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POLYNOMIAL EXPANSIVENESS AND ADMISSIBILITY OF WEIGHTED LEBESGUE SPACES

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Abstract. The paper investigates the interaction between the notions of expansiveness and admissibility. We consider a polynomially bounded discrete evolution family and define an admissibility notion via the solvability of an associated difference equation. Using the admissibility of weighted Lebesgue spaces of sequences, we give a characterization of discrete evolution families which are polynomially expansive and also those which are expansive in the ordinary sense. Thereafter, we apply the main results in order to infer continuous-time characterizations for the notions of expansiveness through the solvability of an associated integral equation.

Keywords: polynomial expansiveness; evolution family

MSC 2020: 34E05, 34D05

1. Introduction

Throughout this paper, we denote by $\mathbb{X}$ a complex Banach space, and by $L(\mathbb{X})$ the algebra of all bounded linear operators on $\mathbb{X}$. The norm on $\mathbb{X}$ and $L(\mathbb{X})$ will be denoted as $\|\cdot\|$. Let $\mathbb{Z}$, $\mathbb{R}$ denote the sets of integer, real numbers, respectively. For $A \subseteq \mathbb{R}$ we write $A_{\geqslant\delta} = \{x \in A\colon\ x \geqslant \delta\}$. Denote $\Delta = \{(t,s)\colon\ t \geqslant s \geqslant 0\}$.

Consider the homogeneous equation

$$x'(t) = A(t)y(t), \quad t \geqslant 0 \tag{1.1}$$

and the inhomogeneous equation

$$y'(t) = A(t)y(t) + h(t), \quad t \geqslant 0, \tag{1.2}$$

where $A(t)$ is in general an unbounded linear operator on $\mathbb{X}$ for all fixed $t \geqslant 0$. Equation (1.1) is called *well-posed* if we assume the existence, uniqueness and continuous dependence of solutions on initial condition. Note that if $A(t)$ is a bounded linear operator for all fixed $t \geqslant 0$, then the well-posedness is always ensured.

In the qualitative theory of differential equations and dynamical systems, a central problem is to study the interaction between the solvability of equation (1.2) and the asymptotic behaviors of solutions of equation (1.1). Historically, this problem has the origin in the pioneering work of Perron in 1930, see [13].

Theorem 1.1 ([13]). *Let $\mathbb{X} = \mathbb{R}^d$. If equation* (1.2) *admits at least one bounded solution for each bounded continuous function $h(\cdot)$, then each bounded solution of equation* (1.1) *goes to zero as $t \to \infty$.*

Many results related to differential equations can carry over quite easily to corresponding results for difference equations. Similarities between differential equations and difference equations have been recognized and exploited very regularly. Thus, shortly after the publication of the work [13], a corresponding study for the case of discrete time was undertaken by Ta Li (see [8]) in which a similar method was done to obtain analogous results for difference equations. In both papers, a central interest is the relationship, for linear equations, between the solvability of the inhomogeneous equation for every bounded perturbation and a certain form of behavior of the solutions of the homogeneous equation. The assumption in Perron's theorem is known as the *admissibility* of function spaces in which we take $h(\cdot)$ and look for $y(\cdot)$. Note that the admissibility can be considered as an operator-theoretic property of the differential operator $\mathcal{D}y(t) = y'(t) - A(t)y(t)$ in suitable function spaces.

Thanks to the contributions of many research groups, the study of Perron's theorem reached far-reaching extensions and generalizations. A more systematic study was done by Massera and Schäffer, whose results are presented in their book, see [9]. Daleckij and Krein showed in [4] that exponential asymptotic behaviors of equation (1.1) can be characterized in terms of the subjectiveness of the operator $\mathcal{D}$. Obtained in [7] is an extension to the infinite-dimensional setting for equations defined on the whole line. We also refer the reader to paper [3] extending Ta Li's theorem to infinite-dimensional Banach spaces. Some different characterizations can be found in papers [2], [10], [16].

It should be noted that the well-posedness is equivalent to the existence of an evolution family solving equation (1.1). Therefore, the admissibility is often known in more general context such as the existence of mild-solutions. For the purpose of the present work, it is necessary to recall these terminologies.

Definition 1.2. A family $\{U(t,s)\}_{(t,s)\in\Delta}$ is called an *evolution family* if the following conditions hold for every $t \geqslant r \geqslant s \geqslant 0$:

(1) $U(t,t) = I$, $U(t,s) = U(t,r)U(r,s)$;
(2) for each $x \in \mathbb{X}$, the mapping $(t,s) \mapsto U(t,s)x$ is continuous.

Definition 1.3. An evolution family $\{U(t,s)\}_{(t,s)\in\Delta}$ is called
(1) *exponentially bounded* if there exist constants M, ω such that

$$\|U(t,s)\| \leqslant M\mathrm{e}^{\omega(t-s)}, \quad t \geqslant s \geqslant 0;$$

(2) *exponentially stable* if it is exponentially bounded with $\omega < 0$;
(3) *exponentially unstable* if there exist constants $K, \alpha > 0$ such that

$$\|U(t,s)x\| \geqslant K\mathrm{e}^{\alpha(t-s)}\|x\|, \quad t \geqslant s \geqslant 0;$$

(4) *exponentially expansive* if it is exponentially unstable and the operator $U(t,s)$ is surjective for all fixed $t \geqslant s \geqslant 0$.

If there exists an evolution family $\{U(t,s)\}_{(t,s)\in\Delta}$ associated with equation (1.1), then the *mild-solution* of equation (1.2) is defined by

$$y(t) = U(t,s)y(s) + \int_s^t U(t,\tau)h(\tau)\,\mathrm{d}\tau, \quad t \geqslant s \geqslant 0. \tag{1.3}$$

In recent years, homogeneous equations were investigated in the unified setting of evolution families and instead of using the differential operator $\mathcal{D}$, one started to study the associated integral equation (1.3). A significant contribution in this direction was made by Minh et al. in [12] who obtained the following characterization for the exponential expansiveness.

Theorem 1.4 ([12]). *Let $\{U(t,s)\}_{(t,s)\in\Delta}$ be an exponentially bounded evolution family and denote $\mathcal{C}(\mathbb{R}_{\geqslant r}) = \{f\colon \mathbb{R}_{\geqslant r} \to \mathbb{X}$ is continuous with $\lim_{t\to\infty} f(t) = 0\}$. Then $\{U(t,s)\}_{(t,s)\in\Delta}$ is exponentially expansive if and only if the pair $(\mathcal{C}(\mathbb{R}_{\geqslant 0}), \mathcal{C}(\mathbb{R}_{\geqslant 0}))$ is admissible to equation (1.3); meaning for every $r \geqslant 0$ and every $h \in \mathcal{C}(\mathbb{R}_{\geqslant r})$ there is a unique $u \in \mathcal{C}(\mathbb{R}_{\geqslant r})$ such that the pair (h,u) verifies equation (1.3).*

Using discrete-time arguments, Megan et al. in [11] gave another proof for Theorem 1.4. A version of Theorem 1.4 in terms of *unweighted* Lebesgue spaces $\ell^p(\mathbb{Z}_{\geqslant 0})$ was carried out in [17]. Recall that

$$\ell^\infty(\mathbb{Z}_{\geqslant\delta}) = \{\mathbf{x}\colon \mathbb{Z}_{\geqslant\delta} \to \mathbb{X} \text{ is bounded}\},$$
$$\ell^p(\mathbb{Z}_{\geqslant\delta}) = \Big\{\mathbf{x}\colon \mathbb{Z}_{\geqslant\delta} \to \mathbb{X}\colon \sum_{j\geqslant\delta} \|\mathbf{x}(j)\|^p < \infty\Big\}.$$

In recent years, there has been a growing concern in the polynomial asymptotic behaviors of solutions of evolution equations in Banach spaces. The interesting part of a polynomial behavior lies in the fact that it is a weaker requirement than the corresponding exponential behavior. Alternatively speaking, an exponentially stable evolution family is always polynomially stable, but the converse implication fails to hold. We refer the reader to paper [6] for some examples that are polynomially stable but that are not exponentially stable. In addition, paper [6] detects a relationship between polynomial stability and *weighted* Lebesgue spaces

$$\ell^p_w(\mathbb{Z}_{\geqslant\delta}) = \Big\{\mathbf{x}\colon\ \mathbb{Z}_{\geqslant\delta} \to \mathbb{X}\colon\ \sum_{j\geqslant\delta} j^{-1}\|\mathbf{x}(j)\|^p < \infty\Big\}, \quad p \in [1,\infty),\ \delta > 0.$$

This relationship can be stated as follows: a polynomially bounded evolution family $\{U(t,s)\}_{(t,s)\in\Delta}$ is polynomially stable if and only if there exists $p \geqslant 1$ such that for each $x \in \mathbb{X}$ the mapping $j \mapsto U(js,s)x$ lies in the *weighted* Lebesgue space $\ell^p_w(\mathbb{Z}_{\geqslant 1})$ in a uniform way.

Although the study of Perron's theorem for exponential behaviors continued to develop rapidly, what deals with polynomial behaviors, as far as we know, seems not to have received proper concern. In [5], Hai presented versions of Perron's theorem for polynomial stability or polynomial expansiveness. It turns out that polynomial behaviors are related to the existence of solutions of the integral equation

$$u(t) = U(t,s)u(s) + \int_s^t \tau^{-1}U(t,\tau)f(\tau)\,\mathrm{d}\tau, \quad t \geqslant s > 0. \tag{1.4}$$

It is clear that the equation above can be obtained from equation (1.3) by taking $h(t) = t^{-1}f(t)$; in other words, it refers to the mild-solution of equation (1.2) when $h(t) = t^{-1}f(t)$.

In this paper, we investigate a similar problem for the admissibility of *weighted* Lebesgue spaces with the expected conclusion that the weighted space setting is more delicate to delineate. Namely, we consider a polynomially bounded *discrete evolution family* and define an admissibility concept with respect to an associated difference equation such that the input and output spaces are weighted Lebesgue spaces

$$\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant\delta}) := \begin{cases} \ell^p_w(\mathbb{Z}_{\geqslant\delta}) & \text{if } 1 \leqslant p < \infty, \\ \ell^\infty(\mathbb{Z}_{\geqslant\delta}) & \text{if } p = \infty. \end{cases}$$

We give a criterion for the existence of a polynomial expansiveness through the admissibility of $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}))$ with the exponents satisfying $1 \leqslant p \leqslant q \leqslant \infty$ and $(p,q) \neq (1,\infty)$ (see Theorem 3.8). Meanwhile, the admissibility of $(\ell^1_w(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$ is equivalent to the expansiveness in the ordinary sense (see Theorem 3.10). For $p > q$, we indicate in Example 3.9 that the polynomial expansiveness cannot imply the admissibility of $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}))$.

Thereafter, we develop continuous-time versions of Theorems 3.8 and 3.10. It should be emphasised that our proofs do not rely completely on the traditional method of constructing input and output functions. Namely, we apply Theorems 3.8 and 3.10 in order to infer continuous-time characterizations for expansiveness of *continuous evolution families* via the solvability of equation (1.4). This helps to avoid repeating analogous arguments twice, once for discrete time and once for continuous time. In Theorem 4.3, we employ the admissibility in continuous time to characterize polynomially bounded *continuous evolution families* which are polynomially expansive. Meanwhile, Theorem 4.5 is a characterization for the expansiveness in the ordinary sense.

2. Preparation

2.1. Polynomial expansiveness. Denote $\Omega_{\geqslant\kappa} := \{(m,n) \in \mathbb{Z}\times\mathbb{Z}\colon\ m \geqslant n \geqslant \kappa\}$. To simplify notation we only write Ω in the case, where $\kappa = 0$. We get a closer look at the concepts used in the entire paper.

Definition 2.1. A family $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ of bounded linear operators is called a *discrete evolution family* if for every $(m,n), (n,r) \in \Omega$ it satisfies

(1) $\mathcal{A}(r,r) = I$;
(2) $\mathcal{A}(m,r) = \mathcal{A}(m,n)\mathcal{A}(n,r)$.

Definition 2.2. A discrete evolution family $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ is called

(1) *polynomially bounded* if there exist positive constants M, ω such that

$$\|\mathcal{A}(m,n)\| \leqslant M m^\omega n^{-\omega}, \quad (m,n) \in \Omega_{\geqslant 1}; \tag{2.1}$$

(2) *polynomially unstable* if there exist positive constants K, α such that

$$\|\mathcal{A}(m,n)x\| \geqslant K m^\alpha n^{-\alpha}\|x\|, \quad (m,n) \in \Omega_{\geqslant 1};$$

(3) *polynomially expansive* if it is polynomially unstable and the operator $\mathcal{A}(m,n)$ is surjective for all fixed $(m,n) \in \Omega_{\geqslant 1}$;
(4) *expansive in the ordinary sense* if it is polynomially expansive with $\alpha = 0$.

Definition 2.3. An evolution family $\{U(t,s)\}_{(t,s)\in\Delta}$ is called

(1) *polynomially unstable* if there exist constants $K, \alpha > 0$ such that

$$\|U(t,s)x\| \geqslant K t^\alpha s^{-\alpha}\|x\|, \quad t \geqslant s \geqslant 1,\ x \in \mathbb{X}\setminus\{0\};$$

(2) *polynomially expansive* if it is polynomially unstable and the operator $U(t,s)$ is surjective for all fixed $t \geqslant s \geqslant 1$;
(3) *expansive in the ordinary sense* if it is polynomially expansive with $\alpha = 0$.

Note that paper [1] gives necessary and sufficient conditions in terms of generalized exponents for the existence of polynomial behavior.

Remark 2.4. It turns out that a polynomial behavior is a weaker requirement than the corresponding exponential behavior.
(1) The relations between polynomial and exponential stability are given in the following diagram

$$\begin{tikzcd} \text{polynomial stability} \arrow[r, Rightarrow] & \text{polynomial boundedness} \arrow[d, Rightarrow] \\ \text{exponential stability} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{exponential boundedness.} \end{tikzcd}$$

(2) Exponential expansiveness $\Longrightarrow$ polynomial expansiveness.
It should be noted that the converse implications fail to hold.

Remark 2.5. If $\{U(t,s)\}_{(t,s)\in\Delta}$ is a continuous evolution family, then

$$\mathcal{A}\colon \Omega \to L(\mathbb{X}), \quad \mathcal{A}(m,n) = U(m,n),\ (m,n) \in \Omega$$

is a discrete evolution family.

2.2. Banach spaces of sequences or functions and admissibility. It is well-known that $\ell^p_w(\mathbb{Z}_{\geqslant\delta})$, $\ell^\infty(\mathbb{Z}_{\geqslant\delta})$, $\ell^p(\mathbb{Z}_{\geqslant\delta})$ are Banach sequence spaces endowed with the norms

$$\|\mathbf{x}\|_{w,p} := \Big(\sum_{j\geqslant\delta} j^{-1}\|\mathbf{x}(j)\|^p\Big)^{1/p},$$

$$\|\mathbf{x}\|_\infty := \sup\{\|\mathbf{x}(m)\|\colon\ m \in \mathbb{Z}_{\geqslant\delta}\}, \quad \|\mathbf{x}\|_{u,p} := \Big(\sum_{j\geqslant\delta} \|\mathbf{x}(j)\|^p\Big)^{1/p},$$

respectively.

Lemma 2.6. *If $\mathbf{u} \in \ell^p_w(\mathbb{Z}_{\geqslant\delta}) \cap \ell^\infty(\mathbb{Z}_{\geqslant\delta})$, then $\mathbf{u} \in \ell^q_w(\mathbb{Z}_{\geqslant\delta})$ for all fixed $q \geqslant p$.*

P r o o f.

$$\sum_{j\geqslant\delta} j^{-1}\|\mathbf{u}(j)\|^q \leqslant \|\mathbf{u}\|_\infty^{q-p}\sum_{j\geqslant\delta} j^{-1}\|\mathbf{u}(j)\|^p = \|\mathbf{u}\|_\infty^{q-p}\|\mathbf{u}\|^p_{w,p} < \infty.$$

□

Obtained in the lemma below are elementary inequalities that will be used to estimate the norm of sequences in $\ell^p_w(\mathbb{Z}_{\geqslant\delta})$.

Lemma 2.7. *The following inequalities hold for every $\alpha > 0$, $m \in \mathbb{Z}_{\geqslant 1}$ and $(n,\kappa) \in \Omega$.*

(1) $m^\alpha \sum\limits_{j=m}^{\infty} j^{-\alpha-1} \leqslant 2^{\alpha+1}\alpha^{-1}$;

(2) $\sum\limits_{n=\kappa}^{j} n^{\alpha-1} \leqslant 2\alpha^{-1}j^\alpha$;

(3) $\lambda^{-1}(\lambda-1) \leqslant \sum\limits_{j=m}^{\lambda m} j^{-1} \leqslant 2\ln\lambda$.

P r o o f. We prove the first item and the remaining items are left to the reader. Since $\lfloor\tau\rfloor = j$ for $j \leqslant \tau < j+1$, we have

$$\begin{aligned} m^\alpha \sum_{j=m}^{\infty} j^{-\alpha-1} &= m^\alpha \sum_{j=m}^{\infty} \int_j^{j+1} \lfloor\tau\rfloor^{-\alpha-1}\,\mathrm{d}\tau \leqslant m^\alpha \sum_{j=m}^{\infty} \int_j^{j+1} 2^{\alpha+1}\tau^{-\alpha-1}\,\mathrm{d}\tau \\ &= m^\alpha \int_m^\infty 2^{\alpha+1}\tau^{-\alpha-1}\,\mathrm{d}\tau = 2^{\alpha+1}\alpha^{-1}. \end{aligned}$$

□

For $\delta \in \mathbb{Z}_{\geqslant 1}$, we consider the difference equation

$$\mathbf{u}(m) - \mathcal{A}(m,n)\mathbf{u}(n) = \sum_{j=n}^{m-1} \frac{1}{j}\mathcal{A}(m,j)\mathbf{f}(j), \quad (m,n) \in \Omega_{\geqslant\delta}. \tag{2.2}$$

Definition 2.8. The pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}))$ is called *admissible to equation* (2.2) if for each $k \in \mathbb{Z}_{\geqslant 1}$ and each $\mathbf{f} \in \ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant k})$ there exists a unique $\mathbf{u} \in \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant k})$ such that equation (2.2) holds for all $(m,n) \in \Omega_{\geqslant 1}$.

Remark 2.9. Similarly, we also can define the admissibility of *unweighted* Lebesgue spaces.

Obtained in the example below is the negative conclusion that the admissibility of *unweighted Lebesgue spaces* cannot imply the polynomial expansiveness.

Example 2.10. Let $\mathbb{X} = \mathbb{R}$ and $\mathcal{A}(m,n) = 1$. It is clear that $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ is not polynomially expansive. We will prove that the pair $(\ell^p(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$ is admissible for any $p \in (1,\infty)$. It is enough to prove that for every $\mathbf{f} \in \ell^p(\mathbb{Z}_{\geqslant\kappa})$ the sequence

$$\mathbf{u}\colon\ \mathbb{Z}_{\geqslant\kappa} \to \mathbb{X}, \quad \mathbf{u}(n) = -\sum_{j=n}^{\infty} j^{-1}\mathbf{f}(j)$$

belongs to $\ell^\infty(\mathbb{Z}_{\geqslant\kappa})$. Indeed, we can choose $r \in (1,\infty)$ with $1/r + 1/p = 1$. By Hölder's inequality, we estimate

$$\|\mathbf{u}(n)\| \leqslant \left(\sum_{j=n}^{\infty} j^{-r}\right)^{1/r} \|\mathbf{f}\|_{u,p} \leqslant \left(\sum_{j=1}^{\infty} j^{-r}\right)^{1/r} \|\mathbf{f}\|_{u,p} < \infty,$$

as wanted.

The following observation is useful for finding solutions of equation (2.2).

Lemma 2.11. *Let $\beta \in \mathbb{Z}_{\geqslant 1}$ and $x \in \mathbb{X}$. For every $\zeta \in \ell^1(\mathbb{Z}_{\geqslant\beta})$, the pair $(\mathbf{f}_\zeta, \mathbf{u}_\zeta)$ given by*

$$\text{(2.3)} \qquad \mathbf{f}_\zeta, \mathbf{u}_\zeta \colon \mathbb{Z}_{\geqslant\beta} \to \mathbb{X}, \quad \mathbf{f}_\zeta(n) = -n\zeta(n)\mathcal{A}(n,\beta)x, \quad \mathbf{u}_\zeta(n) = \sum_{j=n}^{\infty} \zeta(j)\mathcal{A}(n,\beta)x$$

verifies equation (2.2) *for every* $(m,n) \in \Omega_{\geqslant\beta}$.

Remark 2.12. If the pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}))$ is admissible, then we consider the operator

$$\mathcal{O}^{p,q}_\delta \colon \ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}) \to \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \quad \mathcal{O}^{p,q}_\delta(\mathbf{f}) = \mathbf{u},$$

where the pair $(\mathbf{f}, \mathbf{u})$ satisfies the difference equation (2.2). An elementary proof shows that the operator $\mathcal{O}^{p,q}_\delta$ is closed, and so it must be bounded.

To develop continuous-time versions, we need weighted Lebesgue spaces of functions and they are defined as follows:

$$L^p_w(\mathbb{R}_{\geqslant\delta}) = \left\{ f \colon \mathbb{R}_{\geqslant\delta} \to \mathbb{X},\ t^{-1/p} f(t) \text{ is measurable with } \int_\delta^\infty t^{-1}\|f(t)\|^p \,\mathrm{d}t < \infty \right\}$$

for $p \in [1,\infty)$, $\delta > 0$ and

$$L^\infty(\mathbb{R}_{\geqslant\delta}) = \left\{ f \colon \mathbb{R}_{\geqslant\delta} \to \mathbb{X} \text{ is measurable with } \operatorname*{ess\,sup}_{t\geqslant\delta} \|f(t)\| < \infty \right\}.$$

To simplify the presentation, we denote

$$L^p_{\mathrm{wt}}(\mathbb{R}_{\geqslant\delta}) = \begin{cases} L^p_w(\mathbb{R}_{\geqslant\delta}) & \text{if } p \in [1,\infty), \\ L^\infty(\mathbb{R}_{\geqslant\delta}) & \text{if } p = \infty. \end{cases}$$

Definition 2.13. The pair $(L^p_{\mathrm{wt}}(\mathbb{R}_{\geqslant 1}), L^q_{\mathrm{wt}}(\mathbb{R}_{\geqslant 1}))$ is called *admissible to equation* (1.4) if for each $\delta \in \mathbb{R}_{\geqslant 1}$ and each $f \in L^p_{\mathrm{wt}}(\mathbb{R}_{\geqslant\delta})$ there exists a unique $u \in L^q_{\mathrm{wt}}(\mathbb{R}_{\geqslant\delta})$ such that the pair (f,u) verifies equation (1.4) for all $t \geqslant s \geqslant \delta$.

We end this section with a continuous version of Lemma 2.11.

Lemma 2.14. *Let $\beta > 0$ and $x \in \mathbb{X}$. For every $\zeta \in L^1(\mathbb{R}_{\geqslant\beta})$, the pair (f_ζ, u_ζ) given by*

$$f_\zeta, u_\zeta\colon \mathbb{R}_{\geqslant\beta} \to \mathbb{X}, \quad f_\zeta(t) = -t\zeta(t)U(t,\beta)x, \quad u_\zeta(t) = \int_t^\infty \zeta(\tau)\,\mathrm{d}\tau U(t,\beta)x \tag{2.4}$$

verifies equation (1.4) *for every $t \geqslant s \geqslant \beta$.*

3. Discrete time

3.1. Outputs in $\ell^\infty(\mathbb{Z}_{\geqslant1})$. Obtained in this section is the interesting conclusion that outputs always lie in $\ell^\infty(\mathbb{Z}_{\geqslant1})$ for any admissibility.

Proposition 3.1. *Let $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ be a discrete evolution family, which is polynomially bounded (that is* (2.1) *holds). Let $p, q \in [1,\infty]$. If the pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant1}), \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant1}))$ is admissible, then so is $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant1}), \ell^\infty(\mathbb{Z}_{\geqslant1}))$.*

Proof. Let $m, \kappa \in \mathbb{Z}_{\geqslant1}$, $n \in \{m,\dots,2m\}$ and $\mathbf{f} \in \ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant\kappa})$. Then there exists a unique $\mathbf{u} \in \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant\kappa})$ such that the pair $(\mathbf{f},\mathbf{u})$ verifies equation (2.2). In particular, we have

$$\mathbf{u}(2m) = \mathcal{A}(2m,n)\mathbf{u}(n) + \sum_{j=n}^{2m-1} j^{-1}\mathcal{A}(2m,j)\mathbf{f}(j),$$

which implies, by (2.1), that

$$\|\mathbf{u}(2m)\| \leqslant M2^\omega\|\mathbf{u}(n)\| + M2^\omega \sum_{j=n}^{2m-1} j^{-1}\|\mathbf{f}(j)\|. \tag{3.1}$$

If $p = 1$, then the above gives

$$\|\mathbf{u}(2m)\| \leqslant M2^\omega\|\mathbf{u}(n)\| + M2^\omega\|\mathbf{f}\|_{w,1}.$$

If $p = \infty$, then it follows from (3.1) that

$$\|\mathbf{u}(2m)\| \leqslant M2^\omega\|\mathbf{u}(n)\| + M2^\omega \sum_{j=n}^{2m-1} j^{-1}\|\mathbf{f}\|_\infty \leqslant M2^\omega\|\mathbf{u}(n)\| + M2^{\omega+1}\ln 2\|\mathbf{f}\|_\infty,$$

where the last inequality uses Lemma 2.7, statement (3). If $1 < p < \infty$, then we can find $r > 1$ with $1/r + 1/p = 1$. This r can be used to rewrite (3.1) as

$$\begin{aligned}\|\mathbf{u}(2m)\| &\leqslant M2^{\omega}\|\mathbf{u}(n)\| + M2^{\omega}\sum_{j=n}^{2m-1} j^{-1/r}j^{-1/p}\|\mathbf{f}(j)\| \\ &\leqslant M2^{\omega}\|\mathbf{u}(n)\| + M2^{\omega}\Big(\sum_{j=n}^{2m-1} j^{-1}\Big)^{1/r}\Big(\sum_{j=n}^{2m-1} j^{-1}\|\mathbf{f}(j)\|^{p}\Big)^{1/p} \\ &\leqslant M2^{\omega}\|\mathbf{u}(n)\| + M2^{\omega}(2\ln 2)^{1/r}\|\mathbf{f}\|_{w,p}.\end{aligned}$$

In the three cases, there always exist constants $A, B > 0$ such that

$$\|\mathbf{u}(2m)\| \leqslant A\|\mathbf{u}(n)\| + B.$$

Then

$$\|\mathbf{u}(2m)\|\sum_{n=m}^{2m} n^{-1} \leqslant A\sum_{n=m}^{2m} n^{-1}\|\mathbf{u}(n)\| + B\sum_{n=m}^{2m} n^{-1}.$$

We again use Lemma 2.7, statement (3) to obtain that

$$\|\mathbf{u}(2m)\| \leqslant 2A\sum_{n=m}^{2m} n^{-1}\|\mathbf{u}(n)\| + 4B\ln 2. \tag{3.2}$$

There are two cases of the exponent q. If $q = 1$, then from (3.2) we immediately have

$$\|\mathbf{u}(2m)\| \leqslant 2A\|\mathbf{u}\|_{w,1} + 4B\ln 2.$$

If $q > 1$, then we can find $s > 1$ with $1/s + 1/q = 1$. It results from (3.2) that

$$\begin{aligned}\|\mathbf{u}(2m)\| &\leqslant 2A\sum_{n=m}^{2m} n^{-1/s}n^{-1/q}\|\mathbf{u}(n)\| + 4B\ln 2 \\ &\leqslant 2A\Big(\sum_{n=m}^{2m} n^{-1}\Big)^{1/s}\Big(\sum_{n=m}^{2m} n^{-1}\|\mathbf{u}(n)\|^{q}\Big)^{1/q} + 4B\ln 2 \\ &\leqslant 2A(2\ln 2)^{1/s}\|\mathbf{u}\|_{w,q} + 4B\ln 2.\end{aligned}$$

These reveal that $\sup\{\|\mathbf{u}(2m)\|\colon\ m \in \mathbb{Z}_{\geqslant\kappa}\} < \infty$. Note that

$$\mathbf{u}(2m+1) = \mathcal{A}(2m+1, 2m)\mathbf{u}(2m) + (2m)^{-1}\mathcal{A}(2m+1, 2m)\mathbf{f}(2m),$$

which implies, by (2.1), that

$$\begin{aligned}\|\mathbf{u}(2m+1)\| &\leqslant M2^{\omega}(\|\mathbf{u}(2m)\| + (2m)^{-1}\|\mathbf{f}(2m)\|)\\ &= M2^{\omega}(\|\mathbf{u}(2m)\| + (2m)^{-1+1/p}(2m)^{-1/p}\|\mathbf{f}(2m)\|)\\ &\leqslant M2^{\omega}(\|\mathbf{u}(2m)\| + 2^{-1+1/p}\|\mathbf{f}\|_{w,p})\end{aligned}$$

and so the supremum $\sup\{\|\mathbf{u}(2m+1)\|\colon\ m \in \mathbb{Z}_{\geqslant\kappa}\} < \infty$. The proof is complete. □

3.2. Necessary conditions. The following result isolates necessary conditions for discrete evolution families to be polynomially expansive and for those to be expansive in the ordinary sense.

Proposition 3.2. *Let $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ be a discrete evolution family, which is polynomially bounded (that is* (2.1) *holds). Let $1 \leqslant p \leqslant q \leqslant \infty$. Then the following assertions hold.*

(1) *If $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ is expansive in the ordinary sense, then the pair $(\ell^1_w(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$ is admissible to equation* (2.2).

(2) *If $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ is polynomially expansive, then the pair $(\ell^p_{\rm wt}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\rm wt}(\mathbb{Z}_{\geqslant 1}))$ is admissible to equation* (2.2).

P r o o f. The proof of the first item is left to the reader. The second item is proved as follows. Let $\kappa \in \mathbb{Z}_{\geqslant 1}$ and $\mathbf{f} \in \ell^p_{\rm wt}(\mathbb{Z}_{\geqslant\kappa})$. We define the sequence

$$\mathbf{u}\colon\ \mathbb{Z}_{\geqslant\kappa} \to \mathbb{X}, \quad \mathbf{u}(n) := -\sum_{j=n}^{\infty} j^{-1}\mathcal{A}(j,n)^{-1}\mathbf{f}(j). \tag{3.3}$$

A direct computation shows that the pair $(\mathbf{f},\mathbf{u})$ verifies equation (2.2) and

$$\|\mathbf{u}(n)\| \leqslant Kn^{\alpha}\sum_{j=n}^{\infty} j^{-\alpha-1}\|\mathbf{f}(j)\|. \tag{3.4}$$

To show that $\mathbf{u} \in \ell^q_{\rm wt}(\mathbb{Z}_{\geqslant\kappa})$ we make use of Lemma 2.6. To that aim, we consider three cases of the exponent p.

Case 1: If $p = \infty$, then it follows from (3.4) that

$$\|\mathbf{u}(n)\| \leqslant K\|\mathbf{f}\|_\infty n^{\alpha}\sum_{j=n}^{\infty} j^{-\alpha-1} \leqslant 2^{\alpha+1}\alpha^{-1}K\|\mathbf{f}\|_\infty \quad \text{(by Lemma 2.7 (1))}.$$

Case 2: If $p = 1$, then it follows from (3.4) that

$$\|\mathbf{u}(n)\| \leqslant K\sum_{j=n}^{\infty} j^{-1}\|\mathbf{f}(j)\| \leqslant K\|\mathbf{f}\|_{w,1},$$

which gives $\mathbf{u} \in \ell^{\infty}(\mathbb{Z}_{\geqslant\kappa})$. Also by (3.4), we have

$$\begin{aligned}\sum_{n=\kappa}^{\infty} n^{-1}\|\mathbf{u}(n)\| \leqslant K\sum_{n=\kappa}^{\infty} n^{\alpha-1}\sum_{j=n}^{\infty} j^{-\alpha-1}\|\mathbf{f}(j)\| &= K\sum_{j=\kappa}^{\infty} j^{-\alpha-1}\|\mathbf{f}(j)\|\sum_{n=\kappa}^{j} n^{\alpha-1}\\ &\leqslant 2\alpha^{-1}K\|\mathbf{f}\|_{w,1} \quad \text{(by Lemma 2.7 (2))}\end{aligned}$$

and consequently $\mathbf{u} \in \ell^1_w(\mathbb{Z}_{\geqslant\kappa})$.

Case 3: If $1 < p < \infty$, then we can choose $r > 1$ with $1/r + 1/p = 1$. For choosing $a, b > 0$ with $a + b = 1$, inequality (3.4) is equivalent to

$$\begin{aligned}\|\mathbf{u}(n)\| &\leqslant Kn^{\alpha}\sum_{j=n}^{\infty} j^{-b\alpha-1/r} j^{-a\alpha-1/p}\|\mathbf{f}(j)\|\\ &\leqslant Kn^{\alpha}\Big(\sum_{j=n}^{\infty} j^{-rb\alpha-1}\Big)^{1/r}\Big(\sum_{j=n}^{\infty} j^{-pa\alpha-1}\|\mathbf{f}(j)\|^p\Big)^{1/p}\\ &\leqslant Kn^{a\alpha}2^{b\alpha+1/r}(rb\alpha)^{-1/r}\Big(\sum_{j=n}^{\infty} j^{-pa\alpha-1}\|\mathbf{f}(j)\|^p\Big)^{1/p} \quad \text{(by Lemma 2.7 (1))},\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{n=\kappa}^{\infty} n^{-1}\|\mathbf{u}(n)\|^p &\leqslant K^p 2^{pb\alpha+p/r}(rb\alpha)^{-p/r}\sum_{n=\kappa}^{\infty} n^{pa\alpha-1}\sum_{j=n}^{\infty} j^{-pa\alpha-1}\|\mathbf{f}(j)\|^p\\ &= K^p 2^{pb\alpha+p/r}(rb\alpha)^{-p/r}\sum_{j=\kappa}^{\infty} j^{-pa\alpha-1}\|\mathbf{f}(j)\|^p\sum_{n=\kappa}^{j} n^{pa\alpha-1}\\ &\leqslant K^p 2^{pb\alpha+p/r+1}(rb\alpha)^{-p/r}(pa\alpha)^{-1}\sum_{j=\kappa}^{\infty} j^{-1}\|\mathbf{f}(j)\|^p\\ &\qquad\qquad \text{(by Lemma 2.7 \ (2))},\end{aligned}$$

which in turns gives $\mathbf{u} \in \ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant\kappa})$.

The remaining task is to prove the uniqueness. Suppose that we can take $\mathbf{v} \in \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant\kappa})$ such that $(\mathbf{f}, \mathbf{v})$ also verifies equation (2.2). Then $\mathbf{u}(m) - \mathbf{v}(m) = \mathcal{A}(m,\kappa)(\mathbf{u}(\kappa) - \mathbf{v}(\kappa))$. It follows from the polynomial expansiveness that

$$\|\mathbf{u}(m) - \mathbf{v}(m)\| \geqslant Km^{\alpha}\kappa^{-\alpha}\|\mathbf{u}(\kappa) - \mathbf{v}(\kappa)\|.$$

From the above inequality and the condition $\mathbf{u}, \mathbf{v} \in \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant\kappa})$, we must have $\mathbf{u}(\kappa) - \mathbf{v}(\kappa) = 0$, which gives $\mathbf{u} \equiv \mathbf{v}$. $\square$

3.3. Steps toward sufficient conditions. The natural question arises whether, in the polynomial setting, the converse statement of Proposition 3.2 holds. To give the answer to this question, we need the auxiliary results below. Each result isolates a certain form of behavior of discrete evolution families.

3.3.1. Bijectivity. It turns out that the admissibility of the pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}))$ can imply the bijectivity of discrete evolution operators.

Proposition 3.3. *If the pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}))$ is admissible, then for every $\beta \in \mathbb{Z}_{\geqslant 1}$ the operator $\mathcal{A}(\beta, 1)$ is bijective.*

P r o o f. Let $\beta \in \mathbb{Z}_{\geqslant 1}$. The conclusion is trivial if $\beta = 1$. In the rest, we only consider $\beta > 1$.

Injectivity. Let $x \in \mathbb{X}$ with the property that $\mathcal{A}(\beta, 1)x = 0$. We must show that $x = 0$. Indeed, let us define the sequences $\mathbf{u}_1, \mathbf{u}_2 \colon \mathbb{Z}_{\geqslant 1} \to \mathbb{X}$ by setting

$$\mathbf{u}_1(j) := 0, \quad \mathbf{u}_2(j) := \mathcal{A}(j, 1)x, \quad j \in \mathbb{Z}_{\geqslant 1}.$$

It is clear that $\mathbf{u}_1, \mathbf{u}_2 \in \ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1})$, and furthermore both $(0, \mathbf{u}_1)$ and $(0, \mathbf{u}_2)$ verify equation (2.2). It follows from the uniqueness of admissibility that $\mathbf{u}_1 = \mathbf{u}_2$, and so $x = \mathbf{u}_2(1) = \mathbf{u}_1(1) = 0$.

Surjectivity. Take an arbitrary $y \in \mathbb{X}$. We must show that there exists $x \in \mathbb{X}$ such that $\mathcal{A}(\beta, 1)x = y$. We also consider the sequences

$$\mathbf{f}_1(j) := \begin{cases} -y & \text{if } j = \beta, \\ 0 & \text{if } j > \beta, \end{cases} \qquad \mathbf{u}(j) := \begin{cases} \beta^{-1}y & \text{if } j = \beta, \\ 0 & \text{if } j > \beta. \end{cases}$$

A direct computation shows that the pair $(\mathbf{f}_1, \mathbf{u}) \in (\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant \beta}), \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant \beta}))$ verifies equation (2.2) for every $(m, n) \in \Omega_{\geqslant \beta}$. The sequence $\mathbf{f}_1$ can be extended to $\mathbb{Z}_{\geqslant 1}$ by setting $\mathbf{f}_1(j) = 0$ for $j \in \{1, \ldots, \beta - 1\}$. Since the pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}))$ is admissible, we can find a unique $\mathbf{v} \in \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1})$ such that the pair $(\mathbf{f}_1, \mathbf{v})$ satisfies equation (2.2). Due to the uniqueness of the output, $\mathbf{u}$ must be the restriction of $\mathbf{v}$ on $\mathbb{Z}_{\geqslant \beta}$, that is $\mathbf{u} \equiv \mathbf{v}\big|_{\mathbb{Z}_{\geqslant \beta}}$. In particular, we have $y = \beta \mathbf{u}(\beta) = \beta \mathbf{v}(\beta) = \beta \mathcal{A}(\beta, 1)\mathbf{v}(1) = \mathcal{A}(\beta, 1)[\beta \mathbf{v}(1)]$, which completes the proof. $\square$

Thus, we can prove the result below, whose proof makes use of the property that $\mathcal{A}(\alpha, 1) = \mathcal{A}(\alpha, \beta)\mathcal{A}(\beta, 1)$.

Proposition 3.4. *If the pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}))$ is admissible, then the operator $\mathcal{A}(\alpha, \beta)$ is bijective for all fixed $(\alpha, \beta) \in \Omega_{\geqslant 1}$.*

3.3.2. Below boundedness. It turns out that discrete evolution operators are bounded from below if the pair $(\ell^p_{\rm wt}(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$ is admissible and moreover, the boundedness is uniform.

Proposition 3.5. *Let $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ be a discrete evolution family, which is polynomially bounded (that is (2.1) holds) and let $p \in [1,\infty]$. If the pair $(\ell^p_{\rm wt}(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$ is admissible, then there exists a constant $L>0$ such that*

$$\|\mathcal{A}(\alpha,\beta)x\| \geqslant L\|x\|, \quad x\in\mathbb{X},\ (\alpha,\beta)\in\Omega_{\geqslant 1}.$$

P r o o f. Let $x\in\mathbb{X}$ and $\alpha,\beta\in\mathbb{Z}_{\geqslant 1}$ such that $\alpha>\beta$. Consider the map $\zeta\colon\ \mathbb{Z}_{\geqslant 1}\to\mathbb{R}$ given by

$$\zeta(n) = \begin{cases} n^{-1} & \text{if } n\in\{\alpha+1,\dots,2\alpha\},\\ 0, & \text{otherwise.}\end{cases}$$

Let $\mathbf{f}_\zeta$, $\mathbf{u}_\zeta\colon\ \mathbb{Z}_{\geqslant\beta}\to\mathbb{X}$ be the sequences given by (2.3), that is

$$\mathbf{f}_\zeta(n) = \begin{cases} -\mathcal{A}(n,\beta)x, & n\in\{\alpha+1,\dots,2\alpha\},\\ 0, & \text{otherwise,}\end{cases}$$

$$\mathbf{u}_\zeta(n) = \begin{cases} \displaystyle\sum_{j=\alpha+1}^{2\alpha} j^{-1}\mathcal{A}(n,\beta)x, & \beta\leqslant n\leqslant\alpha,\\ \displaystyle\sum_{j=n}^{2\alpha} j^{-1}\mathcal{A}(n,\beta)x, & \alpha<n\leqslant 2\alpha,\\ 0, & \text{otherwise.}\end{cases}$$

By (2.1), we estimate

$$\|\mathcal{A}(n,\beta)x\| \leqslant Mn^\omega\alpha^{-\omega}\|\mathcal{A}(\alpha,\beta)x\| \leqslant M2^\omega\|\mathcal{A}(\alpha,\beta)x\|, \quad n\in\{\alpha+1,\dots,2\alpha\}, \tag{3.5}$$

which implies, by the first inequality in (3.5), that

$$\begin{aligned}\sum_{n=\alpha+1}^{2\alpha} n^{-1}\|\mathcal{A}(n,\beta)x\|^p &\leqslant M^p\|\mathcal{A}(\alpha,\beta)x\|^p\sum_{n=\alpha+1}^{2\alpha} n^{-1+p\omega}\alpha^{-p\omega}\\ &\leqslant M^p\|\mathcal{A}(\alpha,\beta)x\|^p(p\omega)^{-1}2^{p\omega+1},\end{aligned} \tag{3.6}$$

where the last inequality uses Lemma 2.7, statement (2). From (3.5) and (3.6), the function $\mathbf{f}_\zeta\in\ell^p_{\rm wt}(\mathbb{Z}_{\geqslant\beta})$ with the norm

$$\|\mathbf{f}_\zeta\|_{w,p} \leqslant \begin{cases} M2^\omega\|\mathcal{A}(\alpha,\beta)x\| & \text{if } p=\infty,\\ M(p\omega)^{-1/p}2^{\omega+1/p}\|\mathcal{A}(\alpha,\beta)x\| & \text{if } p<\infty.\end{cases} \tag{3.7}$$

Let us define the function $\hat{\mathbf{f}} \in \ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1})$ by setting

$$\hat{\mathbf{f}}(j) = \begin{cases} \mathbf{f}_\zeta(j), & j \in \mathbb{Z}_{\geqslant \alpha}, \\ 0, & j \in \{1, 2, \dots, \alpha\}. \end{cases}$$

Since the pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$ is admissible, there exists $\widehat{\mathbf{u}} \in \ell^\infty(\mathbb{Z}_{\geqslant 1})$ such that the pair $(\hat{\mathbf{f}}, \widehat{\mathbf{u}})$ verifies equation (2.2), and consequently

$$\widehat{\mathbf{u}}(\alpha) - \mathcal{A}(\alpha, \beta)\widehat{\mathbf{u}}(\beta) = \sum_{j=\beta}^{\alpha-1} j^{-1}\mathcal{A}(\alpha, j)\hat{\mathbf{f}}(j) = 0.$$

The last equality is equivalent to the fact that

$$\mathcal{A}(\alpha, \beta)\widehat{\mathbf{u}}(\beta) = \widehat{\mathbf{u}}(\alpha) = \mathbf{u}_\zeta(\alpha) = \sum_{j=\alpha+1}^{2\alpha} j^{-1}\mathcal{A}(\alpha, \beta)x,$$

which implies, as $\mathcal{A}(\alpha, \beta)$ is injective, that

$$\widehat{\mathbf{u}}(\beta) = \sum_{j=\alpha+1}^{2\alpha} j^{-1}x.$$

Thus, we can write

$$\|\widehat{\mathbf{u}}\|_\infty \geqslant \|\widehat{\mathbf{u}}(\beta)\| \geqslant \|x\| \sum_{j=\alpha+1}^{2\alpha} \frac{1}{2\alpha} = \frac{1}{2}\|x\|. \tag{3.8}$$

It was discussed in Remark 2.12 that the operator $\mathcal{O}^{p,\infty}_\delta$ is bounded, and so we can write

$$\|\widehat{\mathbf{u}}\|_\infty \leqslant \|\mathcal{O}^{p,\infty}_\delta\| \, \|\hat{\mathbf{f}}\|_{w,p}. \tag{3.9}$$

From (3.7), (3.8), (3.9), we obtain the desired conclusion. □

3.3.3. Contraction. We show that the inverse of a discrete evolution operator is a contraction operator provided that either $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$ or $(\ell^1_w(\mathbb{Z}_{\geqslant 1}), \ell^q_w(\mathbb{Z}_{\geqslant 1}))$ is admissible.

For the admissibility of the pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$, we obtain the following result.

Proposition 3.6. *Let $1 < p \leqslant \infty$. Let $\{\mathcal{A}(m, n)\}_{(m,n)\in\Omega}$ be a discrete evolution family which is polynomially bounded (that is (2.1) holds). If the pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$ is admissible, then there exists $\lambda \in \mathbb{Z}_{\geqslant 1}$ such that*

$$\|\mathcal{A}(\lambda\beta, \beta)x\| \geqslant 2\|x\|, \quad \beta \in \mathbb{Z}_{\geqslant 1}, \ x \in \mathbb{X}. \tag{3.10}$$

P r o o f. Let $x \in \mathbb{X}$, $\beta \in \mathbb{Z}_{\geqslant 1}$ and $\lambda \in \mathbb{Z}_{>1}$. Consider the map $\zeta\colon \mathbb{Z}_{\geqslant\beta} \to \mathbb{R}$ given by

$$\zeta(n) = \begin{cases} n^{-1} & \text{if } \beta+1 \leqslant n \leqslant \lambda\beta, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbf{f}_\zeta$, $\mathbf{u}_\zeta\colon \mathbb{Z}_{\geqslant\beta} \to \mathbb{X}$ be the sequences given by (2.3), that is

$$\mathbf{f}_\zeta(n) = \begin{cases} -\mathcal{A}(n,\beta)x & \text{if } \beta+1 \leqslant n \leqslant \lambda\beta, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{u}_\zeta(n) = \begin{cases} \Big(\sum\limits_{j=\beta+1}^{\lambda\beta} j^{-1}\Big)x & \text{if } n=\beta, \\ 0 & \text{if } n \geqslant \lambda\beta+1, \\ \Big(\sum\limits_{j=n}^{\lambda\beta} j^{-1}\Big)\mathcal{A}(n,\beta)x & \text{if } \beta+1 \leqslant n \leqslant \lambda\beta. \end{cases}$$

It results from Proposition 3.5 that

$$\|\mathcal{A}(n,\beta)x\| \leqslant L^{-1}\|\mathcal{A}(\lambda\beta,\beta)x\|, \quad \beta+1 \leqslant n \leqslant \lambda\beta, \tag{3.11}$$

which gives

$$\begin{aligned}\sum_{n=\beta+1}^{\lambda\beta} n^{-1}\|\mathcal{A}(n,\beta)x\|^p &\leqslant L^{-p}\|\mathcal{A}(\lambda\beta,\beta)x\|^p \sum_{n=\beta+1}^{\lambda\beta} n^{-1} \\ &\leqslant L^{-p}\|\mathcal{A}(\lambda\beta,\beta)x\|^p 2\ln\lambda,\end{aligned} \tag{3.12}$$

where the last inequality uses Lemma 2.7, (3). Inequalities (3.11) and (3.12) reveal that the function $\mathbf{f}_\zeta \in \ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant\beta})$ with

$$\|\mathbf{f}_\zeta\|_{w,p} \leqslant \begin{cases} L^{-1}\|\mathcal{A}(\lambda\beta,\beta)x\| & \text{if } p=\infty, \\ L^{-1}\|\mathcal{A}(\lambda\beta,\beta)x\| 2^{1/p}(\ln\lambda)^{1/p} & \text{if } p<\infty. \end{cases} \tag{3.13}$$

Let us define the sequence $\hat{\mathbf{f}} \in \ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1})$ by setting

$$\hat{\mathbf{f}}(j) = \begin{cases} \mathbf{f}_\zeta(j), & j \in \mathbb{Z}_{\geqslant\beta}, \\ 0, & \text{otherwise.} \end{cases}$$

Since the pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$ is admissible, there exists $\widehat{\mathbf{u}} \in \ell^\infty(\mathbb{Z}_{\geqslant 1})$ such that the pair $(\hat{\mathbf{f}}, \widehat{\mathbf{u}})$ verifies equation (2.2), and furthermore $\widehat{\mathbf{u}}(\beta) = \mathbf{u}_\zeta(\beta)$. These allow us to write

$$\|\widehat{\mathbf{u}}\|_\infty \geqslant \|\widehat{\mathbf{u}}(\beta)\| = \sum_{j=\beta+1}^{\lambda\beta} j^{-1}\|x\| \geqslant \ln\Big(\frac{\lambda}{2}\Big)\|x\|,$$

where the last inequality uses the fact that $j^{-1} \geqslant \ln(1+j^{-1})$. It was discussed in Remark 2.12 that the operator $\mathcal{O}_\delta^{p,\infty}$ is bounded, and so

$$\ln\Bigl(\frac{\lambda}{2}\Bigr)\|x\| \leqslant \|\mathcal{O}_\delta^{p,\infty}\|\,\|\hat{\mathbf{f}}\|_{w,p} \leqslant \begin{cases} \|\mathcal{O}_\delta^{p,\infty}\|L^{-1}\|\mathcal{A}(\lambda\beta,\beta)x\| & \text{if } p=\infty,\\ \|\mathcal{O}_\delta^{p,\infty}\|L^{-1}\|\mathcal{A}(\lambda\beta,\beta)x\|2^{1/p}(\ln\lambda)^{1/p} & \text{if } p<\infty. \end{cases}$$

As $p>1$, the constant λ can be chosen so that inequality (3.10) holds. □

For the admissibility of the pair $(\ell_w^1(\mathbb{Z}_{\geqslant 1}),\ell_w^q(\mathbb{Z}_{\geqslant 1}))$, we also obtain the same conclusion.

Proposition 3.7. *Let* $1\leqslant q<\infty$. *Let* $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ *be a discrete evolution family, which is polynomially bounded (that is* (2.1) *holds). If the pair* $(\ell_w^1(\mathbb{Z}_{\geqslant 1}),\ell_w^q(\mathbb{Z}_{\geqslant 1}))$ *is admissible, then there exists* $\lambda\in\mathbb{Z}_{\geqslant 1}$ *such that* (3.10) *holds.*

P r o o f. Let $x\in\mathbb{X}\setminus\{0\}$ and $\lambda_2\geqslant\lambda_1\geqslant 2$. Note that Proposition 3.4 reveals that there always exists the operator $\mathcal{A}(\cdot,\cdot)^{-1}$. Thus, we can consider the sequences

$$\mathbf{f}\colon\ \mathbb{Z}_{\geqslant 1}\to\mathbb{X},\quad \mathbf{f}(n)=\begin{cases}\|\mathcal{A}(n,\lambda_1)x\|^{-1}\mathcal{A}(n,\lambda_1)x & \text{if } n\in\{\lambda_1,\dots,\lambda_2\},\\ 0, & \text{otherwise,}\end{cases}$$

$$\mathbf{u}\colon\ \mathbb{Z}_{\geqslant 1}\to\mathbb{X},$$

$$\mathbf{u}(n)=\begin{cases}-\displaystyle\sum_{j=\lambda_1}^{\lambda_2} j^{-1}\|\mathcal{A}(j,\lambda_1)x\|^{-1}\mathcal{A}(\lambda_1,n)^{-1}x & \text{if } n\in\{1,\dots,\lambda_1-1\},\\ -\displaystyle\sum_{j=n}^{\lambda_2} j^{-1}\|\mathcal{A}(j,\lambda_1)x\|^{-1}\mathcal{A}(n,\lambda_1)x, & \text{if } n\in\{\lambda_1,\dots,\lambda_2\},\\ 0, & \text{otherwise.}\end{cases}$$

It can be proved that the pair $(\mathbf{f},\mathbf{u})\in(\ell_w^1(\mathbb{Z}_{\geqslant 1}),\ell_w^q(\mathbb{Z}_{\geqslant 1}))$ and furthermore

$$\begin{aligned}\|\mathbf{f}\|_{w,1} &= \sum_{n=\lambda_1}^{\lambda_2} n^{-1}\\ \|\mathbf{u}\|_{w,q} &\geqslant \Bigl(\sum_{n=1}^{\lambda_1-1} n^{-1}\|\mathbf{u}(n)\|^q\Bigr)^{1/q}\\ &\geqslant \sum_{j=\lambda_1}^{\lambda_2} j^{-1}\|\mathcal{A}(j,\lambda_1)x\|^{-1}\Bigl(\sum_{n=1}^{\lambda_1-1} n^{-1}\|\mathcal{A}(\lambda_1,n)^{-1}x\|^q\Bigr)^{1/q}.\end{aligned}$$

It was discussed in Remark 2.12 that the operator $\mathcal{O}_\delta^{1,q}$ is bounded, and so we can write

$$\sum_{j=\lambda_1}^{\lambda_2} j^{-1}\|\mathcal{A}(j,\lambda_1)x\|^{-1}\Bigl(\sum_{n=1}^{\lambda_1-1} n^{-1}\|\mathcal{A}(\lambda_1,n)^{-1}x\|^q\Bigr)^{1/q} \leqslant \|\mathcal{O}_\delta^{1,q}\|\sum_{n=\lambda_1}^{\lambda_2} n^{-1}.$$

In particular, with $\lambda_2 = 2\lambda_1$ we have

$$\sum_{j=\lambda_1}^{2\lambda_1} j^{-1}\|\mathcal{A}(j,\lambda_1)x\|^{-1}\Big(\sum_{n=1}^{\lambda_1-1} n^{-1}\|\mathcal{A}(\lambda_1,n)^{-1}x\|^q\Big)^{1/q} \leqslant \|\mathcal{O}_\delta^{1,q}\| \sum_{n=\lambda_1}^{2\lambda_1} n^{-1}$$
$$\leqslant \|\mathcal{O}_\delta^{1,q}\| 2\ln 2 \quad \text{(by Lemma 2.7 (3))},$$

which implies, as $\|\mathcal{A}(j,\lambda_1)x\| \leqslant M2^\omega\|x\|$ for $j \in \{\lambda_1,\dots,2\lambda_1\}$, that

$$\|\mathcal{O}_\delta^{1,q}\| 2\ln 2 \geqslant M^{-1}2^{-\omega}\|x\|^{-1}\sum_{j=\lambda_1}^{2\lambda_1} j^{-1}\Big(\sum_{n=1}^{\lambda_1-1} n^{-1}\|\mathcal{A}(\lambda_1,n)^{-1}x\|^q\Big)^{1/q}$$
$$\geqslant M^{-1}2^{-\omega-1}\|x\|^{-1}\Big(\sum_{n=1}^{\lambda_1-1} n^{-1}\|\mathcal{A}(\lambda_1,n)^{-1}x\|^q\Big)^{1/q} \quad \text{(by Lemma 2.7 (3))}.$$

The last inequality can be used to show that there exists a constant C such that

$$\sum_{n=\lambda_3}^{r\lambda_3} n^{-1}\|\mathcal{A}(r\lambda_3,n)^{-1}x\|^q \leqslant C\|x\|^q, \quad r,\lambda_3 \in \mathbb{Z}_{\geqslant 1},\ x \in \mathbb{X}\setminus\{0\}.$$

Since $\mathcal{A}(n,\lambda_3)\mathcal{A}(r\lambda_3,\lambda_3)^{-1} = \mathcal{A}(r\lambda_3,n)^{-1}$, by Propositions 3.4 and 3.5 we estimate

$$L\|\mathcal{A}(r\lambda_3,\lambda_3)^{-1}x\| \leqslant \|\mathcal{A}(r\lambda_3,n)^{-1}x\|,$$

and so

$$C\|x\|^q \geqslant \sum_{n=\lambda_3}^{r\lambda_3} n^{-1}\|\mathcal{A}(r\lambda_3,n)^{-1}x\|^q \geqslant L^q\|\mathcal{A}(r\lambda_3,\lambda_3)^{-1}x\|^q \sum_{n=\lambda_3}^{r\lambda_3} n^{-1}$$
$$\geqslant L^q\|\mathcal{A}(r\lambda_3,\lambda_3)^{-1}x\|^q \ln r,$$

where the last inequality uses the fact that $n^{-1} \geqslant \ln(1+n^{-1})$. Thus, the constant r can be chosen so that inequality (3.10) holds. □

With all preparation in place, we can now state and prove discrete characterizations of discrete evolution families which are polynomially expansive and also those which are expansive in the ordinary sense.

3.4. Polynomial expansiveness. We start with the polynomial expansiveness.

Theorem 3.8. *Let $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ be a discrete evolution family which is polynomially bounded (that is* (2.1) *holds). Let $1 \leqslant p \leqslant q \leqslant \infty$ with $(p,q) \neq (1,\infty)$. Then $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ is polynomially expansive if and only if the pair $(\ell^p_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant 1}))$ is admissible to equation* (2.2).

P r o o f. The necessity follows from Proposition 3.2, statement (2). The remaining task is to prove the sufficiency. By Propositions 3.6, 3.7, and 3.1, there exists $\lambda \in \mathbb{Z}_{\geqslant 1}$ such that

$$\|\mathcal{A}(\lambda\beta,\beta)x\| \geqslant 2\|x\|, \quad \beta \in \mathbb{Z}_{\geqslant 1},\ x \in \mathbb{X}.$$

Let $t \in \mathbb{Z}_{\geqslant 1}$, $\beta \in \mathbb{Z}_{\geqslant 1}$ and $x \in \mathbb{X}$. Set $Y := \{j \in \mathbb{Z}_{\geqslant 0} \colon t\lambda^{-j} \geqslant 1\}$. Denoting $k := \max Y$, we have

$$\lambda^k \leqslant t < \lambda^{k+1}.$$

Since $\mathcal{A}(\lambda^{k+1}\beta,\beta)x = \mathcal{A}(\lambda^{k+1}\beta,t\beta)\mathcal{A}(t\beta,\beta)x$, we estimate

$$\|\mathcal{A}(\lambda^{k+1}\beta,\beta)x\| \leqslant M\left(\frac{\lambda^{k+1}}{t}\right)^{\omega}\|\mathcal{A}(t\beta,\beta)x\| \leqslant M\lambda^{\omega}\|\mathcal{A}(t\beta,\beta)x\|.$$

The last inequality gives

$$\begin{aligned}\|\mathcal{A}(t\beta,\beta)x\| &\geqslant M^{-1}\lambda^{-\omega}\|\mathcal{A}(\lambda^{k+1}\beta,\beta)x\| \geqslant M^{-1}\lambda^{-\omega}2^{k+1}\|x\|\\ &\geqslant M^{-1}\lambda^{-\omega}2^{\ln t/\ln\lambda}\|x\|,\end{aligned}$$

as wanted. $\square$

The polynomial expansiveness cannot ensure the admissibility of the pair $(\ell^p_w(\mathbb{Z}_{\geqslant 1}), \ell^q_w(\mathbb{Z}_{\geqslant 1}))$ for all exponents $p, q \in [1,\infty]$, as shows the example below.

Example 3.9. Let $\mathbb{X} = \mathbb{R}$. The evolution family

$$\mathcal{A}(m,n)x = \frac{m+1}{n+1}x, \quad x \in \mathbb{R},\ (m,n) \in \Omega$$

is polynomially bounded and moreover, it is polynomially expansive. For further details about examples that reach this property, i.e. being exponentially/polynomially bounded and unstable or expansive as an equivalent form, the reader can refer to the papers [14], [15]. The pair $(\ell^2_w(\mathbb{Z}_{\geqslant 1}), \ell^1_w(\mathbb{Z}_{\geqslant 1}))$ is not admissible since the sequence $\mathbf{f}(n) = 1/\ln(n+1)$ belongs to $\ell^2_w(\mathbb{Z}_{\geqslant 1})$ and there is no $\mathbf{u} \in \ell^1_w(\mathbb{Z}_{\geqslant 1})$ such that the pair $(\mathbf{f}, \mathbf{u})$ verifies equation (2.2).

3.5. Ordinary expansiveness.

Theorem 3.10. *Let $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ be a discrete evolution family which is polynomially bounded (that is* (2.1) *holds). Then $\{\mathcal{A}(m,n)\}_{(m,n)\in\Omega}$ is expansive in the ordinary sense if and only if the pair $(\ell^1_w(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$ is admissible to equation* (2.2).

P r o o f. The necessity uses Proposition 3.2, statement (1), while the sufficiency follows from Propositions 3.4 and 3.5. □

Note that there are different characterizations of instability in the papers due to Minh et al., see [12], and Slyusarchuk, see [18].

4. Continuous time

In this section, we characterize continuous evolution families which are expansive in terms of the solvability of equation (1.4). Our proof makes use of discrete characterizations proved in the previous section. This use is motivated by Remark 2.5 which says that if $\{U(t,s)\}_{(t,s)\in\Delta}$ is a continuous evolution family, then the operators $\mathcal{A}(m,n)=U(m,n)$, where $(m,n)\in\Omega$, form a discrete evolution family.

4.1. Necessary conditions. This section establishes necessary conditions for the concepts of expansiveness.

Proposition 4.1. *Let $\{U(t,s)\}_{(t,s)\in\Delta}$ be a polynomially bounded evolution family (that is* (2.1) *holds). Let $1\leqslant p\leqslant q\leqslant\infty$. Then the following assertions hold.*

(1) *If $\{U(t,s)\}_{(t,s)\in\Delta}$ is expansive in the ordinary sense, then the pair $(L^1_w(\mathbb{R}_{\geqslant 1}), L^\infty(\mathbb{R}_{\geqslant 1}))$ is admissible to equation* (1.4).
(2) *If $\{U(t,s)\}_{(t,s)\in\Delta}$ is polynomially expansive, then the pair $(L^p_{\rm wt}(\mathbb{R}_{\geqslant 1}), L^q_{\rm wt}(\mathbb{R}_{\geqslant 1}))$ is admissible to equation* (1.4).

P r o o f. Let $r\in\mathbb{R}_{\geqslant 1}$. For $f\in L^p_{\rm wt}(\mathbb{R}_{\geqslant r})$, we define the function

$$u\colon\ \mathbb{R}_{\geqslant r}\to\mathbb{X},\quad f(t)=-\int_t^\infty \tau^{-1}U(\tau,t)^{-1}f(\tau)\,\mathrm{d}\tau. \tag{4.1}$$

Using the same arguments as in the proof of Proposition 3.2, we obtain the desired conclusion. □

4.2. Admissibility in continuous time. In what follows, the symbol $\lfloor t\rfloor$ stands for the greatest integer less than or equal to t. We isolate a necessary condition for the admissibility in continuous time.

Proposition 4.2. *Let $\{U(t,s)\}_{(t,s)\in\Delta}$ be a polynomially bounded evolution family (that is* (2.1) *holds). Let $1\leqslant p\leqslant q\leqslant\infty$. If the pair $(L^p_{\rm wt}(\mathbb{R}_{\geqslant 1}), L^q_{\rm wt}(\mathbb{R}_{\geqslant 1}))$ is admissible to equation* (1.4), *then* (i) *the pair $(\ell^p_{\rm wt}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\rm wt}(\mathbb{Z}_{\geqslant 1}))$ is admissible to equation* (2.2) *and* (ii) *the operator $U(t,s)$ is surjective for all fixed $t\geqslant s\geqslant 1$.*

P r o o f. Consider the function

$$\varphi\colon \mathbb{R}_{\geqslant 0} \to [0,2], \quad \varphi(t) = \begin{cases} 4t & \text{if } t \in [0,1/2], \\ 4-4t & \text{if } t \in [1/2,1], \\ 0, & \text{otherwise.} \end{cases}$$

To show that the operator $U(t,s)$ is surjective for all fixed $t \geqslant s \geqslant 1$, it is enough to prove that the operator $U(\beta,1)$ is surjective for all fixed $\beta \geqslant 1$. Indeed, take an arbitrary $\beta > 1$ and $x \in \mathbb{X}$. Consider the function

$$\zeta\colon \mathbb{R}_{\geqslant 1} \to \mathbb{X}, \quad \zeta(t) = \begin{cases} \varphi(t-\beta) & \text{if } \beta \leqslant t \leqslant \beta+1, \\ 0, & \text{otherwise.} \end{cases}$$

Let f_ζ, u_ζ be the functions given by (2.4). We can use Lemma 2.14 to show that the pair (f_ζ, u_ζ) verifies equation (1.4) for every $t \geqslant s \geqslant \beta$. Note that f_ζ can be extended to $\mathbb{R}_{\geqslant 1}$ by setting $f_\zeta(t) = 0$ for $t \in [1,\beta]$ and then $f_\zeta \in L^p_{\rm wt}(\mathbb{R}_{\geqslant 1})$. Since the pair $(L^p_{\rm wt}(\mathbb{R}_{\geqslant 1}), L^q_{\rm wt}(\mathbb{R}_{\geqslant 1}))$ is admissible, there is $v \in L^q_{\rm wt}(\mathbb{R}_{\geqslant 1})$ such that the pair (f_ζ, v) satisfies equation (1.4) for every $t \geqslant s \geqslant 1$. In particular,

$$v(\beta) - U(\beta,1)v(1) = \int_1^\beta \tau^{-1} U(\beta,\tau) f_\zeta(\tau)\,\mathrm{d}\tau = 0,$$

which gives $v(\beta) = U(\beta,1)v(1)$. On the other hand, a simple computation shows that $u_\zeta \in L^q_{\rm wt}(\mathbb{R}_{\geqslant 1})$. Due to the admissibility of $(L^p_{\rm wt}(\mathbb{R}_{\geqslant 1}), L^q_{\rm wt}(\mathbb{R}_{\geqslant 1}))$, the equality $u_\zeta(t) = v(t)$ holds for $t \in \mathbb{R}_{\geqslant \beta}$. Thus, we can write $x = u_\zeta(\beta) = v(\beta) = U(\beta,1)v(1)$ and so $U(\beta,1)$ is surjective.

We prove that the pair $(\ell^p_{\rm wt}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\rm wt}(\mathbb{Z}_{\geqslant 1}))$ is admissible. Indeed, take an arbitrary $\kappa \in \mathbb{Z}_{\geqslant 1}$ and $\mathbf{f} \in \ell^p_{\rm wt}(\mathbb{Z}_{\geqslant \kappa})$. We define the function

$$g\colon \mathbb{R}_{\geqslant \kappa} \to \mathbb{X}, \quad g(t) = t\lfloor t\rfloor^{-1}\varphi(t - \lfloor t\rfloor)U(t,\lfloor t\rfloor)\mathbf{f}(\lfloor t\rfloor).$$

First, we state the following.

Claim: The function $g \in L^p_{\rm wt}(\mathbb{R}_{\geqslant \kappa})$.

Indeed, since $t < \lfloor t\rfloor + 1$, we have

$$\|g(t)\| \leqslant 2\varphi(t-\lfloor t\rfloor)\|U(t,\lfloor t\rfloor)\mathbf{f}(\lfloor t\rfloor)\| \leqslant 4\|U(t,\lfloor t\rfloor)\mathbf{f}(\lfloor t\rfloor)\| \leqslant M2^{\omega+2}\|\mathbf{f}(\lfloor t\rfloor)\|.$$

There are two cases of the exponent p. If $p = \infty$, it follows immediately from above that $g \in L^\infty(\mathbb{R}_{\geqslant 1})$. If $p < \infty$, then

$$\begin{aligned} \int_\kappa^\infty t^{-1}\|g(t)\|^p\,\mathrm{d}t &\leqslant M^p 2^{p\omega+2p}\sum_{j=\kappa}^\infty \int_j^{j+1} t^{-1}\|\mathbf{f}(\lfloor t\rfloor)\|^p\,\mathrm{d}t \\ &= M^p 2^{p\omega+2p}\sum_{j=\kappa}^\infty \|\mathbf{f}(j)\|^p \ln(1+j^{-1}) \leqslant M^p 2^{p\omega+2p}\|\mathbf{f}\|^p_{w,p} < \infty, \end{aligned}$$

which completes the claim.

Since the pair $(L^p_{\mathrm{wt}}(\mathbb{R}_{\geqslant 1}), L^q_{\mathrm{wt}}(\mathbb{R}_{\geqslant 1}))$ is admissible, there exists a unique $v \in L^q_{\mathrm{wt}}(\mathbb{R}_{\geqslant \kappa})$ such that the pair (g, v) verifies equation (1.4) for every $t \geqslant s \geqslant \kappa$. In particular, with $(m, n) \in \Omega_{\geqslant \kappa}$, this equation reduces to

$$\begin{aligned} v(m) - U(m,n)v(n) &= \sum_{j=n}^{m-1} \int_j^{j+1} \lfloor \tau \rfloor^{-1} \varphi(\tau - \lfloor \tau \rfloor) U(m, \lfloor \tau \rfloor) \mathbf{f}(\lfloor \tau \rfloor) \, \mathrm{d}\tau \\ &= \sum_{j=n}^{m-1} j^{-1} U(m,j) \mathbf{f}(j) \int_j^{j+1} \varphi(\tau - j) \, \mathrm{d}\tau = \sum_{j=n}^{m-1} j^{-1} U(m,j)\mathbf{f}(j). \end{aligned}$$

The last equality above prompts to set the sequence

$$\mathbf{u}\colon \mathbb{Z}_{\geqslant \kappa} \to \mathbb{X}, \quad \mathbf{u}(n) = v(n).$$

We have demonstrated that the pair $(\mathbf{f}, \mathbf{u})$ verifies equation (2.2) for every $(m, n) \in \Omega_{\geqslant \kappa}$. The remaining task is to show that $\mathbf{u} \in \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant \kappa})$. To that aim, we still use the fact that the pair (g, v) verifies equation (1.4) for every $t \geqslant s \geqslant \kappa$. In particular, for $s \in [n, n+1]$, where $n \in \mathbb{Z}_{\geqslant \kappa}$, we have

$$\begin{aligned} v(n+1) &= U(n+1, s)v(s) + \int_s^{n+1} \tau^{-1} U(n+1, \tau) g(\tau) \, \mathrm{d}\tau \\ &= U(n+1, s)v(s) + \int_s^{n+1} \lfloor \tau \rfloor^{-1} \varphi(\tau - \lfloor \tau \rfloor) U(n+1, \lfloor \tau \rfloor) \mathbf{f}(\lfloor \tau \rfloor) \, \mathrm{d}\tau, \end{aligned}$$

which infers, by (2.1), that

$$\begin{aligned} \|v(n+1)\| &\leqslant M 2^{\omega} \left(\|v(s)\| + \int_n^{n+1} \lfloor \tau \rfloor^{-1} \varphi(\tau - \lfloor \tau \rfloor) \|\mathbf{f}(\lfloor \tau \rfloor)\| \, \mathrm{d}\tau \right) \\ &= M 2^{\omega} (\|v(s)\| + n^{-1} \|\mathbf{f}(n)\|). \end{aligned}$$

If $q = \infty$, then we obtain the desired assertion. If $q < \infty$, then we can use the fact that $(a+b)^q \leqslant 2^{q-1}(a^q + b^q)$ to get

$$\|v(n+1)\|^q \leqslant M^q 2^{q\omega + q - 1} (\|v(s)\|^q + n^{-q} \|\mathbf{f}(n)\|^q).$$

Multiplying both sides by s^{-1} and then integrating with respect to s over the closed interval $[n, n+1]$, we get

$$\|v(n+1)\|^q \ln(1+n^{-1}) \leqslant M^q 2^{q\omega+q-1} \left(\int_n^{n+1} s^{-1} \|v(s)\|^q \, \mathrm{d}s + n^{-q} \|\mathbf{f}(n)\|^q \ln(1+n^{-1}) \right).$$

Hence,

$$\frac{\|v(n+1)\|^q}{n+1} \leqslant M^q 2^{q\omega+q-1}\left(\left[\ln\left(1+\frac{1}{n}\right)^{n+1}\right]^{-1}\int_n^{n+1} s^{-1}\|v(s)\|^q\,\mathrm{d}s + \frac{\|\mathbf{f}(n)\|^q}{(1+n)n^q}\right).$$

Note that as $\lim_{n\to\infty} \ln(1+1/n)^{n+1} = 1$ for n large enough the inequality

$$\ln(1+1/n)^{n+1} > 2^{-1}$$

holds. Meanwhile,

$$n^{-1-q}\|\mathbf{f}(n)\|^q \leqslant n^{-1}\|\mathbf{f}(n)\|^p\|\mathbf{f}\|_{w,p}^{q-p}.$$

We continue

$$\frac{\|v(n+1)\|^q}{n+1} \leqslant M^q 2^{q\omega+q-1}\left(2\int_n^{n+1} s^{-1}\|v(s)\|^q\,\mathrm{d}s + n^{-1}\|\mathbf{f}(n)\|^p\|\mathbf{f}\|_{w,p}^{q-p}\right).$$

As such, there exist constants A, B such that

$$(n+1)^{-1}\|v(n+1)\|^q \leqslant A\int_n^{n+1} s^{-1}\|v(s)\|^q\,\mathrm{d}s + Bn^{-1}\|\mathbf{f}(n)\|^p$$

and so

$$\begin{aligned}\sum_{n\geqslant\kappa}(n+1)^{-1}\|v(n+1)\|^q &\leqslant A\sum_{n\geqslant\kappa}\int_n^{n+1} s^{-1}\|v(s)\|^q\,\mathrm{d}s + B\sum_{n\geqslant\kappa} n^{-1}\|\mathbf{f}(n)\|^p\\ &\leqslant A\|v\|_{w,q} + B\|\mathbf{f}\|_{w,p} < \infty.\end{aligned}$$

These reveal that $\mathbf{u} \in \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant\kappa})$.

To prove the uniqueness, it is enough to show that if the sequence $\mathbf{w} \in \ell^q_{\mathrm{wt}}(\mathbb{Z}_{\geqslant\kappa})$ satisfies

$$\mathbf{w}(m) - U(m,n)\mathbf{w}(n) = 0, \quad (m,n) \in \Omega_{\geqslant\kappa},$$

then $\mathbf{w} = 0$. Indeed, for such sequence $\mathbf{w}$, let us define the function

$$\varphi\colon\ \mathbb{R}_{\geqslant\kappa} \to \mathbb{X}, \quad \varphi(t) = U(t,\lfloor t\rfloor)\mathbf{w}(\lfloor t\rfloor).$$

It can be checked that $\varphi \in L^q_{\mathrm{wt}}(\mathbb{R}_{\geqslant\kappa})$. We have

$$U(t,s)\varphi(s) = U(t,\lfloor s\rfloor)\mathbf{w}(\lfloor s\rfloor) = U(t,\lfloor t\rfloor)U(\lfloor t\rfloor,\lfloor s\rfloor)\mathbf{w}(\lfloor s\rfloor) = U(t,\rfloor t\rfloor)\mathbf{w}(\lfloor t\rfloor) = \varphi(t),$$

which means that the pair $(0,\varphi)$ verifies equation (1.4) for every $t \geqslant s \geqslant \kappa$. Due to the uniqueness, we must have $\varphi = 0$ and so $\mathbf{w} = 0$. $\square$

Now we are ready to state and prove continuous-time versions of Theorems 3.8 and 3.10.

4.3. Polynomial expansiveness. For the polynomial expansiveness, we obtain a continuous-time version of Theorem 3.8.

Theorem 4.3. *Let $\{U(t,s)\}_{(t,s)\in\Delta}$ be a polynomially bounded evolution family (that is* (2.1) *holds). Let $1 \leqslant p \leqslant q \leqslant \infty$ with $(p,q) \neq (1,\infty)$. Then the following assertions are equivalent.*

(1) *$\{U(t,s)\}_{(t,s)\in\Delta}$ is polynomially expansive.*
(2) *The pair $(L^p_{\rm wt}(\mathbb{R}_{\geqslant 1}), L^q_{\rm wt}(\mathbb{R}_{\geqslant 1}))$ is admissible to equation* (1.4).
(3) *The pair $(\ell^p_{\rm wt}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\rm wt}(\mathbb{Z}_{\geqslant 1}))$ is admissible to equation* (2.2) *and the operator $U(t,s)$ is surjective for all fixed $t \geqslant s \geqslant 1$.*

P r o o f. The implication (1) $\Rightarrow$ (2) follows from Proposition 4.1, while the implication (2) $\Rightarrow$ (3) derives from Proposition 4.2.

(3) $\Rightarrow$ (1). Let $t \geqslant s \geqslant 1$. Since the pair $(\ell^p_{\rm wt}(\mathbb{Z}_{\geqslant 1}), \ell^q_{\rm wt}(\mathbb{Z}_{\geqslant 1}))$ is admissible, there exist constants $K, \alpha > 0$ such that

$$\|U(m,n)x\| \geqslant K m^{\alpha} n^{-\alpha}\|x\|, \quad (m,n)\in\Omega_{\geqslant 1},\ x\in\mathbb{X}\setminus\{0\}.$$

By (2.1), we have

$$M2^{\omega}\|U(t,1)x\| \geqslant \|U(\lfloor t\rfloor+1,1)x\| \geqslant K(\lfloor t\rfloor+1)^{\alpha}\|x\| \geqslant Kt^{\alpha}\|x\|.$$

Hence, the operator $U(t,1)$ is injective and using the hypothesis it must be invertible for all $t \geqslant 1$. Since $U(t,1) = U(t,s)U(s,1)$, we deduce that the operator $U(t,s)$ is also invertible for all $t \geqslant s \geqslant 1$. Thus, there is $y \in \mathbb{X}$ with $x = U(s,\lfloor s\rfloor)y$, which infers, by (2.1), that

$$\|x\| \leqslant M2^{\omega}\|y\|. \tag{4.2}$$

There are two cases of t. If $t \leqslant \lfloor s\rfloor + 1$, then also by (2.1) we have

$$\begin{aligned} M2^{\omega}\|U(t,s)x\| &\geqslant \|U(\lfloor s\rfloor+1,s)x\| = \|U(\lfloor s\rfloor+1,\lfloor s\rfloor)y\| \\ &\geqslant K(\lfloor s\rfloor+1)^{\alpha}\lfloor s\rfloor^{-\alpha}\|y\| \geqslant Kt^{\alpha}s^{-\alpha}\|y\| \\ &\geqslant Kt^{\alpha}s^{-\alpha}M^{-1}2^{-\omega}\|x\| \quad (\text{by } (4.2)). \end{aligned}$$

If $t \geqslant \lfloor s\rfloor + 1$, then also by (2.1) we have

$$\begin{aligned} M2^{\omega}\|U(t,s)x\| &\geqslant \|U(\lfloor t\rfloor+1,s)x\| = \|U(\lfloor t\rfloor+1,\lfloor s\rfloor)y\| \\ &\geqslant K(\lfloor t\rfloor+1)^{\alpha}\lfloor s\rfloor^{-\alpha}\|y\| \geqslant Kt^{\alpha}s^{-\alpha}\|y\| \\ &\geqslant Kt^{\alpha}s^{-\alpha}M^{-1}2^{-\omega}\|x\| \quad (\text{by } (4.2)). \end{aligned}$$

Both cases show that $\{U(t,s)\}_{(t,s)\in\Delta}$ is polynomially expansive. $\square$

The polynomial expansiveness cannot ensure the admissibility of the pair $(L_w^p(\mathbb{R}_{\geqslant 1}), L_w^q(\mathbb{R}_{\geqslant 1}))$ for all exponents $p, q \in [1, \infty]$ as shows the example below.

Example 4.4. Let $\mathbb{X} = \mathbb{R}$. It is clear to see that the evolution family

$$U(t,s)x = \frac{t+1}{s+1}x, \quad x \in \mathbb{R}, \ (t,s) \in \Delta$$

is polynomially bounded and moreover it is polynomially expansive. The pair $(L_w^2(\mathbb{R}_{\geqslant 1}), L_w^1(\mathbb{R}_{\geqslant 1}))$ is not admissible since the function $f(t) = 1/\ln(t+1)$ belongs to $L_w^2(\mathbb{R}_{\geqslant 1})$ and there is no $u \in L_w^1(\mathbb{R}_{\geqslant 1})$ such that the pair (f, u) verifies equation (1.4).

4.4. Ordinary expansiveness. We end this paper by a continuous-time version of Theorem 3.10.

Theorem 4.5. *Let* $\{U(t,s)\}_{(t,s)\in\Delta}$ *be a polynomially bounded evolution family (that is* (2.1) *holds). Then the following assertions are equivalent.*

(1) $\{U(t,s)\}_{(t,s)\in\Delta}$ *is expansive in the ordinary sense.*
(2) *The pair* $(L_w^1(\mathbb{R}_{\geqslant 1}), L^\infty(\mathbb{R}_{\geqslant 1}))$ *is admissible to equation* (1.4).
(3) *The pair* $(\ell_w^1(\mathbb{Z}_{\geqslant 1}), \ell^\infty(\mathbb{Z}_{\geqslant 1}))$ *is admissible to equation* (2.2) *and the operator* $U(t,s)$ *is surjective for all fixed* $t \geqslant s \geqslant 1$.

P r o o f. The implication (1) $\Rightarrow$ (2) follows from Proposition 4.1, while the implication (2) $\Rightarrow$ (3) derives from Proposition 4.2. The implication (3) $\Rightarrow$ (1) is proved similarly as Theorem 4.3. $\square$

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References

[1] *L. Barreira, C. Valls*: Polynomial growth rates. Nonlinear Anal., Theory Methods Appl. *71* (2009), 5208–5219. zbl MR doi
[2] *F. Bătăran, C. Preda, P. Preda*: Extending some results of L. Barreira and C. Valls to the case of linear skew-product semiflows. Result. Math. *72* (2017), 965–978. zbl MR doi
[3] *C. V. Coffman, J. J. Schäffer*: Dichotomies for linear difference equations. Math. Ann. *172* (1967), 139–166. zbl MR doi
[4] *Ju. L. Daletskij, M. G. Krejn*: Stability of Solutions of Differential Equations in Banach Space. Translations of Mathematical Monographs 43. American Mathematical Society, Providence, 1974. zbl MR doi
[5] *P. V. Hai*: On the polynomial stability of evolution families. Appl. Anal. *95* (2016), 1239–1255. zbl MR doi
[6] *P. V. Hai*: Polynomial stability of evolution cocycles and Banach function spaces. Bull. Belg. Math. Soc.—Simon Stevin *26* (2019), 299–314. zbl MR doi

[7] *B. M. Levitan, V. V. Zhikov*: Almost Periodic Functions and Differential Equations. Cambridge University Press, Cambridge, 1982. zbl MR

[8] *T. Li*: Die Stabilitätsfrage bei Differenzengleichungen. Acta Math. *63* (1934), 99–141. (In German.) zbl MR doi

[9] *J. L. Massera, J. J. Schäffer*: Linear Differential Equations and Function Spaces. Pure and Applied Mathematics 21. Academic Press, New York, 1966. zbl MR

[10] *M. Megan, A. L. Sasu, B. Sasu*: Theorems of Perron type for uniform exponential dichotomy of linear skew-product semiflows. Bull. Belg. Math. Soc.—Simon Stevin *10* (2003), 1–21. zbl MR doi

[11] *M. Megan, B. Sasu, A. L. Sasu*: Exponential expansiveness and complete admissibility for evolution families. Czech. Math. J. *54* (2004), 739–749. zbl MR doi

[12] *N. V. Minh, F. Räbiger, R. Schnaubelt*: Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line. Integral Equations Oper. Theory *32* (1998), 332–353. zbl MR doi

[13] *O. Perron*: Die Stabilitätsfrage bei Differentialgleichungen. Math. Z. *32* (1930), 703–728. (In German.) zbl MR doi

[14] *I.-L. Popa, T. Ceauşu, M. Megan*: Nonuniform power instability and Lyapunov sequences. Appl. Math. Comput. *247* (2014), 969–975. zbl MR doi

[15] *I.-L. Popa, M. Megan, T. Ceauşu*: Exponential dichotomies for linear discrete-time systems in Banach spaces. Appl. Anal. Discrete Math. *6* (2012), 140–155. zbl MR doi

[16] *C. Preda, P. Preda, A.-P. Petre*: On the uniform exponential stability of linear skew-product three-parameter semiflows. Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér. *54* (2011), 269–279. zbl MR

[17] *B. Sasu*: New criteria for exponential expansiveness of variational difference equations. J. Math. Anal. Appl. *327* (2007), 287–297. zbl MR doi

[18] *V. E. Slyusarchuk*: Instability of difference equations with respect to the first approximation. Differ. Uravn. *22* (1986), 722–723. (In Russian.) zbl MR

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