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Yu Liu; Panyue Zhou

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GORENSTEIN DIMENSION OF ABELIAN CATEGORIES ARISING FROM CLUSTER TILTING SUBCATEGORIES

Yu Liu, Chengdu, Panyue Zhou, Yueyang

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Abstract. Let $\mathscr{C}$ be a triangulated category and $\mathscr{X}$ be a cluster tilting subcategory of $\mathscr{C}$. Koenig and Zhu showed that the quotient category $\mathscr{C}/\mathscr{X}$ is Gorenstein of Gorenstein dimension at most one. But this is not always true when $\mathscr{C}$ becomes an exact category. The notion of an extriangulated category was introduced by Nakaoka and Palu as a simultaneous generalization of exact categories and triangulated categories. Now let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives, and $\mathscr{X}$ a cluster tilting subcategory of $\mathscr{C}$. We show that under certain conditions, the quotient category $\mathscr{C}/\mathscr{X}$ is Gorenstein of Gorenstein dimension at most one. As an application, this result generalizes the work by Koenig and Zhu.

Keywords: extriangulated category; abelian category; cluster tilting subcategory; Gorenstein dimension

MSC 2020: 18G80, 18E10

1. Introduction

Koenig and Zhu in [2] provided a general framework passing from triangulated categories to abelian categories by factoring out cluster tilting subcategories. More precisely, let $\mathscr{C}$ be a triangulated category and $\mathscr{X}$ a cluster tilting subcategory of $\mathscr{C}$. They showed that the quotient category $\mathscr{C}/\mathscr{X}$ is an abelian category and that it is Gorenstein of Gorenstein dimension at most one. Demonet and Liu in [1] gave a way to construct abelian categories from some exact categories. More precisely, let $\mathcal{B}$ be

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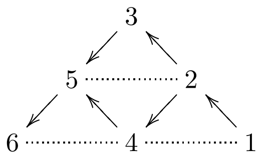
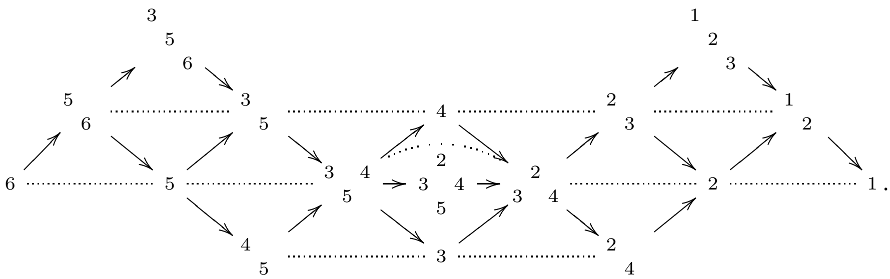
an exact category and $\mathscr{X}$ be a cluster tilting subcategory of $\mathcal{B}$. They showed that the quotient category $\mathcal{B}/\mathscr{X}$ is an abelian category. Hence, it is quite natural to ask whether this abelian quotient category $\mathcal{B}/\mathscr{X}$ is Gorenstein of Gorenstein dimension at most one. Unfortunately, this result is not always true for an exact category. See the following example.

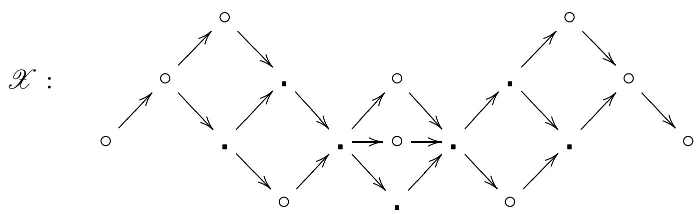
Example 1.1. We revisit Example 3.2 presented in [3]. Let Λ be the k-algebra given by the quiver



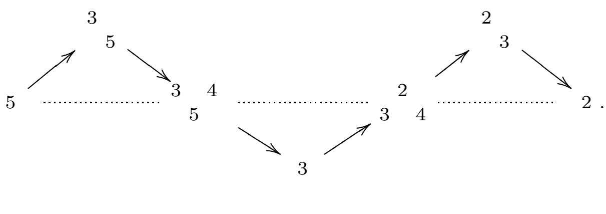
with mesh relations, where k is a field. The AR-quiver of $\mathcal{B} := \operatorname{mod}\Lambda$ is given by



We denote by "∘" in the AR-quiver the indecomposable objects which belong to a subcategory and by "▪" the objects which do not. Put



where $\mathscr{X}$ is a cluster tilting subcategory of $\mathcal{B}$. Then $\mathcal{B}/\mathscr{X} \simeq (\operatorname{mod}\Omega\mathscr{X}/\mathcal{P})$, where $\mathcal{P}$ is the full subcategory of projective objects and Ω is a *syzygy functor*, and its quiver is the following:



It is not Gorenstein of Gorenstein dimension at most one. Note that the non-projective injective object 2 has projective dimension 3.

Recently, the notion of an extriangulated category was introduced by Nakaoka and Palu in [5] as a simultaneous generalization of exact categories and triangulated categories. Cluster tilting theory gives a way to construct abelian categories from some extriangulated categories. Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives, and $\mathscr{X}$ a cluster tilting subcategory of $\mathscr{C}$. Then the quotient category $\mathscr{C}/\mathscr{X}$ is an abelian category, see [4], [7]. We know that a module category can be viewed as an extriangulated category with enough projectives and enough injectives. Hence the abelian quotient category $\mathscr{C}/\mathscr{X}$ is not Gorenstein of Gorenstein dimension at most one in general, see Example 1.1.

Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives, and let be $\mathscr{X}$ a subcategory of $\mathscr{C}$. We denote the full subcategory of projective objects in $\mathscr{C}$ by $\mathcal{P}$. Dually, the full subcategory of injective objects in $\mathscr{C}$ is denoted by $\mathcal{I}$. We denote $\Omega\mathscr{X} = \mathrm{CoCone}(\mathcal{P}, \mathscr{X})$, that is to say, $\Omega\mathscr{X}$ is the full subcategory of $\mathscr{C}$ consisting of objects ΩX such that there exists an $\mathbb{E}$-triangle:

$$\Omega X \xrightarrow{a} P \xrightarrow{b} X \dashrightarrow,$$

with $P \in \mathcal{P}$ and $X \in \mathscr{X}$. We call $\Omega\mathscr{X}$ the *syzygy* of $\mathscr{X}$. Dually we define the *cosyzygy* of $\mathscr{X}$ by $\Sigma\mathscr{X} = \mathrm{Cone}(\mathscr{X}, \mathcal{I})$. Namely, $\Sigma\mathscr{X}$ is the full subcategory of $\mathscr{C}$ consisting of objects ΣX such that there exists an $\mathbb{E}$-triangle:

$$X \xrightarrow{c} I \xrightarrow{d} \Sigma X \dashrightarrow$$

with $I \in \mathcal{I}$ and $X \in \mathscr{X}$. For more details, see [4], Definition 4.2 and Proposition 4.3.

Our main result is as follows, which gives sufficient conditions on the quotient category $\mathscr{C}/\mathscr{X}$, which is Gorenstein of Gorenstein dimension at most one, where $\mathscr{C}$ is an extriangulated category with enough projectives and enough injectives and $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$.

Theorem 1.2 (see Theorem 3.7 for more details). *Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives. Suppose that $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$ and $\mathcal{A}$ is the abelian quotient category $\mathscr{C}/\mathscr{X}$. Then:*

(1) *The category $\mathcal{A}$ has enough projective objects and enough injective objects.*

(2) *If $\Sigma(\Omega\mathscr{X}) \subseteq \mathscr{X}$ and $\Omega(\Sigma\mathscr{X}) \subseteq \mathscr{X}$, then the category $\mathcal{A}$ is Gorenstein of Gorenstein dimension at most one.*

Note that any triangulated category can be viewed as an extriangulated category with enough projectives and enough injectives. In this case, the condition

$\Sigma(\Omega\mathscr{X}) \subseteq \mathscr{X}$ and $\Omega(\Sigma\mathscr{X}) \subseteq \mathscr{X}$ is automatically satisfied. As an application, our result generalizes the work by Koenig and Zhu, see [2], Theorem 4.3.

The article is organised as follows: in Section 2, we review some elementary definitions and facts that we need to use. In Section 3, we prove the main result of this article.

2. Preliminaries

Throughout this article, if $\mathscr{X}$ is a subcategory of an additive category $\mathscr{C}$, then we always assume that $\mathscr{X}$ is a full subcategory which is closed under isomorphisms, direct sums and direct summands.

We recall some definitions and basic properties of extriangulated categories from [5]. Let $\mathscr{C}$ be an additive category. Suppose that $\mathscr{C}$ is equipped with a biadditive functor

$$\mathbb{E}\colon \mathscr{C}^{\rm op} \times \mathscr{C} \to Ab,$$

where Ab is the category of abelian groups. For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. Thus, formally, an $\mathbb{E}$-extension is a triplet (A, δ, C). Let (A, δ, C) be an $\mathbb{E}$-extension. Since $\mathbb{E}$ is a bifunctor, for any $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$, we have $\mathbb{E}$-extensions

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A') \quad \text{and} \quad \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A).$$

We abbreviate denote them by $a_*\delta$ and $c^*\delta$. For any $A, C \in \mathscr{C}$, the zero element $0 \in \mathbb{E}(C, A)$ is called the *spilt* $\mathbb{E}$-*extension*.

Definition 2.1 ([5], Definition 2.3). Let $(A, \delta, C), (A', \delta', C')$ be any pair of $\mathbb{E}$-extensions. A *morphism*

$$(a, c)\colon (A, \delta, C) \to (A', \delta', C')$$

of $\mathbb{E}$-extensions is a pair of morphisms $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C, C')$ in $\mathscr{C}$, satisfying the equality $a_*\delta = c^*\delta'$. Simply, we denote it as $(a, c)\colon \delta \to \delta'$.

Definition 2.2 ([5], Definition 2.6). Let $\delta = (A, \delta, C)$, $\delta' = (A', \delta C')$ be any pair of $\mathbb{E}$-extensions. Let

$$C \xrightarrow{\iota_C} C \oplus C' \xleftarrow{\iota_{C'}} C' \quad \text{and} \quad A \xleftarrow{p_A} A \oplus A' \xrightarrow{p_{A'}} A'$$

be coproduct and product in $\mathscr{C}$, respectively. Remark that, by the biadditivity of $\mathbb{E}$, we have a natural isomorphism,

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A').$$

Let $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ be the element corresponding to $(\delta, 0, 0, \delta')$ through the above isomorphism. This is the unique element which satisfies

$$\mathbb{E}(\iota_C, p_A)(\delta \oplus \delta') = \delta, \quad \mathbb{E}(\iota_C, p_{A'})(\delta \oplus \delta') = 0,$$
$$\mathbb{E}(\iota_{C'}, p_A)(\delta \oplus \delta') = 0, \quad \mathbb{E}(\iota_{C'}, p_{A'})(\delta \oplus \delta') = \delta'.$$

If $A = A'$ and $C = C'$, then the sum $\delta + \delta' \in \mathbb{E}(C, A)$ of $\delta, \delta' \in \mathbb{E}(C, A)$ is obtained by

$$\delta + \delta' = \mathbb{E}(\Delta_C, \nabla_A)(\delta \oplus \delta'),$$

where $\Delta_C = \binom{1}{1} \colon C \to C \oplus C$, $\nabla_A = (1, 1) \colon A \oplus A \to A$.

Definition 2.3 ([5], Definitions 2.7 and 2.8)**.** Let $A, C \in \mathscr{C}$ be any pair of objects. Sequences of morphisms in $\mathscr{C}$

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A \xrightarrow{x'} B' \xrightarrow{y'} C$$

are said to be *equivalent* if there exists an isomorphism $b \in \mathscr{C}(B, B')$ which makes the following diagram commutative:

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & \simeq \downarrow b & & \| \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C. \end{array}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

For any $A, C \in \mathscr{C}$, we denote $0 = \left[A \xrightarrow{\binom{1}{0}} A \oplus C \xrightarrow{(0,1)} C\right]$.

For any two equivalence classes, we denote as

$$[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'].$$

Definition 2.4 ([5], Definition 2.9)**.** Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ if it satisfies the following condition:

▷ Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions, with

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \quad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

Then, for any morphism $(a, c)\colon \delta \to \delta'$, there exists $b \in \mathscr{C}(B, B')$ which makes the following diagram commutative:

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C'. \end{array}$$

In the above situation, we say that the triplet (a, b, c) realizes (a, c).

Definition 2.5 ([5], Definition 2.10)**.** A realization $\mathfrak{s}$ of $\mathbb{E}$ is called *additive* if it satisfies the following conditions.

(1) For any $A, C \in \mathscr{C}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C, A)$ satisfies $\mathfrak{s}(0) = 0$.

(2) For any pair of $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$,

$$\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$$

holds.

Definition 2.6 ([5], Definition 2.12)**.** A triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is called an *externally triangulated category* (or *extriangulated category* for short) if it satisfies the following conditions:

(ET1) $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to Ab$ is a biadditive functor.

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions, realized as

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \quad s(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

For any commutative square

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

in $\mathscr{C}$, there exists a morphism $(a, c)\colon \delta \to \delta'$ satisfying $cy = y'b$.

(ET3)$^{\mathrm{op}}$ Dual of (ET3).

(ET4) Let (A, δ, D) and (B, δ', F) be $\mathbb{E}$-extensions realized by

$$A \xrightarrow{f} B \xrightarrow{f'} D \quad \text{and} \quad B \xrightarrow{g} C \xrightarrow{g'} F,$$

respectively. Then there exists an object $E \in \mathscr{C}$, a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\ & & F & = & F \end{array}$$

in $\mathscr{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which together satisfy the following compatibilities.

(i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $f'_*\delta'$,

(ii) $d^*\delta'' = \delta$,

(iii) $f_*\delta'' = e^*\delta'$.

(ET4)$^{\rm op}$ Dual of (ET4).

Remark 2.7. We know that both exact categories and triangulated categories are extriangulated categories (see [5], Example 2.13) and extension-closed subcategories of extriangulated categories are again extriangulated (see [5], Remark 2.18). Moreover, there exist extriangulated categories which are neither exact categories nor triangulated categories, see [5], Proposition 3.30 and [6], Example 4.14.

We use the following terminology.

Definition 2.8 ([5], Definitions 2.15, 2.19, 3.23 and 3.25). Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

(1) A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation* if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. In this case, x is called an *inflation* and y is called a *deflation.*

(2) If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in \mathbb{E}(C, A)$, we call the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ an $\mathbb{E}$-*triangle*, and write it in the following way:

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} .$$

(3) Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. If a triplet (a, b, c) realizes $(a, c)\colon \delta \to \delta'$, then we write it as

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} & \end{array}$$

and call (a, b, c) a *morphism of* $\mathbb{E}$-*triangles.*

(4) An object $P \in \mathscr{C}$ is called *projective* if for any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and any morphism $c \in \mathscr{C}(P, C)$, there exists $b \in \mathscr{C}(P, B)$ satisfying $yb = c$. We denote the full subcategory of projective objects by $\mathcal{P} \subseteq \mathscr{C}$. Dually, the full subcategory of injective objects is denoted by $\mathcal{I} \subseteq \mathscr{C}$.

(5) We say that $\mathscr{C}$ *has enough projective objects* if for any object $C \in \mathscr{C}$ there exists an $\mathbb{E}$-triangle $A \xrightarrow{x} P \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ satisfying $P \in \mathcal{P}$. We can define the notion of having enough injectives dually.

Definition 2.9 ([7], Definition 2.10). Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathscr{X}$ a subcategory of $\mathscr{C}$.

▷ $\mathscr{X}$ is called *rigid* if $\mathbb{E}(\mathscr{X}, \mathscr{X}) = 0$;

▷ $\mathscr{X}$ is called *cluster tilting* if it satisfies the following conditions:
  (a) $\mathscr{X}$ is a functorially finite in $\mathscr{C}$;
  (b) $M \in \mathscr{X}$ if and only if $\mathbb{E}(M, \mathscr{X}) = 0$;
  (c) $M \in \mathscr{X}$ if and only if $\mathbb{E}(\mathscr{X}, M) = 0$.

By the definition of a cluster tilting subcategory, we can conclude:

Lemma 2.10. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with enough projectives and enough injectives.*
(i) *If $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$, then $\mathcal{P} \subseteq \mathscr{X}$ and $\mathcal{I} \subseteq \mathscr{X}$.*
(ii) *$\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$ if and only if*
  (1) *$\mathscr{X}$ is rigid;*
  (2) *for any $C \in \mathscr{C}$, there is an $\mathbb{E}$-triangle $C \xrightarrow{a} X_1 \xrightarrow{b} X_2 \overset{\delta}{\dashrightarrow}$, where $X_1, X_2 \in \mathscr{X}$;*
  (3) *for any $C \in \mathscr{C}$, there is an $\mathbb{E}$-triangle $X_3 \xrightarrow{c} X_4 \xrightarrow{d} C \overset{\eta}{\dashrightarrow}$, where $X_3, X_4 \in \mathscr{X}$.*

Proof. (i) This follows from Proposition 3.24 and its dual in [5].

(ii) Assume that $\mathscr{X}$ is cluster tilting. It is obvious that $\mathscr{X}$ is rigid. For any $C \in \mathscr{C}$, since $\mathscr{X}$ is contravariantly finite in $\mathscr{C}$, then there is a right $\mathscr{X}$-approximation $u\colon X \to C$ of C. Since $\mathscr{C}$ has enough projectives, there is an $\mathbb{E}$-triangle $A \xrightarrow{v} P \xrightarrow{w} C \overset{\theta}{\dashrightarrow}$, where $P \in \mathcal{P}$. It follows that $w\colon P \to C$ is a deflation. Put $X_4 := X \oplus P \in \mathscr{X}$ since $P \in \mathcal{P} \subseteq \mathscr{X}$ and $d := (u, w)$. By [5], Corollary 3.16, we have that $d\colon X_4 \to C$ is a deflation. Thus there is an $\mathbb{E}$-triangle

$$X_3 \xrightarrow{c} X_4 \xrightarrow{d} C \overset{\eta}{\dashrightarrow}. \tag{2.1}$$

Applying the functor $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},-)$ to the $\mathbb{E}$-triangle (2.1), we have the following exact sequence:

$$\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},X_4)\xrightarrow{\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},d)}\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},C)\longrightarrow\mathbb{E}(\mathscr{X},X_3)\longrightarrow\mathbb{E}(\mathscr{X},X_4)=0.$$

Since $u\colon X\to C$ is a right $\mathscr{X}$-approximation of C, it is easy to see that $d\colon X\to C$ is also a right $\mathscr{X}$-approximation of C. It follows that $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},d)$ is an epimorphism. Thus $\mathbb{E}(\mathscr{X},X_3)=0$ implies $X_3\in\mathscr{X}$ since $\mathscr{X}$ is cluster tilting.

Similarly, one can show that for any $C\in\mathscr{C}$, there is an $\mathbb{E}$-triangle $C\xrightarrow{a}X_1\xrightarrow{b}X_2\overset{\delta}{\dashrightarrow}$, where $X_1,X_2\in\mathscr{X}$.

Now we assume that $\mathscr{X}$ satisfies the conditions (1), (2) and (3). For any $C\in\mathscr{C}$, there is an $\mathbb{E}$-triangle

$$X_3\xrightarrow{c}X_4\xrightarrow{d}C\overset{\eta}{\dashrightarrow},$$

where $X_3,X_4\in\mathscr{X}$. Applying the functor $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},-)$ to this $\mathbb{E}$-triangle, we have the following exact sequence:

$$\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},X_4)\xrightarrow{\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},d)}\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},C)\longrightarrow\mathbb{E}(\mathscr{X},X_3).$$

Since $\mathscr{X}$ is rigid, we have $\mathbb{E}(\mathscr{X},X_3)=0$. This shows that $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X},d)$ is an epimorphism. Thus $d\colon X\to C$ is a right $\mathscr{X}$-approximation of C. Hence $\mathscr{X}$ is contravariantly finite in $\mathscr{C}$. Similarly, we can show that $\mathscr{X}$ is covariantly finite in $\mathscr{C}$. So $\mathscr{X}$ is functorially finite in $\mathscr{C}$.

Since $\mathscr{X}$ is rigid, we obtain that $\mathbb{E}(M,\mathscr{X})=0$ for any $M\in\mathscr{X}$. Now we suppose $\mathbb{E}(M,\mathscr{X})=0$. Since $M\in\mathscr{C}$, then there is an $\mathbb{E}$-triangle

$$X_5\xrightarrow{f}X_6\xrightarrow{g}M\overset{\varphi}{\dashrightarrow},$$

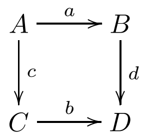
where $X_5,X_6\in\mathscr{X}$. It follows that $\varphi\in\mathbb{E}(M,X_5)=0$. By [5], Corollary 3.5, we get that g is a retraction. Thus M is a direct summand of X_6 implies $M\in\mathscr{X}$ since $X_6\in\mathscr{X}$.

Similarly, we can show that $M\in\mathscr{X}$ if and only if $\mathbb{E}(\mathscr{X},M)=0$. $\square$

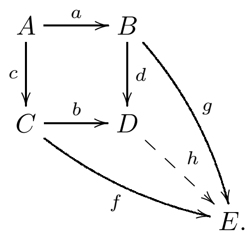
Let $\mathscr{C}$ be an additive category and $\mathscr{X}$ a subcategory of $\mathscr{C}$. We denote by $\mathscr{C}/\mathscr{X}$ the category whose objects are objects of $\mathscr{C}$ and whose morphisms are elements of $\mathrm{Hom}_{\mathscr{C}}(A,B)/\mathscr{X}(A,B)$ for $A,B\in\mathscr{C}$, where $\mathscr{X}(A,B)$ is the subgroup of $\mathrm{Hom}_{\mathscr{C}}(A,B)$ consisting of morphisms which factor through an object in $\mathscr{X}$. The category is called the *quotient category* of $\mathscr{C}$ by $\mathscr{X}$. For any morphism $f\colon A\to B$ in $\mathscr{C}$, we denote by $\overline{f}$ the image of f under the natural quotient functor $\mathscr{C}\to\mathscr{C}/\mathscr{X}$.

Theorem 2.11 ([7], Theorem 3.4 and [4], Theorem 3.2)**.** *Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives, and $\mathscr{X}$ a cluster tilting subcategory of $\mathscr{C}$. The quotient category $\mathscr{C}/\mathscr{X}$ is an abelian category.*

A commutative square



in $\mathscr{C}$ is called *weak pushout* if two morphisms $f \in \mathrm{Hom}_{\mathscr{C}}(C, E)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(B, E)$ satisfy $ga = fc$, there exists $h \in \mathrm{Hom}_{\mathscr{C}}(D, E)$ which makes the following diagram commutative:



Lemma 3.1. *Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives. Suppose that $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$ and $\mathcal{A}$ is the abelian quotient category $\mathscr{C}/\mathscr{X}$. Then any object $C \in \mathscr{C}$ admits an epimorphism $\overline{\beta}\colon \Omega X \to C$ for some $X \in \mathscr{X}$ in $\mathcal{A}$. Dually any object $C \in \mathscr{C}$ admits a monomorphism $\overline{\alpha}\colon C \to \Sigma X$ for some $X \in \mathscr{X}$ in $\mathcal{A}$.*

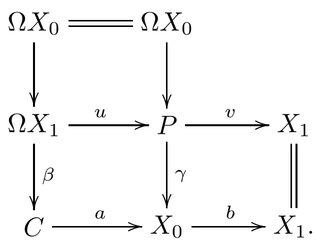
P r o o f. We only prove the first statement. The second statement is dual.

Since $\mathscr{X}$ is cluster tilting, there is an $\mathbb{E}$-triangle $C \xrightarrow{a} X_0 \xrightarrow{b} X_1 \dashrightarrow$, where $X_0, X_1 \in \mathscr{X}$. By definition ΩX_0 admits an $\mathbb{E}$-triangle

$$\Omega X_0 \xrightarrow{u} P \xrightarrow{v} X_0 \dashrightarrow. \tag{3.1}$$

By $(\mathrm{ET4})^{\mathrm{op}}$, we have the following commutative diagram made of $\mathbb{E}$-triangles

(3.2)



We claim that $\overline{\beta}\colon \Omega X_1 \to C$ is an epimorphism in $\mathcal{A}$. In fact, assume that $\bar{c}\colon C \to B$ is any morphism in $\mathcal{A}$ such that $\bar{c} \circ \overline{\beta} = 0$. Then $c\beta$ factors through $\mathscr{X}$. Applying the functor $\operatorname{Hom}_{\mathscr{C}}(-,\mathscr{X})$ to the $\mathbb{E}$-triangle (3.1), we have the following exact sequence:

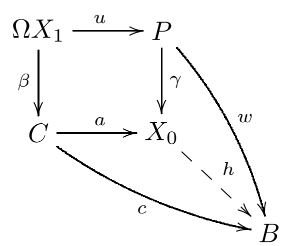
$$\operatorname{Hom}_{\mathscr{C}}(P,\mathscr{X}) \xrightarrow{\operatorname{Hom}_{\mathscr{C}}(u,\mathscr{X})} \operatorname{Hom}_{\mathscr{C}}(\Omega X_0,\mathscr{X}) \longrightarrow \mathbb{E}(X_0,\mathscr{X}) = 0.$$

This shows that u is a left $\mathscr{X}$-approximation of ΩX_1. It follows that there exists a morphism $w\colon P \to B$ such that $c\beta = wu$.

By [5], Lemma 3.13, the lower-left square in the diagram (3.2)

$$\begin{array}{ccc} \Omega X_1 & \xrightarrow{u} & P \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ C & \xrightarrow{a} & X_0 \end{array}$$

is a weak pushout. Thus there exists a morphism $h\colon X_0 \to B$ which makes the following diagram commutative:



which implies $\bar{c} = 0$. Hence $\overline{\beta}$ is an epimorphism in $\mathcal{A}$. $\square$

The following lemma can be found in [4], Proposition 1.20.

Lemma 3.2. *Let $\mathscr{C}$ be an extriangulated category and $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$ be any $\mathbb{E}$-triangle in $\mathscr{C}$. Assume that $x\colon A \to D$ is any morphism in $\mathscr{C}$. Then there exists a commutative diagram*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \overset{\delta}{\dashrightarrow} & \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} & & \Vert & & \\ D & \xrightarrow{a} & F & \xrightarrow{b} & C & \overset{x_*\delta}{\dashrightarrow} & \end{array}$$

of $\mathbb{E}$-triangles in $\mathscr{C}$, and moreover

$$A \xrightarrow{\binom{f}{x}} B \oplus D \xrightarrow{(y,-a)} F \overset{b^*\delta}{\dashrightarrow}$$

becomes an $\mathbb{E}$-triangle in $\mathscr{C}$.

Lemma 3.3. *Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives. Suppose that $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$ and $\mathcal{A}$ is the abelian quotient category $\mathscr{C}/\mathscr{X}$.*

(1) *If $f\colon A \to B$ is a morphism in $\mathscr{C}$, then there exists an inflation $\alpha = \binom{f}{a}\colon A \to X_0 \oplus B$ in $\mathscr{C}$ such that $\overline{\alpha} = \overline{f}$.*

(2) *If $f\colon A \to B$ is a morphism in $\mathscr{C}$, then there exists a deflation $\beta = (f, -b)\colon X_1 \oplus A \to B$ in $\mathscr{C}$ such that $\overline{\beta} = \overline{f}$.*

Proof. We only show the first one, the second is dual. Since $\mathscr{X}$ is cluster tilting, there exists an $\mathbb{E}$-triangle

$$A \xrightarrow{a} X_0 \xrightarrow{b} X_1 \dashrightarrow,$$

where $X_0, X_1 \in \mathscr{X}$. By Lemma 3.2, we get the following commutative diagram made of $\mathbb{E}$-triangles

$$\begin{array}{ccccccc} A & \xrightarrow{a} & X_0 & \xrightarrow{b} & X_1 & \dashrightarrow & \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle y} & & \| & & \\ B & \xrightarrow{c} & C & \xrightarrow{d} & X_1 & \dashrightarrow & . \end{array}$$

Moreover, $A \xrightarrow{\alpha=\binom{f}{a}} X_0 \oplus B \xrightarrow{(y,-c)} C \dashrightarrow$ is an $\mathbb{E}$-triangle in $\mathscr{C}$.

This shows that α is an inflation and $\overline{\alpha} = \overline{f}$. □

Lemma 3.4. *Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives. Suppose that $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$. Then $\Omega\mathscr{X}$ and $\Sigma\mathscr{X}$ are closed under direct summands.*

Proof. See the proof of Lemma 5.9 in [4]. □

Remark 3.5. Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives. If $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$, then $\Omega\mathscr{X}/\mathscr{X} = \Omega\mathscr{X}/\mathcal{P}$ and $\Sigma\mathscr{X}/\mathscr{X} = \Sigma\mathscr{X}/\mathcal{I}$. For convenience, we denote $\Omega\overline{\mathscr{X}} := \Omega\mathscr{X}/\mathscr{X}$ and $\Sigma\underline{\mathscr{X}} := \Sigma\mathscr{X}/\mathscr{X}$.

Proof. We only prove $\Omega\mathscr{X}/\mathscr{X} = \Omega\mathscr{X}/\mathcal{P}$. By duality, we have $\Sigma\mathscr{X}/\mathscr{X} = \Sigma\mathscr{X}/\mathcal{I}$.

We first prove that a morphism $f\colon \Omega X \to C$ factors through $\mathcal{P}$ with $X \in \mathscr{X}$ if and only if it factors through $\mathscr{X}$. Since $\mathcal{P} \subseteq \mathscr{X}$, we only need to prove that f factors through $\mathscr{X}$ implies it factors through $\mathcal{P}$. Assume that f factors through $\mathscr{X}$, namely that there exist morphisms $u\colon \Omega X \to X_2$ and $v\colon X_2 \to C$ with $X_2 \in \mathscr{X}$ such that $f = vu$. By the definition of $\Omega\mathscr{X}$, we have the following $\mathbb{E}$-triangle:

$$\Omega X \xrightarrow{a} P \xrightarrow{b} X \dashrightarrow,$$

where $P \in \mathcal{P}$. Since $\mathscr{X}$ is cluster tilting, there exists an $\mathbb{E}$-triangle:

$$X_0 \xrightarrow{c} X_1 \xrightarrow{d} C \dashrightarrow,$$

where $X_0, X_1 \in \mathscr{X}$. Since $\mathbb{E}(\mathscr{X}, \mathscr{X}) = 0$, we have that d is a right $\mathscr{X}$-approximation of C. Then there exists a morphism $w\colon X_2 \to X_1$ such that $v = dw$ and then $f = dwu$. Since a is a left $\mathscr{X}$-approximation of ΩX, there exists a morphism $h\colon P \to X_1$ such that $wu = ha$. It follows that $f = (dh)a$. This shows that f factors through $\mathcal{P}$.

Thus by definition we have $\Omega\mathscr{X}/\mathscr{X} = \Omega\mathscr{X}/\mathcal{P}$. $\square$

Lemma 3.6. *Let $\mathscr{C}$ be an extriangulated category with enough projectives and enough injectives. Suppose that $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$ and $\mathcal{A}$ is the abelian quotient category $\mathscr{C}/\mathscr{X}$. Then an object M of $\mathcal{A}$ is a projective object if and only if $M \in \Omega\overline{\mathscr{X}}$. Dually an object N of $\mathcal{A}$ is an injective object if and only if $N \in \Sigma\underline{\mathscr{X}}$.*

Proof. We prove the first statement only, the second one is obtained dually.

Let $\overline{g}\colon B \to C$ be an epimorphism in $\mathcal{A}$ and $\overline{\beta}\colon \Omega X \to C$ be any morphism in $\mathscr{C}$, where $X \in \mathscr{X}$. By Lemma 3.3, we can assume that it admits an $\mathbb{E}$-triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow.$$

Since $\mathscr{X}$ is cluster tilting, there exists an $\mathbb{E}$-triangle

$$B \xrightarrow{a} X_0 \xrightarrow{b} X_1 \dashrightarrow,$$

where $X_0, X_1 \in \mathscr{X}$. By (ET4), we get the following commutative diagram made of $\mathbb{E}$-triangles:

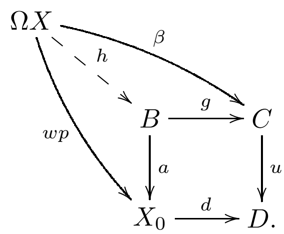
(3.3)
$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \| & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle u} \\ A & \xrightarrow{c} & X_0 & \xrightarrow{d} & D \\ & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle v} \\ & & X_1 & = & X_1. \end{array}$$

It follows that $ug = da$ and then $\bar{u} \circ \overline{g} = 0$. Since $\overline{g}$ is an epimorphism, we have $\bar{u} = 0$. By definition, ΩX admits an $\mathbb{E}$-triangle $\Omega X \xrightarrow{p} P \xrightarrow{q} X \dashrightarrow$, where $P \in \mathcal{P}$. Since $\bar{u} \circ \overline{\beta} = 0$, then $u\beta$ factors through $\mathscr{X}$. As $\mathbb{E}(\mathscr{X}, \mathscr{X}) = 0$, we obtain that p

is a left $\mathscr{X}$-approximation of ΩX. Thus there exists a morphism $r\colon P \to D$ such that $rp = u\beta$. Since P is a projective object, there exists a morphism $w\colon P \to X_0$ such that $r = dw$. It follows that $d(wp) = u\beta$. By the dual of [5], Lemma 3.13, the upper-right square in the diagram (3.3)

$$\begin{array}{ccc} B & \xrightarrow{\ g\ } & C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle u} \\ X_0 & \xrightarrow{\ d\ } & D \end{array}$$

is a weak pullback. Thus there exists a morphism $h\colon \Omega X \to B$ which makes the following diagram commutative:



Hence $\overline{\beta} = \overline{g} \circ \bar{h}$. This shows that ΩX is a projective object in $\mathcal{A}$.

Conversely, assume that M is a projective object in $\mathcal{A}$; by Lemma 3.1, there exists an epimorphism $\overline{\beta}\colon \Omega X \to M$ for some $X \in \mathscr{X}$ in $\mathcal{A}$. Thus M is a direct summand of ΩX in $\mathcal{A}$. Hence by Lemma 3.4, we imply that M lies in $\Omega\overline{\mathscr{X}}$. $\square$

Recall that an abelian category with enough projectives and injectives is called *Gorenstein* if all projective objects of this category have finite injective dimension, and all injective objects have finite projective dimension. The maximum of the injective dimensions of projectives and the projective dimensions of injectives is called *Gorenstein dimension* of the category.

Theorem 3.7. *Let $\mathscr{C}$ be an extriangulated category with enough projective objects and enough injective objects. Suppose that $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$ and $\mathcal{A}$ is the abelian quotient category $\mathscr{C}/\mathscr{X}$. Then:*

(1) *The category $\mathcal{A}$ has enough projective objects and enough injective objects.*

(2) *If $\Sigma(\Omega\mathscr{X}) \subseteq \mathscr{X}$ and $\Omega(\Sigma\mathscr{X}) \subseteq \mathscr{X}$, then the category $\mathcal{A}$ is Gorenstein of Gorenstein dimension at most one.*

P r o o f. (1) This follows from Lemmas 3.1 and 3.6.

(2) Let ΣX be any injective object in $\mathcal{A}$. Since $\mathscr{X}$ is cluster tilting, there exists an $\mathbb{E}$-triangle

$$\Sigma X \xrightarrow{a} X_0 \xrightarrow{b} X_1 \dashrightarrow,$$

where $X_0, X_1 \in \mathscr{X}$. By the definition of $\Omega\mathscr{X}$, we have the following $\mathbb{E}$-triangle:

$$\Sigma X_0 \xrightarrow{u} P_0 \xrightarrow{v} X_0 \dashrightarrow,$$

where $P_0 \in \mathcal{P}$. By (ET4)$^{\mathrm{op}}$, we have the following commutative diagram made of $\mathbb{E}$-triangles:

(3.4)
$$\begin{array}{ccccc}
\Omega X_0 & = & \Omega X_0 & & \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle u} & & \\
\Omega X_1 & \xrightarrow{c} & P_0 & \xrightarrow{d} & X_1 \dashrightarrow \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle v} & & \| \\
\Sigma X & \xrightarrow{a} & X_0 & \xrightarrow{b} & X_1 \dashrightarrow . \\
\downarrow & & \downarrow & & \\
\end{array}$$

By the definition of $\Omega\mathscr{X}$, we have the following $\mathbb{E}$-triangle $\Omega(\Sigma X) \xrightarrow{x} P_1 \xrightarrow{y} \Sigma X \dashrightarrow$, where $P_1 \in \mathcal{P}$. By the dual of [5], Proposition 3.17, we obtain the following commutative diagram made of $\mathbb{E}$-triangles:

(3.5)
$$\begin{array}{ccccc}
 & & \Omega X_0 & = & \Omega X_0 \\
 & & \downarrow{\scriptstyle \binom{0}{1}} & & \downarrow{\scriptstyle p} \\
\Omega(\Sigma X) & \xrightarrow{\binom{x}{h}} & P_1 \oplus \Omega X_0 & \xrightarrow{(-p',\ p)} & \Omega X_1 \dashrightarrow \\
\| & & \downarrow{\scriptstyle (1,0)} & & \downarrow{\scriptstyle q} \\
\Omega(\Sigma X) & \xrightarrow{x} & P_1 & \xrightarrow{y} & \Sigma X \dashrightarrow . \\
 & & \downarrow & & \downarrow \\
\end{array}$$

We claim that

$$\Omega(\Sigma X) \xrightarrow{\bar{h}} \Omega X_0 \xrightarrow{\bar{p}} \Omega X_1 \xrightarrow{\bar{q}} \Sigma X \to 0$$

is an exact sequence in $\mathcal{A}$. In fact, in the diagram (3.5) we obtain that $qp = 0$ and

$$(-p', p)\begin{pmatrix} x \\ h \end{pmatrix} = 0$$

which implies $\bar{q} \circ \bar{p} = 0$ and $\bar{p} \circ \bar{h} = 0$. This shows that $\operatorname{Im}(\bar{p}) \subseteq \operatorname{Ker}(\bar{q})$ and $\operatorname{Im}(\bar{h}) \subseteq \operatorname{Ker}(\bar{p})$.

Now we show that $\operatorname{Ker}(\bar{q}) \subseteq \operatorname{Im}(\bar{p})$.

Let $\overline{\alpha}\colon M \to \Omega X$ be any morphism in $\mathcal{A}$ such that $\bar{q} \circ \overline{\alpha} = 0$. Then $q\alpha$ factors through $\mathscr{X}$. By Remark 3.5, we know that $q\alpha$ factors through $\mathcal{P}$. That is to say, there exist morphisms $s\colon M \to P_2$ and $t\colon P_2 \to \Sigma X$ such that $q\alpha = ts$, where $P_2 \in \mathcal{P}$. Since P_2 is a projective object, there exists a morphism $\beta\colon P_2 \to \Omega X_1$ such that $q\beta = t$ and then

$$q(\alpha - \beta s) = q\alpha - q\beta s = q\alpha - ts = 0.$$

Thus there exists a morphism $\gamma\colon M \to \Omega X_0$ such that $\alpha - \beta s = p\gamma$ and then $\alpha = \beta s + p\gamma$. It follows that $\overline{\alpha} = \bar{p} \circ \overline{\gamma}$ which implies $\operatorname{Ker}(\bar{q}) \subseteq \operatorname{Im}(\bar{p})$.

Now we show that $\operatorname{Ker}(\bar{p}) \subseteq \operatorname{Im}(\bar{h})$.

Let $\bar{l}\colon N \to \Omega X_0$ be any morphism in $\mathcal{A}$ such that $\bar{p} \circ \bar{l} = 0$. Then pl factors through $\mathscr{X}$. By Remark 3.5, we know that pl factors through $\mathcal{P}$. That is to say, there exist morphisms $f\colon N \to P_3$ and $g\colon P_3 \to \Omega X_1$ such that $pl = gf$, where $P_3 \in \mathcal{P}$. Since P_3 is a projective object, there exists a morphism $\binom{m}{n}\colon P_3 \to P_1 \oplus \Omega X_0$ such that

$$g = (-p', p)\begin{pmatrix} m \\ n \end{pmatrix} = -p'm + pn$$

and then

$$(-p', p)\begin{pmatrix} mf \\ nf - l \end{pmatrix} = (-p'm + pn)f - pl = 0.$$

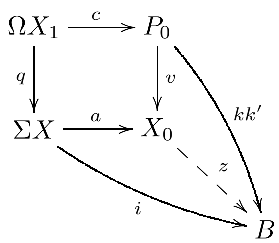
Thus there exists a morphism $w\colon N \to \Omega(\Sigma X)$ such that $\binom{x}{h}w = \binom{mf}{nf-l}$ and then $l = nf - hw$. It follows that $\bar{l} = \bar{h} \circ (-\overline{w})$ which implies $\operatorname{Ker}(\bar{p}) \subseteq \operatorname{Im}(\bar{h})$.

Now we show that $\bar{q}$ is an epimorphism in $\mathcal{A}$.

Let $\bar{i}\colon \Sigma X \to L$ be any morphism in $\mathcal{A}$ such that $\bar{i} \circ \bar{q} = 0$. Then iq factors through $\mathscr{X}$, namely, there exist morphisms $j\colon \Omega X_1 \to X_2$ and $k\colon X_2 \to L$ such that $iq = kj$. Since $\mathbb{E}(\mathscr{X}, \mathscr{X}) = 0$, we have that c is a left $\mathscr{X}$-approximation of ΩX_1. Thus there exists a morphism $k'\colon P_0 \to X_2$ such that $k'c = j$. It follows that $iq = (kk')c$. By [5], Lemma 3.13, the lower-left square in the diagram (3.4)

$$\begin{array}{ccc} \Omega X_1 & \xrightarrow{c} & P_0 \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle v} \\ \Sigma X & \xrightarrow{a} & X_0 \end{array}$$

is a weak pushout. Thus there exists a morphism $z\colon X_0 \to L$ which makes the following diagram commutative



which implies $\bar{i} = 0$. Hence $\bar{q}$ is an epimorphism in $\mathcal{A}$. This shows that $\Omega(\Sigma X) \xrightarrow{\bar{h}} \Omega X_0 \xrightarrow{\bar{p}} \Omega X_1 \xrightarrow{\bar{q}} \Sigma X \to 0$ is an exact sequence in $\mathcal{A}$.

In the diagram (3.4), we obtain that $aq = vc$ and $u = cp$. In the diagram (3.5), we obtain that $y = -qp'$ and $p'x = ph$. Thus we have that $ay = -aqp' = v(-cp')$ and $(-cp')x = -cph = -uh$. Hence we have the following commutative diagram of $\mathbb{E}$-triangles

$$\begin{array}{ccccccc}
\Omega(\Sigma X) & \xrightarrow{x} & P_1 & \xrightarrow{y} & \Sigma X & \dashrightarrow & \\
\downarrow{\scriptstyle -h} & & \downarrow{\scriptstyle -cp'} & & \downarrow{\scriptstyle a} & & \\
\Omega X_0 & \xrightarrow{u} & P_0 & \xrightarrow{v} & X_0 & \dashrightarrow & .
\end{array}$$

By the definition of Ω, we have $\Omega a = -h$ and then $\bar{h} = -\Omega\bar{a}$. Since $\Omega a\colon\ \Omega(\Sigma X) \to \Omega X_0$ and $\Omega(\Sigma\mathscr{X}) \subseteq \mathscr{X}$, we have $\Omega\bar{a} = 0$ in $\mathcal{A}$. Namely, $\bar{h} = 0$ in $\mathcal{A}$. So we obtain that $0 \to \Omega X_0 \xrightarrow{\bar{p}} \Omega X_1 \xrightarrow{\bar{q}} \Sigma X \to 0$ is an exact sequence in $\mathcal{A}$. This shows that any injective object ΣX in $\mathcal{A}$ has projective dimension at most one.

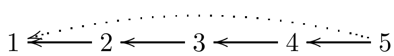
Dually, we can show that any projective object in $\mathcal{A}$ has injective dimension at most one.

Therefore $\mathcal{A}$ is Gorenstein of Gorenstein dimension at most one. □

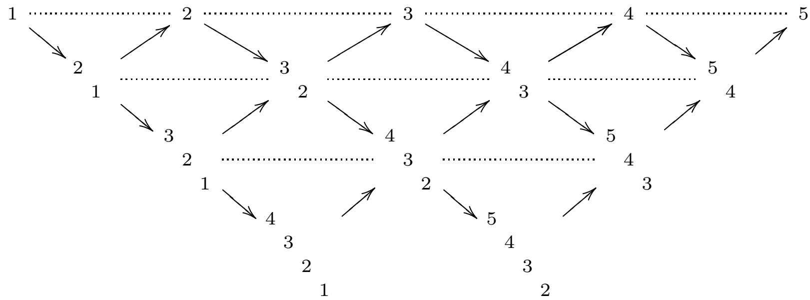
We conclude this section with two examples illustrating our result. Since $\mathcal{A}$ is Gorenstein of Gorenstein dimension at most one, it is either hereditary or of infinite global dimension.

In the following examples, we denote by "∘" in a quiver the objects which belong to a subcategory and by "•" the objects which do not.

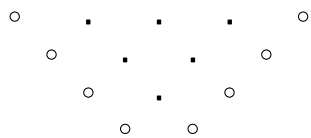
Example 3.8. Let Λ be the path algebra of the following quiver



then we obtain the AR-quiver of $\mathscr{C} :=$ mod Λ.

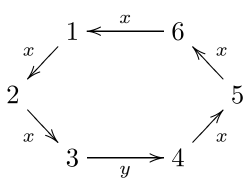


It is straightforward to verify that the subcategory

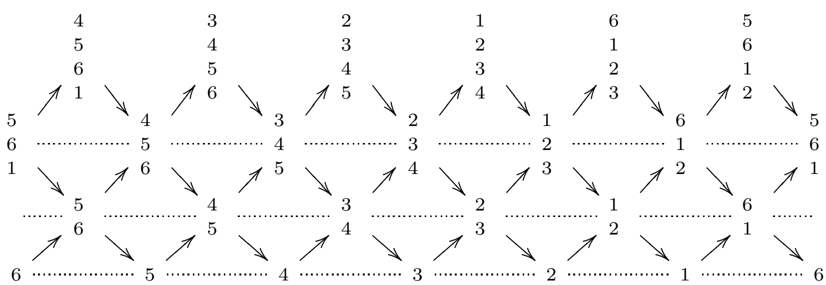
$\mathscr{X} =$ 

is a cluster tilting subcategory of $\mathscr{C}$. Note that $\Sigma(\Omega\mathscr{X}) \subseteq \mathscr{X}$ and $\Omega(\Sigma\mathscr{X}) \subseteq \mathscr{X}$. By Theorem 3.7, we have that $\mathscr{C}/\mathscr{X}$ is Gorenstein of Gorenstein dimension at most one. Moreover, it is hereditary.

Example 3.9. Let Λ be the k-algebra given by the quiver

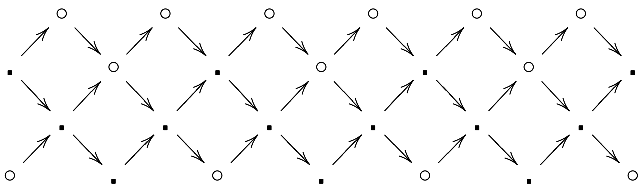


with relations $x^4 = 0$, the AR-quiver of $\mathscr{C} :=$ mod Λ is given by:



The first column and the last column are identical. It is straightforward to verify that the subcategory

$\mathscr{X}$:



is a cluster tilting subcategory of $\mathscr{C}$. Note that $\Sigma(\Omega\mathscr{X}) \subseteq \mathscr{X}$ and $\Omega(\Sigma\mathscr{X}) \subseteq \mathscr{X}$. By Theorem 3.7, we have that $\mathscr{C}/\mathscr{X}$ is Gorenstein of Gorenstein dimension at most one. Moreover, it is of infinite global dimension.

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Authors' addresses: Yu Liu, School of Mathematics, Southwest Jiaotong University, 610031, 111 N 1st Section, 2nd Ring Rd, Sha Xi Mei Shi Yi Tiao Jie, Jinniu District, Chengdu, Sichuan, P. R. China, e-mail: `liuyu86@swjtu.edu.cn`; Panyue Zhou (corresponding author), College of Mathematics, Hunan Institute of Science and Technology, Xueyuan Rd, 414006, Yueyang, Hunan, P. R. China, e-mail: `panyuezhou@163.com`.