\beta_1}(t) \\ &\geqslant (\delta\Pi(t))^{\alpha_i-\beta_1}\Big(\frac{\Pi(t)w^{1/\gamma}(t)}{2}\Big)^{\beta_1} = \frac{\delta^{\alpha_i-\beta_1}}{2^{\beta_1}}\Pi^{\alpha_i}(t)w^{\beta_1/\gamma}(t) \quad \text{for } t \geqslant t_2. \end{aligned}$$

Since $w'(t) = -\sum\limits_{i=1}^{m} q_i(t)x^{\alpha_i}(\sigma_i(t)) \leqslant 0$, $t \geqslant t_2$, that is, w is non-increasing, the last inequality becomes

$$x^{\alpha_i}(\sigma_i(\eta)) \geqslant \frac{\delta^{\alpha_i-\beta_1}}{2^{\beta_1}}\Pi^{\alpha_i}(\sigma_i(\eta))w^{\beta_1/\gamma}(\sigma_i(\eta)) \geqslant \frac{\delta^{\alpha_i-\beta_1}}{2^{\beta_1}}\Pi^{\alpha_i}(\sigma_i(\eta))w^{\beta_1/\gamma}(\eta).$$

Therefore

$$(w^{1-\beta_1/\gamma}(t))' = \Big(1 - \frac{\beta_1}{\gamma}\Big)w^{-\beta_1/\gamma}(t)w'(t). \tag{2.8}$$

Integrating (2.8) from t_2 to t and then using the fact that $w > 0$, we find

$$\begin{aligned} \infty > w^{1-\beta_1/\gamma}(t_2) &\geqslant \Big(1 - \frac{\beta_1}{\gamma}\Big)\int_{t_2}^{t} -w^{-\beta_1/\gamma}(\eta)w'(\eta)\,\mathrm{d}\eta \\ &= \Big(1 - \frac{\beta_1}{\gamma}\Big)\int_{t_2}^{t} w^{-\beta_1/\gamma}(\eta)\Big(\sum_{i=1}^{m} q_i(\eta)x^{\alpha_i}(\sigma_i(\eta))\Big)\,\mathrm{d}\eta \\ &\geqslant \frac{1}{2^{\beta_1}\delta^{(\beta_1-\alpha_i)}}\Big(1 - \frac{\beta_1}{\gamma}\Big)\int_{t_2}^{t}\sum_{i=1}^{m} q_i(\eta)\Pi^{\alpha_i}(\sigma_i(\eta))\,\mathrm{d}\eta, \end{aligned}$$

which contradicts (2.6) as $t \to \infty$ and completes the proof of sufficiency for eventually positive solutions. For an eventually negative solution x, we introduce the variables $y = -x$ so that we can apply the above process for the solution y.

Next we show the necessity part by a contrapositive argument. Let (2.6) do not hold. Then it is possible to find a $t_1 > 0$ such that

$$\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\Pi^{\alpha_i}(\sigma_i(\zeta))\,\mathrm{d}\zeta \leqslant \varepsilon\delta^{-\alpha_i} \tag{2.9}$$

for all $\eta \geqslant t_1$ and $\delta, \varepsilon > 0$ satisfying the relation

$$(2\varepsilon)^{1/\gamma} = (1 - p_0)\delta, \tag{2.10}$$

so that $0 < \varepsilon^{1/\gamma} = (1 - p_0)\delta/2^{1/\gamma} < \delta$. Define the set of continuous functions

$$M = \{x \in C([0,\infty))\colon\ \varepsilon^{1/\gamma}(\Pi(t) - \Pi(t_1)) \leqslant x(t) \leqslant \delta(\Pi(t) - \Pi(t_1)),\ t \geqslant t_1\}$$

and define the operator Φ on M by

$$
\begin{aligned}
(\Phi x)(t) =& 0 && \text{if } t \leqslant t_1,\\
(\Phi x)(t) =& -p(t)x(\tau(t)) \\
&+ \int_{t_1}^{t} \Big(\frac{1}{r(\eta)}\Big(\varepsilon + \int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta) x^{\alpha_i}(\sigma_i(\zeta))\,\mathrm{d}\zeta\Big)\Big)^{1/\gamma} \,\mathrm{d}\eta && \text{if } t > t_1.
\end{aligned}
$$

We need to show that Φ has a fixed point which is our required solution of (1.1).

First we estimate $(\Phi x)(t)$ from below. For $x \in M$ we have $0 \leqslant \varepsilon^{1/\gamma}\big(\Pi(t)-\Pi(t_1)\big) \leqslant x(t)$, and by (A2) and (A3) we have

$$
(\Phi x)(t) \geqslant 0 + \int_{t_1}^{t} \Big(\frac{1}{r(\eta)}(\varepsilon + 0)\Big)^{1/\gamma} \,\mathrm{d}\eta = \varepsilon^{1/\gamma}(\Pi(t) - \Pi(t_1)).
$$

Now we estimate $(\Phi x)(t)$ from above. For x in M and by the definition of M, we have $x^{\alpha_i}(\sigma_i(\eta)) \leqslant (\delta \Pi(\sigma_i(\eta)))^{\alpha_i}$. Therefore, by (2.9),

$$
\begin{aligned}
(\Phi x)(t) &\leqslant p_0 \delta(\Pi(\tau(t)) - \Pi(t_1)) \\
&\quad + \int_{t_1}^{t} \Big(\frac{1}{r(\eta)}\Big(\varepsilon + \delta^{\alpha_i} \int_{\eta}^{\infty} \sum_{i=1}^{m} q_i(\zeta) \Pi^{\alpha_i}(\sigma_i(\zeta))\,\mathrm{d}\zeta\Big)\Big)^{1/\gamma} \,\mathrm{d}\eta \\
&\leqslant p_0 \delta(\Pi(t) - \Pi(t_1)) + (2\varepsilon)^{1/\gamma}(\Pi(t) - \Pi(t_1)) = \delta(\Pi(t) - \Pi(t_1)).
\end{aligned}
$$

Hence, Φ maps M to M.

To find our fixed point for Φ in M, let us define a sequence of functions in M by the recurrence relation

$$
\begin{aligned}
u_0(t) &= 0 && \text{for } t = 0,\\
u_1(t) &= (\Phi u_0)(t) = \begin{cases} 0 & \text{if } t < t_1,\\ \varepsilon^{1/\gamma}(\Pi(t) - \Pi(t_1)) & \text{if } t \geqslant t_1, \end{cases}\\
u_{n+1}(t) &= (\Phi u_n)(t) && \text{for } n \geqslant 1,\ t \geqslant t_1.
\end{aligned}
$$

Note that for each fixed t we have $u_1(t) \geqslant u_0(t)$. Using mathematical induction, it is easy to show that $u_{n+1}(t) \geqslant u_n(t)$. Therefore, the sequence $\{u_n\}$ converges pointwise to a function u. Using the Lebesgue dominated convergence theorem, we can show that u is a fixed point of Φ in M. This shows under assumption (2.9), that there is a non-oscillatory solution that does not converge to zero. $\square$

Corollary 2.1. *Under the assumptions of Theorem* 2.1, *every unbounded solution of* (1.1) *is oscillatory if and only if* (2.6) *holds.*

P r o o f. The proof of the corollary is an immediate consequence of Theorem 2.1. $\square$

Theorem 2.2. *Assume that there exists a constant β_2, the quotient of odd positive integers such that $\gamma < \beta_2 < \alpha_i$. If* (A1)–(A5) *hold and $r(t)$ is non-decreasing, then every solution of* (1.1) *either oscillates or converges to zero if and only if*

$$\int_0^\infty \left(\frac{1}{r(\eta)} \int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\,\mathrm{d}\zeta\right)^{1/\gamma} \mathrm{d}\eta = \infty. \tag{2.11}$$

P r o o f. We prove the sufficiency by contradiction. Initially, we assume that x is an eventually positive solution not converging to zero. So, Lemma 2.1 holds and z satisfies any one of the two cases (i) and (ii). In Lemma 2.2, Case 1 leads to $\lim_{t\to\infty} x(t) = 0$, which is a contradiction.

For Case 2, $z(t) > 0$ is increasing for $t \geqslant t_1$ and

$$x^{\alpha_i}(t) \geqslant z^{\alpha_i}(t) \geqslant z^{\alpha_i-\beta_2}(t) z^{\beta_2}(t) \geqslant z^{\alpha_i-\beta_2}(t_1) z^{\beta_2}(t)$$

implies that

$$x^{\alpha_i}(\sigma_i(t)) \geqslant z^{\alpha_i-\beta_2}(t_1) z^{\beta_2}(\sigma_i(t)) \quad \text{for } t \geqslant t_2 > t_1. \tag{2.12}$$

Using (2.5), (2.12) and $\sigma_i(t) \geqslant \sigma_0(t)$, we have

$$\begin{aligned} r(t)(z'(t))^\gamma &\geqslant z^{\alpha_i-\beta_2}(t_1)\left(\int_t^\infty \sum_{i=1}^m q_i(\eta)\,\mathrm{d}\eta\right) z^{\beta_2}(\sigma_i(t)) \\ &\geqslant z^{\alpha_i-\beta_2}(t_1)\left(\int_t^\infty \sum_{i=1}^m q_i(\eta)\,\mathrm{d}\eta\right) z^{\beta_2}(\sigma_0(t)) \end{aligned} \tag{2.13}$$

for $t \geqslant t_2$. Being $r(t)(z'(t))^\gamma$ non-increasing and $\sigma_0(t) \leqslant t$, we have

$$r(\sigma_0(t))(z'(\sigma_0(t)))^\gamma \geqslant r(t)(z'(t))^\gamma.$$

Using the last inequality in (2.13) and then dividing by $z^{\beta_2}(\sigma_0(t)) > 0$, and then operating the power $1/\gamma$ on both sides, we get

$$\frac{z'(\sigma_0(t))}{z^{\beta_2/\gamma}(\sigma_0(t))} \geqslant \left(\frac{z^{\alpha_i-\beta_2}(t_1)}{r(\sigma_0(t))} \int_t^\infty \sum_{i=1}^m q_i(\eta)\,\mathrm{d}\eta\right)^{1/\gamma}$$

for $t \geqslant t_2$. Multiplying the left-hand side by $\sigma_0'(t)/\alpha \geqslant 1$ and integrating from t_2 to t, we find

$$\begin{aligned} &\frac{1}{\alpha}\int_{t_2}^t \frac{z'(\sigma_0(\eta))\sigma_0'(\eta)}{z^{\beta_2/\gamma}(\sigma_0(\eta))}\,\mathrm{d}\eta \\ &\qquad \geqslant z^{(\alpha_i-\beta_2)/\gamma}(t_1) \int_{t_2}^t \left(\frac{1}{r(\sigma_0(\eta))} \int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\,\mathrm{d}\zeta\right)^{1/\gamma} \mathrm{d}\eta, \quad t \geqslant t_2. \end{aligned} \tag{2.14}$$

Since $\gamma < \beta_2$, $r(\sigma_0(\eta)) \leqslant r(\eta)$ and

$$\frac{1}{\alpha(1-\beta_2/\gamma)}(z^{1-\beta_2/\gamma}(\sigma_0(\eta)))_{\eta=t_2}^{t} \leqslant \frac{1}{\alpha(\beta_2/\gamma-1)} z^{1-\beta_2/\gamma}(\sigma_0(t_2)),$$

equation (2.14) becomes

$$\int_{t_2}^{\infty}\Big(\frac{1}{r(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\,\mathrm{d}\zeta\Big)^{1/\gamma}\,\mathrm{d}\eta < \infty,$$

which is a contradiction to (2.11). This contradiction implies that the solution x cannot be eventually positive. The case where x is eventually negative is very similar and we omit it here.

To prove the necessity part, we assume that (2.11) does not hold. For given $\varepsilon = (2/(1-p_0))^{-\alpha_i/\gamma} > 0$, we can find a $t_1 > 0$ such that

$$\int_{t_1}^{\infty}\Big(\frac{1}{r(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\,\mathrm{d}\zeta\Big)^{1/\gamma}\,\mathrm{d}\eta < \varepsilon. \tag{2.15}$$

Consider

$$M = \Big\{x \in C([0,\infty))\colon\ 1 \leqslant x(t) \leqslant \frac{2}{1-p_0} \ \text{ for } t \geqslant t_1\Big\}.$$

Define the operator

$$(\Phi x)(t) = 0 \qquad \text{if } t < t_1,$$

$$(\Phi x)(t) = 1 - p(t)x(\tau(t)) + \int_{t_1}^{t}\Big(\frac{1}{r(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)x^{\alpha_i}(\sigma_i(\zeta))\,\mathrm{d}\zeta\Big)^{1/\gamma}\,\mathrm{d}\eta \quad \text{if } t \geqslant t_1.$$

Indeed, $\Phi x = x$ implies that x is a solution of (1.1).

First we estimate $(\Phi x)(t)$ from below. Let $x \in M$. Then $1 \leqslant x$ implies that $(\Phi x)(t) \geqslant 1$ on $[t_1,\infty)$. Estimating $(\Phi x)(t)$ from above, we let $x \in M$. Then $x \leqslant 2/(1-p_0)$ and thus

$$(\Phi x)(t) \leqslant 1 - p(t)\frac{2}{1-p_0} + \int_{t_1}^{t}\Big(\frac{1}{r(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta)\Big(\frac{2}{1-p_0}\Big)^{\alpha_i}\,\mathrm{d}\zeta\Big)^{1/\gamma}\,\mathrm{d}\eta.$$

By (2.15) and then by the definition of ε, we obtain

$$(\Phi x)(t) \leqslant 1 + \frac{2p_0}{1-p_0} + \Big(\frac{2}{1-p_0}\Big)^{\alpha_i/\gamma}\varepsilon = 1 + \frac{2p_0}{1-p_0} + 1 = \frac{2}{1-p_0}.$$

Therefore Φ maps M to M.

To find a fixed point for Φ in M, we define a sequence of functions by the recurrence relation

$$\begin{aligned} u_0(t) &= 0 && \text{for } t = 0,\\ u_1(t) &= (\Phi u_0)(t) = 1 && \text{for } t \geqslant t_1,\\ u_{n+1}(t) &= (\Phi u_n)(t) && \text{for } n \geqslant 1,\ t \geqslant t_1. \end{aligned}$$

Note that for each fixed t we have $u_1(t) \geqslant u_0(t)$ and we can prove $u_{n+1}(t) \geqslant u_n(t)$ by using the method of induction. Therefore, $\{u_n\}$ converges pointwise to a function u in M. By Lebesgue's dominated convergence theorem, u is a fixed point of Φ and a positive solution to (1.1), which is not converging to zero. This completes the proof of the theorem. $\square$

Corollary 2.2. *Under the assumptions of Theorem* 2.2, *every unbounded solution of* (1.1) *is oscillatory if and only if* (2.11) *holds.*

Example 2.1. Consider the neutral differential equation

$$(2.16)\qquad \begin{aligned} &(\mathrm{e}^{-t}((x(t)-\mathrm{e}^{-t}x(\tau(t)))')^{11/3})'\\ &\quad+\frac{1}{t+1}(x(t-2))^{1/3}+\frac{1}{t+2}(x(t-1))^{5/3}=0. \end{aligned}$$

Here $\gamma = \frac{11}{3}$, $r(t) = \mathrm{e}^{-t}$, $-1 < p(t) = -\mathrm{e}^{-t} \leqslant 0$, $\sigma_1(t) = t-2$, $\sigma_2(t) = t-1$, $\Pi(t) = \int_0^t \mathrm{e}^{3s/11}\,\mathrm{d}s = \frac{11}{3}(\mathrm{e}^{3t/11}-1)$. For $\beta_1 = \frac{7}{3}$, we have $0 < \max\{\alpha_1,\alpha_2\} < \beta_1 < \gamma$, and $x^{\alpha_1-\beta_1} = x^{-2}$ and $x^{\alpha_2-\beta_1} = x^{-2/3}$, which both are decreasing functions. To check (2.6) we have

$$\begin{aligned} \int_0^\infty \sum_{i=1}^m q_i(\eta)\Pi^{\alpha_i}(\sigma_i(\eta))\,\mathrm{d}\eta &\geqslant \int_0^\infty q_1(\eta)\Pi^{\alpha_1}(\sigma_1(\eta))\,\mathrm{d}\eta\\ &= \int_0^\infty \frac{1}{\eta+1}\Big(\frac{11}{3}(\mathrm{e}^{3(\eta-2)/11}-1)\Big)^{1/3}\,\mathrm{d}\eta = \infty, \end{aligned}$$

since the integral approaches ∞ as $\eta \to \infty$. So, all the conditions of Theorem 2.1 hold. Thus, every solution of (2.16) either oscillates or converges to zero.

Example 2.2. Consider the neutral differential equation

$$(2.17)\qquad (((x(t)-\mathrm{e}^{-t}x(\tau(t)))')^{1/3})' + t(x(t-2))^{7/3} + (t+1)(x(t-1))^{11/3} = 0.$$

Here $\gamma = \frac{1}{3}$, $r(t) = 1$, $\sigma_1(t) = t-2$, $\sigma_2(t) = t-1$. For $\beta_2 = \frac{5}{3}$, we have $\min\{\alpha_1,\alpha_2\} > \beta_2 > \gamma$, and $x^{\alpha_1-\beta_2} = x^{2/3}$ and $x^{\alpha_2-\beta_2} = x^2$, which both are increasing functions. To check (2.11) we have

$$\int_{t_1}^{\infty}\left(\frac{1}{r(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m}q_i(\zeta)\,\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\eta \geqslant \int_{t_0}^{\infty}\left(\frac{1}{r(\eta)}\int_{\eta}^{\infty}q_1(\zeta)\,\mathrm{d}\zeta\right)^{1/\gamma}\mathrm{d}\eta$$
$$\geqslant \int_{2}^{\infty}\left(\int_{\eta}^{\infty}\zeta\,\mathrm{d}\zeta\right)^{3}\mathrm{d}\eta=\infty.$$

So, all the conditions of Theorem 2.2 hold. Thus, every solution of (2.17) either oscillates or converges to zero.

R e m a r k 2.1. Based on this work and [5], [6], [8], [13], [15], [17], [19], [18], [25], [28] an open problem that arises is to establish necessary and sufficient conditions for the oscillation of the solutions of the second-order neutral differential equation (1.1) for $p>0$ and $-\infty<p\leqslant -1$.

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