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CONTROLLING PRODUCTS OF CURRENTS BY HIGHER POWERS OF PLURISUBHARMONIC FUNCTIONS

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Abstract. We discuss the existence of the current g^kT, $k \in \mathbb{N}$ for positive and closed currents T and unbounded plurisubharmonic functions g. Furthermore, a new type of weighted Lelong number is introduced under the name of weight k Lelong number.

Keywords: positive current; plurisubharmonic function; plurisubharmonic current

MSC 2020: 32U40

1. Introduction

Mathematical thinking always seeks the uneven cases when the objects seem to be unpredictable. From this point of view, the study of unbounded functions gained its paramount importance.

Let A be a closed subset of an open subset Ω of $\mathbb{C}^n$ and T be a positive closed current on Ω of bi-dimension (p,p). In 1993, Jean-Pierre Demailly in [4] discussed the sufficient conditions on A that make the current gT well defined, where g is a plurisubharmonic function on Ω in the class $L^\infty_{\rm loc}(\Omega \setminus A)$. Namely, in a very elegant fashion, he proved the following assertion.

Theorem 1.1. *Let A be a closed subset of an open subset Ω of $\mathbb{C}^n$ and T be a positive closed current on Ω of bi-dimension (p,p). Suppose that g is a plurisubharmonic function on Ω in the class $L^\infty_{\rm loc}(\Omega \setminus A)$. If the Hausdorff measure $\mathcal{H}_{2p-1}(A)$ vanishes, then gT together with ${\rm dd}^c g \wedge T$ are well defined.*

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Two years later, Jean Fornæss and Nassim Sibony in [5] generalized what Demailly achieved previously, and defined the currents gT and $\mathrm{dd}^c g \wedge T$ when $\mathcal{H}_{2p}(A) = 0$. In both studies, the desired current gT is locally negative and plurisubharmonic (i.e. $\mathrm{dd}^c(gT) \geqslant 0$), and this trait is compatible with the type of currents studied in the literature of this field. Once we consider g with higher powers $k \in \mathbb{N}$, the current $g^k T$ fails to be plurisubharmonic. This might cause further difficulties, since positivity cannot be guaranteed any more. Despite this fact, the authors in [2] proved that the current $g^2 T$ is well defined as soon as $\mathcal{H}_{2p-2}(A) = 0$.

In this paper, we are concerned with higher powers of plurisubharmonic functions. More precisely, our goal is to find sufficient conditions on the unbounded locus of a plurisubharmonic function g that make $g^k T$ well defined.

Theorem 1.2. *Let A be a closed subset of an open subset Ω of $\mathbb{C}^n$ and T be a positive closed current on Ω of bi-dimension (p,p). Suppose that g is a plurisubharmonic function on Ω in the class $L^\infty_{\mathrm{loc}}(\Omega \setminus A)$. If $\mathcal{H}_{2p}(\operatorname{Supp} T \cap A) = 0$ and the trivial extension $\widetilde{g^{k-2}\mathrm{d}g \wedge \mathrm{d}^c g} \wedge T$, $k \in \mathbb{N}$ exists, then $g^k T$ is well defined.*

Furthermore, we discuss a special case where the current $g^k T$ can be obtained with no further restrictions on $\mathrm{d}g \wedge \mathrm{d}^c g \wedge T$.

Theorem 1.3. *Let Ω be an open subset of $\mathbb{C}^n$ and $A \in \Omega$ be the origin. Let T be a positive plurisubharmonic current of bi-dimension (p,p) on Ω. If g is a radial plurisubharmonic function on Ω in the class $L^\infty_{\mathrm{loc}}(\Omega \setminus A)$, then there exists $\alpha > 0$ such that $\mathrm{e}^{-\alpha g} T$ is of finite mass near A. Moreover, the current $g^k T$ is well defined for all $k > 0$.*

Our study also leads to many applications. For instant, when the locus points of g are located in a compact set and $g^{k-2}\mathrm{d}g \wedge \mathrm{d}^c g \wedge T$ is well defined, the weight k Lelong number $\nu(T, g, k)$, which is defined by

$$\lim_{r \to -\infty} \int_{B_g(r)} T \wedge (\mathrm{dd}^c g^k)^p, \quad B_g(r) = \{g < r\},$$

exists. Notice that the weight 1 Lelong number coincides with the classical Demailly-Lelong number. Similarly, the weight k Lelong number is also well defined in the case of radial functions.

2. Preliminaries and notations

Throughout this paper we suppose that A is a closed subset of an open subset Ω of $\mathbb{C}^n$, and denote by $\mathcal{D}_{p,q}(\Omega)$ the space of $\mathcal{C}^\infty$ compactly supported differential forms of bidegree (p,q) on Ω. A form $\varphi \in \mathcal{D}_{p,p}(\Omega)$ is said to be a strongly positive form if φ can be written as

$$\varphi(z) = \sum_{j=1}^{N} \lambda_j(z) \mathrm{i}\alpha_{1,j} \wedge \overline{\alpha}_{1,j} \wedge \ldots \wedge \mathrm{i}\alpha_{p,j} \wedge \overline{\alpha}_{p,j},$$

where $\lambda_j \geqslant 0$ and $\alpha_{s,j} \in \mathcal{D}_{1,0}(\Omega)$. Then $\mathcal{D}_{p,p}(\Omega)$ admits a basis consisting of strongly positive forms. The dual space $\mathcal{D}'_{p,q}(\Omega)$ is the space of currents of bi-dimension (p,q). A current $T \in \mathcal{D}'_{p,p}(\Omega)$ is said to be positive if $\langle T, \varphi \rangle \geqslant 0$ for all strongly positive forms $\varphi \in \mathcal{D}_{p,p}(\Omega)$. For a positive current T, the mass of T over each open subset $V \Subset \Omega$, which is denoted by $\|T\|_V$, is defined as follows:

$$\|T\|_V = \sup\{|T(\varphi)|,\ \varphi \in \mathcal{D}_{p,p}(V),\ \|\varphi\| \leqslant 1\}.$$

Let $\beta = \mathrm{dd}^c|z|^2$ be the Kähler form on $\mathbb{C}^n$ (where $\mathrm{d} = \partial + \overline{\partial}$ and $\mathrm{d}^c = \mathrm{i}(-\partial + \overline{\partial})$, hence $\mathrm{dd}^c = 2\mathrm{i}\partial\overline{\partial}$). Then there exists a constant $C > 0$ depending only on n and p such that

$$T \wedge \frac{\beta^p}{2^p p!}(V) \leqslant \|T\|_V \leqslant CT \wedge \beta^p(V).$$

By $\max_\varepsilon(x_1, x_2)$ we mean the function

$$\max_\varepsilon(x_1, x_2) = \max(x_1, x_2) * \psi_\varepsilon,$$

where ψ_ε is a regularization kernel on $\mathbb{R}^2$ depending only on $\|(x_1, x_2)\|$. Recall that a current T is said to be closed if $\mathrm{d}T = 0$. A current T is said to be *S-plurisubharmonic* (or *S-plurisuperharmonic*) if there exists a positive current S on Ω such that $\mathrm{dd}^c T \geqslant -S$ (or $\mathrm{dd}^c T \leqslant S$). Remember also that for positive and plurisubharmonic (0-plurisubharmonic) current T the Lelong number

$$\nu(T, 0) := \lim_{r \to 0} \frac{1}{r^{2p}} \int_{B(0,r)} T \wedge \beta^p$$

exists.

The trivial extension $\widetilde{T}$. *Let (χ_n) be a smooth bounded sequence which vanishes on a neighborhood of closed subset $A \subset \Omega$ and (χ_n) converges to the characteristic function $\mathbb{1}_{\Omega \setminus A}$ of $\Omega \setminus A$, and let T be a current of order zero defined on $\Omega \setminus A$. If $\chi_n T$ has a limit which does not depend on (χ_n), then this limit is called the trivial extension of T by zero across A and is denoted by $\widetilde{T}$. It is clear that $\widetilde{T}$ exists if and only if T has a locally finite mass across A.*

3. Our main results

We start this section by recalling a version of Ben Massoud-El Mir inequality in the case of S-plurisubharmonic currents.

Lemma 3.1 ([2], Theorem 2.2). *Let A be a closed complete pluripolar subset of Ω and T be a positive and S-plurisuperharmonic current on $\Omega\setminus A$. Let v be a plurisubharmonic function of class $\mathcal{C}^2$, $v\geqslant -1$ on Ω such that $\Omega'=\{z\in\Omega\colon\ v(z)<0\}$ is relatively compact in Ω. Let K be a compact subset of Ω'. Then for every plurisubharmonic function u on Ω' of class $\mathcal{C}^2$ satisfying that $-1\leqslant u<0$ we have*

$$\int_{K\setminus A} T\wedge(\mathrm{dd}^c u)^p\leqslant D_v\int_{\Omega'\setminus A} T\wedge(\mathrm{dd}^c v)^p+\eta_{v,u}\|S\|_{\Omega'} \tag{3.1}$$

for some positive constants D_v and $\eta_{v,u}$.

The result will play a crucial role in the next proof.

P r o o f of Theorem 1.2. Assume that g is negative and k is even. As the problem is local, it is enough to show that g^kT is of locally finite mass near every point z_0 in A. Without loss of generality, one can assume that z_0 is the origin. Clearly, with our choices, the current g^kT is positive and S-plurisuperharmonic, since

$$\begin{aligned}\mathrm{dd}^c(g^kT)&=kg^{k-1}\mathrm{dd}^cg\wedge T+k(k-1)g^{k-2}\mathrm{d}g\wedge\mathrm{d}^cg\wedge T\\&\leqslant k(k-1)[\widetilde{g^{k-2}\mathrm{d}g\wedge\mathrm{d}^cg\wedge T}].\end{aligned} \tag{3.2}$$

But $\mathcal{H}_{2p}(A\cap\operatorname{Supp}T)=0$. Thus, by [3] and [6] there exist a system of coordinates (z',z'') of $\mathbb{C}^p\times\mathbb{C}^{n-p}$ and a polydisk $\Delta^p\times\Delta^{n-p}\subset\mathbb{C}^p\times\mathbb{C}^{n-p}$ such that $(A\cap\operatorname{Supp}T)\cap(\Delta^p\times\partial\Delta^{n-p})=\emptyset$. Moreover, the projection map

$$\pi\colon\ (A\cap\operatorname{Supp}T)\cap(\Delta^p\times\Delta^{n-p})\to\Delta^p$$

is proper, and as $\pi(A\cap\operatorname{Supp}T)$ is closed with a zero Lebesgue measure in Δ^p, one can find an open subset $O\subset\Delta^p\setminus\pi(A\cap\operatorname{Supp}T)$. Therefore the current has locally finite mass on $O\times\Delta^{n-p}$. Let $0<\delta<1$ such that $(A\cap\operatorname{Supp}T)\cap(\Delta^p\times\{z'',\ \delta<|z''|<1\})=\emptyset$, and fix a and t, two real numbers such that $\delta<a<t<1$. Set

$$\varrho_\varepsilon=\max\nolimits_\varepsilon\Big(\pi^*\varrho,\frac{1}{t^2-a^2}(|z''|^2-t^2)\Big), \tag{3.3}$$

where ϱ is a smooth plurisubharmonic function on Δ^p such that $(\mathrm{dd}^c\varrho)^p$ is supported in O. We have $-1 \leqslant \varrho_\varepsilon < 0$ in $t\Delta^n$ and $\varrho_\varepsilon = \pi^*\varrho$ on $\{|z''| \leqslant a\}$, and hence

$$\int_{(t\Delta^n)\setminus A} g^k T \wedge (\mathrm{dd}^c\varrho_\varepsilon)^p = \int_{(t\Delta^p)\times\{|z''|<a\}\setminus A} g^k T \wedge (\mathrm{dd}^c(\pi^*\varrho))^p + \int_{(t\Delta^p)\times\{a\leqslant|z''|<t\}} g^k T \wedge (\mathrm{dd}^c\varrho_\varepsilon)^p.$$

Since $(\mathrm{dd}^c\pi^*\varrho)^p$ is supported in $O\times\Delta^{n-p}$, both integrals on the right-hand side of (3.4) are finite. Therefore, by applying the previous lemma to g^kT for $v = \varrho_\varepsilon$, $u = (|z|^2 - nt^2)/nt^2$ and $S = [\widetilde{g^{k-2}\mathrm{d}g \wedge \mathrm{d}^c g} \wedge T]$, we deduce that $\widetilde{g^kT}$ exists. Now, a decreasing sequence of plurisubharmonic $(g_j)_{j\in\mathbb{N}}$ can be built by regularizing g, and the monotone convergence shows that g^kT is well defined as a limit of g_j^kT. Still the case when k is odd need to be proved. However, in such a situation, we achieve the desired construction by applying the above technique to the current $-g^kT$. This ends the proof. □

In the case of compact obstacles, we do not need to pay any attention to the thickness of A. In fact, we have the following assertion.

Theorem 3.1. *Let A be a compact subset of an open subset Ω of $\mathbb{C}^n$ and T be a positive closed current on Ω of bi-dimension (p,p). Suppose that g is plurisubharmonic function on Ω in the class $L^\infty_{\mathrm{loc}}(\Omega\setminus A)$. If the current $g^{k-2}\mathrm{d}g\wedge\mathrm{d}^c g\wedge T$, $k\in\mathbb{N}$ is well defined, then so is g^kT.*

Proof. By regularizing g, we may assume that g is a negative smooth function. As in the proof of the previous theorem, we emphasize the case when k is even. Take a smooth function χ with compact support $K \subset W \Subset \Omega$ such that $\chi = 1$ on a neighborhood V of A. By applying Stokes' formula, we have

$$(3.4)\quad \int_W \mathrm{dd}^c(\chi|z|^2)\wedge g^kT\wedge\beta^{p-1} = \int_W (\chi|z|^2)\mathrm{dd}^c g^k \wedge T\wedge\beta^{p-1} \leqslant k(k-1)\int_W (\chi|z|^2) g^{k-2}\mathrm{d}g\wedge\mathrm{d}^c g\wedge T\wedge\beta^{p-1}.$$

On the other hand,

$$(3.5)\quad \int_W \mathrm{dd}^c(\chi|z|^2)\wedge g^kT\wedge\beta^{p-1} = \int_{W\setminus V} g^k\mathrm{d}\chi\wedge\mathrm{d}^c|z|^2\wedge T\wedge\beta^{p-1} - \int_{W\setminus V} g^k\mathrm{d}^c\chi\wedge\mathrm{d}|z|^2\wedge T\wedge\beta^{p-1} + \int_{W\setminus V} g^k|z|^2\mathrm{dd}^c\chi\wedge T\wedge\beta^{p-1} + \int_W \chi g^kT\wedge\beta^p.$$

The properties of χ together with relations (3.4) and (3.5) give that

$$\begin{aligned}(3.6)\qquad \int_W \chi g^k T\wedge\beta^p &\leqslant k(k-1)\int_W (\chi|z|^2) g^{k-2}\mathrm{d}g\wedge\mathrm{d}^c g\wedge T\wedge\beta^{p-1}\\ &\quad-\int_{W\setminus O} g^k\mathrm{d}\chi\wedge\mathrm{d}^c|z|^2\wedge T\wedge\beta^{p-1}+\int_{W\setminus O} g^k\mathrm{d}^c\chi\wedge\mathrm{d}|z|^2\wedge T\wedge\beta^{p-1}\\ &\quad-\int_{W\setminus O} g^k|z|^2\mathrm{d}\mathrm{d}^c\chi\wedge T\wedge\beta^{p-1}\end{aligned}$$

for some neighborhood $O\Subset V$ of A. This shows that there exist positive constants μ and λ such that

$$(3.7)\qquad \int_V g^k T\wedge\beta^p\leqslant \mu\|g^{k-2}\mathrm{d}g\wedge\mathrm{d}^c g\wedge T\|_W+\lambda\|g^k T\|_{W\setminus O},$$

and the definition of g^kT is achieved. □

Now, we discuss a special case, where the conditions on g^kT can be relaxed. Namely, we consider the case when g is radial.

P r o o f of Theorem 1.3. As the problem is local, one can assume that $\Omega = B(0,1)$. Now, suppose that u is a radial plurisubharmonic function and μ is a positive Radon measure satisfying that there exists $\alpha>1$ such that $r^{-\alpha}\int_{B(0,r)}\mathrm{d}\mu$ is bounded on $]0,R]$, $R<1$. Hence, there exists $k>0$ such that $\int_{B(0,R)}\mathrm{e}^{-ku}\,\mathrm{d}\mu$ is finite. Indeed, set $\mu(t):=\int_{B(0,t)}\mathrm{d}\mu$, we have

$$\int_{B(0,R)}\mathrm{e}^{-ku}\,\mathrm{d}\mu=\int_0^R\mathrm{e}^{-ku(t)}\,\mathrm{d}\mu(t).$$

As u is plurisubharmonic, one can find a constant $a>0$ such that $-u(t)\leqslant -a\log t$ for all $t\in\,]0,R]$. It follows that

$$\begin{aligned}(3.8)\qquad \int_{B(0,R)}\mathrm{e}^{-ku}\,\mathrm{d}\mu&=\int_0^R\mathrm{e}^{-ku(t)}\,\mathrm{d}\mu(t)\leqslant\lim_{\varepsilon\to0}\int_\varepsilon^R t^{-ak}\,\mathrm{d}\mu\\ &=\lim_{\varepsilon\to0}\Big[R^{-ak}\mu(R)-\varepsilon^{-ak}\mu(\varepsilon)+ak\int_\varepsilon^R t^{-ak-1}\,\mathrm{d}\mu(t)\Big],\end{aligned}$$

and the desired bound is achieved by choosing $k\leqslant(\alpha-1)/a$. However, it is already known that $\int_{B(0,R)}\mathrm{e}^{-ku}T\wedge\beta^p$ is finite, since T is positive plurisubharmonic. Therefore $r^{-2p}\int_{B(0,r)}T\wedge\beta^p$ is bounded as soon as $k<(2p-1)/\nu(u,0)$. In particular, if $u=-\log(-g)$, then $\nu(u,0)=0$, and since $\mathrm{e}^{-ku}=(-g)^k$, it follows that $\int_{B(0,R)}(-g)^kT\wedge\beta^p$ is finite for all $k\in\mathbb{R}^+$. This proves Theorem 1.3. □

Applications of the main results

The above results lead to many interesting generalizations regarding Lelong numbers and the extension of positive currents. More precisely, one can obtain the following.

(1) Let $A = \{z = (z_1, \dots, z_n) \in \Omega;\ z_n = 0\}$ and T be a positive closed current on Ω of bi-dimension (p,p). Suppose that $\mathcal{H}_{2p-2}(\operatorname{Supp} T \cap A) = 0$, then for all positive function ϕ on $\mathbb{R}_+^*$, the current $\phi(|z_n|^2) T \wedge \mathrm{i}\, \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n$ has a trivial extension, so $(\operatorname{Supp} T\, \mathrm{i}\, \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n) \cap A = \emptyset$. Notice that one can replace function z_n by any analytic function f on Ω.

(2) Under the hypotheses of Theorem 3.2, the current $\mathrm{dd^c} g^k \wedge T$ is well defined. This fact allows us to talk about what we call weight k Lelong number which is denoted and defined by

$$\nu(T, g, k) := \lim_{r \to -\infty} \int_{B_g(r)} T \wedge (\mathrm{dd^c} g^k)^p, \quad B_g(r) = \{g < r\}.$$

One can also deduce the existence of the weight k Lelong number for radial functions, thanks to Theorem 1.3.

(3) In addition to the hypotheses of Theorem 1.3, if T is pluriharmonic, then behavior of the annoying term $\mathrm{d}g \wedge \mathrm{d^c} T$ can be predicted. Indeed, by [1], the current $\mathrm{dd^c} g \wedge T$ is well defined. Hence we can define the current $\mathrm{d}g \wedge \mathrm{d^c} T$ on $\mathcal{D}_{p-1,p-1}(\Omega)$ by

$$\mathrm{d}g \wedge \mathrm{d^c} T(\varphi) = \tfrac{1}{2}[\mathrm{dd^c}(gT) - \mathrm{dd^c} g \wedge T](\varphi) \quad \forall\, \varphi \in \mathcal{D}_{p-1,p-1}(\Omega).$$

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