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ON THE VECTORS ASSOCIATED WITH THE ROOTS OF MAX-PLUS CHARACTERISTIC POLYNOMIALS

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Abstract. We discuss the eigenvalue problem in the max-plus algebra. For a max-plus square matrix, the roots of its characteristic polynomial are not its eigenvalues. In this paper, we give the notion of algebraic eigenvectors associated with the roots of characteristic polynomials. Algebraic eigenvectors are the analogues of the usual eigenvectors in the following three senses: (1) An algebraic eigenvector satisfies an equation similar to the equation $A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x}$ for usual eigenvectors. Under a suitable assumption, the equation has a nontrivial solution if and only if λ is a root of the characteristic polynomial. (2) The set of algebraic eigenvectors forms a max-plus subspace called algebraic eigenspace. (3) The dimension of each algebraic eigenspace is at most the multiplicity of the corresponding root of the characteristic polynomial.

Keywords: max-plus algebra; eigenvalue; eigenvector; characteristic polynomial

MSC 2020: 15A18, 15A80

1. Introduction

The max-plus algebra $\mathbb{R} \cup \{-\infty\}$ is a semiring with the following two operations: the conventional max operation $\oplus$ with the identity element $\varepsilon := -\infty$, and the conventional $+$ operation $\otimes$ with the identity element 0. The max-plus algebra, together with the analogous semiring the min-plus algebra $\mathbb{R} \cup \{\infty\}$, has its origin in the shortest path problem or the scheduling problem. It has a wide range of applications in various fields of science and engineering, such as control theory and scheduling of railway systems [3], [8].

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Eigenvalues and eigenvectors of max-plus square matrices are defined as in the conventional linear algebra. For a matrix $A \in \mathbb{R}_{\max}^{n\times n}$, a scalar λ is called an *eigenvalue of* A if there exists a vector $\boldsymbol{x} \neq (\varepsilon, \varepsilon, \ldots, \varepsilon)^\top$, called an *eigenvector*, satisfying $A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x}$. In general, max-plus matrices have a few eigenvalues. The number of the eigenvalues of a matrix cannot exceed the number of strongly connected components of the associated digraph. In particular, an irreducible matrix has exactly one eigenvalue. By contrast, as in the conventional linear algebra, the characteristic polynomial of an n-by-n max-plus matrix admits exactly n roots (counting multiplicities). Cuninghame-Green [6] showed that the maximum root of the characteristic polynomial is always an eigenvalue of the matrix, but other roots are not generally eigenvalues.

The contribution of the present paper is to clarify the role of the roots of the characteristic polynomial of a max-plus matrix in the eigenvalue problem in the max-plus linear algebra. We first note that coefficients of the characteristic polynomial come from the weights of multi-circuits in the associated graph, where a multi-circuit is the union of disjoint elementary circuits in the graph. We call a multi-circuit λ-*maximal* if the corresponding term attains the maximum of the characteristic polynomial when the variable of the polynomial takes the value λ. For a scalar λ and a λ-maximal multi-circuit $\mathcal{C}$, we consider the equation

$$(A_{\setminus\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{x} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus\mathcal{C}}) \otimes \boldsymbol{x}, \tag{1.1}$$

where matrices $A_{\mathcal{C}}, A_{\setminus\mathcal{C}}, E_{\mathcal{C}}$, and $E_{\setminus\mathcal{C}}$ are determined by $\mathcal{C}$ and defined in Section 3.2. This equation is in a sense a generalization of the equation $A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x}$, so we call a vector $\boldsymbol{x}$ satisfying (1.1) an *algebraic eigenvector of* A with respect to λ. The adjective "algebraic" is taken from Akian et al. [1], in which the roots of characteristic polynomials are called *algebraic eigenvalues*. To confirm the validity of our definition of algebraic eigenvectors, we first prove that there exists an algebraic eigenvector with respect to λ if and only if λ is a root of the characteristic polynomial. This holds under the assumption that every essential term of the characteristic polynomial is attained with exactly one permutation. This assumption is not so strong that it is satisfied by generic matrices and is also considered in the settings of the supertropical algebra [10]. We further prove that the definition of algebraic eigenvectors does not depend on the choice of λ-maximal multi-circuits $\mathcal{C}$. This leads to the fact that the set of all algebraic eigenvectors with respect to λ becomes a max-plus subspace, which we call the *algebraic eigenspace*.

In tropical geometry [12], the roots of a polynomial are defined as the values satisfying that the maximum of a polynomial is attained with at least two terms. Thus, the algebraic eigenvalues are defined as the roots of characteristic polynomials in the tropical sense. Hence, it seems natural to define the analogues of eigenvectors by using tropical geometry. In fact, there is an approach from the perspective of supertropical algebra [9], [10], [11], [14] in line with this idea, but it would produce more "eigenvectors" than expected, that is, the number of independent eigenvectors could exceed the multiplicity of the root. The computation of one eigenvector is described in [10], but finding all eigenvectors is difficult. Our definition of algebraic eigenvectors using equality (1.1) is more restrictive and can solve these problems. We show that the computation of all algebraic eigenvectors can be reduced to the usual eigenvalue problem. Further, we prove that the dimension of the algebraic eigenspace is at most the multiplicity of the root of the characteristic polynomial, which is the analogous result to the conventional linear algebra.

2. Preliminaries for the max-plus eigenvalue problem

2.1. Max-plus algebra. Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ be the set of real numbers $\mathbb{R}$ with an extra element $-\infty$. We define two operations, addition $\oplus$ and multiplication $\otimes$, on $\mathbb{R}_{\max}$ in terms of conventional operations by

$$a \oplus b = \max\{a, b\}, \quad a \otimes b = a + b, \quad a, b \in \mathbb{R}_{\max}.$$

Then, $(\mathbb{R}_{\max}, \oplus, \otimes)$ is a commutative semiring called the max-plus algebra or the tropical semiring. Here, $\varepsilon := -\infty$ is the identity element for addition and $e := 0$ is the identity element for multiplication. For details about the max-plus algebra, we refer to [3], [4], [8], [12].

Let $\mathbb{R}_{\max}^{m \times n}$ be the set of $m \times n$ matrices whose entries are in $\mathbb{R}_{\max}$. We denote by $\mathbb{R}_{\max}^{n}$ the set of n-dimensional max-plus column vectors. The arithmetic operations on vectors and matrices are defined as in the conventional linear algebra. For max-plus matrices $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}_{\max}^{m \times n}$, we define the *matrix sum* $A \oplus B = ([A \oplus B]_{ij}) \in \mathbb{R}_{\max}^{m \times n}$ by

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij}.$$

For max-plus matrices $A = (a_{ij}) \in \mathbb{R}_{\max}^{l \times m}$ and $B = (b_{ij}) \in \mathbb{R}_{\max}^{m \times n}$, we define the *matrix product* $A \otimes B = ([A \otimes B]_{ij}) \in \mathbb{R}_{\max}^{l \times n}$ by

$$[A \otimes B]_{ij} = \bigoplus_{k=1}^{m} a_{ik} \otimes b_{kj}.$$

For a max-plus matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{m \times n}$ and a scalar $\alpha \in \mathbb{R}_{\max}$, we define the *scalar multiplication* $\alpha \otimes A = ([\alpha \otimes A]_{ij}) \in \mathbb{R}_{\max}^{m \times n}$ by

$$[\alpha \otimes A]_{ij} = \alpha \otimes a_{ij}.$$

The matrix $E \in \mathbb{R}_{\max}^{n\times n}$ whose diagonal entries are e and other entries are ε is the identity matrix for matrix multiplication.

A subset $U \subset \mathbb{R}_{\max}^{n}$ is called a *subspace* if it is closed with respect to addition $\oplus$ and scalar multiplication $\otimes$. A minimal generating set of a subspace U is called a *basis of* U. In the max-plus algebra, a basis of a subspace is uniquely determined up to scalar multiplication [5], Theorem 18. The number of vectors in a basis is called the *dimension of the subspace.*

For a matrix $A \in \mathbb{R}_{\max}^{n\times n}$, we define the determinant of A by

$$\det A = \bigoplus_{\pi\in S_n} \bigotimes_{i=1}^{n} a_{i\pi(i)},$$

where S_n denotes the symmetric group of order n. A matrix $A \in \mathbb{R}_{\max}^{n\times n}$ is called *nonsingular* if the maximum in $\det A$ is attained at precisely one permutation; otherwise, it is called *singular.* The singularity of max-plus matrices is equivalent to the existence of nontrivial kernels in the sense of tropical geometry.

Theorem 2.1 ([2], Theorem 1.4)**.** *A matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ is singular if and only if there exists a vector $(x_1, x_2, \dots, x_n)^\top \neq (\varepsilon, \varepsilon, \dots, \varepsilon)^\top$ such that the maximum*

$$a_{i1} \otimes x_1 \oplus a_{i2} \otimes x_2 \oplus \ldots \oplus a_{in} \otimes x_n$$

is attained with at least two terms for each $i = 1, 2, \dots, n$.

2.2. Max-plus matrices and graphs. For a matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$, we define a weighted digraph $G(A) = (V, E, w)$ as follows. The vertex set and the edge set are $V = \{1, 2, \dots, n\}$ and $E = \{(i, j) \mid a_{ij} \neq \varepsilon\}$, respectively, and the *weight function* $w\colon E \to \mathbb{R}$ is defined by $w((i, j)) = a_{ij}$ for $(i, j) \in E$. A sequence $P = (i_1, i_2, \dots, i_s)$ of vertices is called a *path* if $(i_p, i_{p+1}) \in E$ for all $p = 1, 2, \dots, s-1$. The number $l(P) := s - 1$ is called the *length* of P and $w(P) := w((i_1, i_2)) + w((i_2, i_3)) + \ldots + w((i_{s-1}, i_s))$ is called the *weight* of P. A path is called a *circuit* if its initial and terminal vertices are identical. A circuit $(i_1, i_2, \dots, i_{s-1}, i_1)$ is called *elementary* if $i_p \neq i_q$ for $1 \leqslant p < q \leqslant s - 1$. For an elementary circuit C, we define the *average weight* of C by $\operatorname{ave}(C) := w(C)/l(C)$.

For $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$, we consider the matrix formal power series of the form

$$A^* := E \oplus A \oplus A^{\otimes 2} \oplus \ldots$$

If there is no circuit with positive weight in $G(A)$, this infinite sum terminates as

$$A^* = E \oplus A \oplus A^{\otimes 2} \oplus \ldots \oplus A^{\otimes n-1},$$

since the (i,j) entry of $A^{\otimes k}$ is identical to the maximum weight of all paths from vertex i to vertex j with length k. In that case, the (i,j) entry of A^* is the maximum weight of all paths from vertex i to vertex j with arbitrary length.

2.3. Eigenvalue problem on the max-plus algebra. For a matrix $A \in \mathbb{R}_{\max}^{n\times n}$, a scalar λ is called an *eigenvalue of* A if there exists a vector $\boldsymbol{x} \neq (\varepsilon, \varepsilon, \ldots, \varepsilon)^\top$ satisfying

$$A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x}.$$

Such nontrivial vector $\boldsymbol{x}$ is called an *eigenvector of A with respect to λ*. In the case where we need to distinguish eigenvalues (eigenvectors) from algebraic eigenvalues (eigenvectors) defined later, we call them geometric eigenvalue (eigenvector). For any eigenvalue λ of A, the set of eigenvectors

$$U(\lambda) = \{\boldsymbol{x} \in \mathbb{R}_{\max}^{n} \mid A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x}\}$$

forms a max-plus subspace, called the *eigenspace* of A with respect to λ. Here, we summarize the results in the literature on the max-plus eigenvalue problem, e.g. [3], [4], [8].

Theorem 2.2. *For a matrix $A \in \mathbb{R}_{\max}^{n\times n}$, the maximum value of the average weights of all elementary circuits in $G(A)$ is the maximum eigenvalue of A.*

Theorem 2.3. *For any eigenvalue λ of $A \in \mathbb{R}_{\max}^{n\times n}$, there exists a circuit in $G(A)$ whose average weight is λ.*

Let λ be the maximum value of the average weights of all elementary circuits in $G(A)$. We define the *critical graph* $G^c(A)$ by the subgraph of $G(A)$ induced by all circuits with average weights λ. We denote by g_k the kth column of $((-\lambda) \otimes A)^*$. Then we have the following theorem.

Theorem 2.4. *A vector g_k is an eigenvector of A with respect to λ if and only if k is a vertex in $G^c(A)$.*

Further, let K be a set of vertices with exactly one vertex from each connected component of $G^c(A)$. Then we have the following theorem.

Theorem 2.5. *The set $\{g_k \mid k \in K\}$ is a basis of the eigenspace $U(\lambda)$.*

2.4. Max-plus characteristic polynomials. A (univariate) polynomial in the max-plus algebra has the form

$$f(t) = c_0 \oplus c_1 \otimes t \oplus c_2 \otimes t^{\otimes 2} \oplus \ldots \oplus c_n \otimes t^{\otimes n}, \quad c_0, c_1, c_2, \ldots, c_n \in \mathbb{R}_{\max}.$$

Max-plus univariate polynomials are piecewise linear functions on $\mathbb{R}_{\max}$. A term $c_k \otimes t^{\otimes k}$ is called *essential* if it contributes to $f(t)$ as a function, that is,

$$c_k \otimes t^{\otimes k} > \bigoplus_{j \neq k} c_j \otimes t^{\otimes j}$$

for some $t \in \mathbb{R}_{\max}$; otherwise, it is called *inessential.* As with standard polynomials over $\mathbb{C}$, each polynomial can be factorized into the product of linear factors:

$$f(t) = (t \oplus r_1)^{\otimes p_1} \otimes (t \oplus r_2)^{\otimes p_2} \otimes \ldots \otimes (t \oplus r_m)^{\otimes p_m}.$$

Then r_i and p_i are called a *root of* $f(t)$ and its *multiplicity*, respectively. In the graph of the piecewise linear function $f(t)$, the roots are the bending points of $f(t)$ and the multiplicities are the differences in the slopes of the lines around the roots.

In this paper, we focus on the characteristic polynomial of a matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n \times n}$. As in the conventional algebra, the *characteristic polynomial* of A is defined by

$$\varphi_A(t) := \det(A \oplus t \otimes E).$$

If we expand the right-hand side, the coefficient of $t^{\otimes k}$ is the maximum weight of the multi-circuits in $G(A)$ with length $n - k$. Here, a multi-circuit means the set of disjoint elementary circuits in $G(A)$ and its length (weight) is the sum of the lengths (weights) of these circuits. The following factorization algorithm is essentially the same as the operations RESOLUTION and RECTIFY in [7], Section IX, but it is reformulated in terms of graph theory.

Algorithm 2.6

Input: A matrix $A \in \mathbb{R}_{\max}^{n \times n}$

Output: The factorization of the characteristic polynomial of A

(1) Set $i := 0$ and $\mathcal{C}_0 = \emptyset$.

(2) Set $i := i + 1$.

  (a) If there is no multi-circuit in $G(A)$ whose length is larger than $l(\mathcal{C}_{i-1})$, then set $m := i$, $\lambda_m := \varepsilon$ and $p_m := n - (p_1 + p_2 + \ldots + p_{i-1})$ and proceed to (3).

  (b) If there exist multi-circuits in $G(A)$ whose lengths are larger than $l(\mathcal{C}_{i-1})$, let $\mathcal{C}_i$ be the multi-circuit $\mathcal{C}$ attaining the maximum value of

$$\frac{w(\mathcal{C}) - w(\mathcal{C}_{i-1})}{l(\mathcal{C}) - l(\mathcal{C}_{i-1})}$$

among them. If there is more than one such multi-circuit, we choose the longest one. We set

$$\lambda_i := \frac{w(\mathcal{C}_i) - w(\mathcal{C}_{i-1})}{l(\mathcal{C}_i) - l(\mathcal{C}_{i-1})}$$

and $p_i := l(\mathcal{C}_i) - l(\mathcal{C}_{i-1})$, and we repeat (2).

(3) We have the characteristic polynomial

$$\varphi_A(t) = (t \oplus \lambda_1)^{\otimes p_1} \otimes (t \oplus \lambda_2)^{\otimes p_2} \otimes \ldots \otimes (t \oplus \lambda_m)^{\otimes p_m}.$$

We define the *relative average* of multi-circuits $\mathcal{C}'$ with respect to $\mathcal{C}$ by

$$\text{r.ave}(\mathcal{C}, \mathcal{C}') = \begin{cases} \dfrac{w(\mathcal{C}') - w(\mathcal{C})}{l(\mathcal{C}') - l(\mathcal{C})} & \text{if } l(\mathcal{C}') > l(\mathcal{C}), \\ \varepsilon & \text{otherwise.} \end{cases}$$

Using this notion, λ_i in Algorithm 2.6 is the maximum value of the relative averages of all multi-circuits with respect to $\mathcal{C}_{i-1}$ in $G(A)$.

As in the conventional algebra, the characteristic polynomial of a matrix is related to the eigenvalue problem.

Theorem 2.7 ([6], Theorem 3)**.** *For a matrix $A \in \mathbb{R}_{\max}^{n\times n}$, the maximum root of its characteristic polynomial is the maximum eigenvalue of A.*

Theorem 2.8 ([1], Fact 3 of Section 5)**.** *All eigenvalues of a matrix $A \in \mathbb{R}_{\max}^{n\times n}$ are roots of its characteristic polynomial.*

3. Algebraic eigenvectors

As we saw in the end of Section 2, the maximum root of the characteristic polynomial is an eigenvalue of A. The other roots, however, may not be eigenvalues. Thus, our concern is to clarify the roles of the roots of the characteristic polynomial that are not maximums. To investigate this problem, we introduce the notion of algebraic eigenvectors associated with the roots of the characteristic polynomial. The adjective "algebraic" is taken from [1], in which the roots of the characteristic polynomial are called algebraic eigenvalues.

3.1. One assumption for generic matrices. For $A \in \mathbb{R}_{\max}^{n\times n}$ and variable t, we define the $2n \times 2n$ matrix

$$\tilde{A}(t) = \begin{pmatrix} A & t \otimes E \\ E & E \end{pmatrix}.$$

We note that $\varphi_A(t) = \det \tilde{A}(t)$ as functions of t. The matrix $\tilde{A}(t)$ also admits a graph theoretical characterization. We say that a permutation $\pi \in S_{2n}$ is *finite with respect to* $\tilde{A}(t) = (\tilde{a}_{ij})$ if $\tilde{a}_{i\pi(i)} \neq \varepsilon$ for $i = 1, 2, \ldots, 2n$. For a multi-circuit $\mathcal{C}$ in $G(A)$, we define a finite permutation $\pi^{\mathcal{C}} \in S_{2n}$ as follows:

$$\pi^{\mathcal{C}}(i) = \begin{cases} (\text{the next vertex of } i \text{ in } \mathcal{C}) & i \in V(\mathcal{C}),\ 1 \leqslant i \leqslant n, \\ i + n & i \notin V(\mathcal{C}),\ 1 \leqslant i \leqslant n, \\ i & i - n \in V(\mathcal{C}),\ n + 1 \leqslant i \leqslant 2n, \\ i - n & i - n \notin V(\mathcal{C}),\ n + 1 \leqslant i \leqslant 2n. \end{cases}$$

The map $\mathcal{C} \mapsto \pi^{\mathcal{C}}$ gives a one to one correspondence between multi-circuits in $G(A)$ and finite permutations with respect to $\tilde{A}(t)$. For $\lambda \neq \varepsilon$, we say that a multi-circuit $\mathcal{C}$ is λ-*maximal* if $\pi^{\mathcal{C}}$ attains the maximum of $\det \tilde{A}(\lambda)$. A multi-circuit $\mathcal{C}$ is ε-*maximal* if $\pi^{\mathcal{C}}$ attains the maximum of $\det \tilde{A}(\bar{\lambda})$ for a sufficiently small finite value $\bar{\lambda}$. Note that the ε-maximal multi-circuit has the maximum length among all multi-circuits in $G(A)$. In Algorithm 2.6, both $\mathcal{C}_{i-1}$ and $\mathcal{C}_i$ are λ_i-maximal multi-circuits.

Lemma 3.1. *If λ is a root of the characteristic polynomial $\varphi_A(t)$, then the matrix $\tilde{A}(\lambda)$ is singular.*

P r o o f. If $\lambda = \varepsilon$ is a root of $\varphi_A(t)$, the graph $G(A)$ has no multi-circuit with length n. Then $\det A$ and $\det \tilde{A}(\varepsilon)$ are both ε, which means $\tilde{A}(\varepsilon)$ is singular.

If $\lambda \neq \varepsilon$ is a root of $\varphi_A(t)$, there exist at least two terms, say $c_{k_1} \otimes t^{\otimes k_1}$ and $c_{k_2} \otimes t^{\otimes k_2}$, $k_1 \neq k_2$, such that

$$c_{k_1} \otimes \lambda^{\otimes k_1} = c_{k_2} \otimes \lambda^{\otimes k_2} = \varphi_A(\lambda) = \det \tilde{A}(\lambda).$$

Both $c_{k_1} \otimes \lambda^{\otimes k_1}$ and $c_{k_2} \otimes \lambda^{\otimes k_2}$ appear in the summand of $\det \tilde{A}(\lambda)$, and $\tilde{A}(\lambda)$ is singular. $\square$

Generally, the converse of the above lemma is not true. However, it holds under the following assumption, which is so weak that it is satisfied by generic matrices.

Assumption 3.2. For a matrix $A \in \mathbb{R}_{\max}^{n\times n}$, we assume that all essential terms of its characteristic polynomial are attained with exactly one permutation. Equivalently, if $c_k \otimes t^{\otimes k}$ is an essential term of $\varphi_A(t)$, there exists exactly one multi-circuit $\mathcal{C}$ with $l(\mathcal{C}) = n - k$ and $w(\mathcal{C}) = c_k$ in $G(A)$.

Proposition 3.3. *Under Assumption* 3.2 *for a matrix* $A \in \mathbb{R}_{\max}^{n\times n}$, λ *is a root of the characteristic polynomial* $\varphi_A(t)$ *if and only if the matrix* $\tilde{A}(\lambda)$ *is singular.*

Proof. The "only if" part has been proved in Lemma 3.1. For the "if" part, suppose that $\tilde{A}(\lambda)$ is singular. If the maximum of $\varphi_A(\lambda)$ is attained with exactly one term, say $c_k \otimes \lambda^{\otimes k}$, then this must be an essential term. From Assumption 3.2, the maximum of $\det \tilde{A}(\lambda)$ is also attained exactly once, which leads to a contradiction. Thus, the maximum of $\varphi_A(\lambda)$ is attained at least twice. Hence, λ is a root of $\varphi_A(t)$. □

In terms of graph theory, λ is a finite root of $\varphi_A(t)$ if and only if there exist at least two λ-maximal multi-circuits of the associated graph. Hereinafter, we proceed with our argument under Assumption 3.2. We note that this kind of assumption also appears in the literature on supertropical algebra [10].

3.2. Definition of algebraic eigenvectors. For a matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n\times n}$ and a multi-circuit $\mathcal{C}$ in $G(A)$, we define four types of matrices, $A_{\mathcal{C}}, A_{\setminus\mathcal{C}}, E_{\mathcal{C}}$ and $E_{\setminus\mathcal{C}}$, as follows:

$$[A_{\mathcal{C}}]_{ij} = \begin{cases} a_{ij} & \text{if } (i,j) \in E(\mathcal{C}), \\ \varepsilon & \text{otherwise}, \end{cases} \qquad [A_{\setminus\mathcal{C}}]_{ij} = \begin{cases} \varepsilon & \text{if } (i,j) \in E(\mathcal{C}), \\ a_{ij} & \text{otherwise}, \end{cases}$$
$$[E_{\mathcal{C}}]_{ij} = \begin{cases} e & \text{if } i = j,\ i \in V(\mathcal{C}), \\ \varepsilon & \text{otherwise}, \end{cases} \qquad [E_{\setminus\mathcal{C}}]_{ij} = \begin{cases} e & \text{if } i = j,\ i \notin V(\mathcal{C}), \\ \varepsilon & \text{otherwise}. \end{cases}$$

Here $V(\mathcal{C})$ and $E(\mathcal{C})$ denote the vertex set and the edge set of $\mathcal{C}$, respectively. Now we present the main result of this paper together with the definition of algebraic eigenvectors.

Theorem 3.4. *Let* $A \in \mathbb{R}_{\max}^{n\times n}$. *Then* $\lambda \in \mathbb{R}_{\max}$ *is an algebraic eigenvalue (i.e., a root of* $\varphi_A(t)$*) if and only if there exists a* λ*-maximal multi-circuit* $\mathcal{C}$ *and a vector* $\boldsymbol{x} \neq (\varepsilon, \varepsilon, \ldots, \varepsilon)^\top$ *such that*

$$(A_{\setminus\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{x} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus\mathcal{C}}) \otimes \boldsymbol{x}.$$

We call such a nontrivial vector $\boldsymbol{x}$ *an algebraic eigenvector of* A *with respect to* λ.

Remark 3.5. If λ is the maximum (geometric) eigenvalue of A, then it coincides with the maximum algebraic eigenvalue and hence $\mathcal{C} = \emptyset$ is λ-maximal. Thus, the equation above will be $A \otimes \boldsymbol{x} = \lambda \otimes \boldsymbol{x}$, which is the same as in the definition of usual (geometric) eigenvalues and eigenvectors. In fact, we will prove later that the other (geometric) eigenvectors of A are also algebraic eigenvectors.

P r o o f. "If part": Suppose there exists a λ-maximal multi-circuit $\mathcal{C}$ and a vector $\boldsymbol{x} \neq (\varepsilon, \varepsilon, \ldots, \varepsilon)^\top$ such that

$$(A_{\setminus\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{x} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus\mathcal{C}}) \otimes \boldsymbol{x}.$$

For $i = 1, 2, \ldots, 2n$, if we evaluate the ith row of

$$\begin{pmatrix} A & \lambda \otimes E \\ E & E \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x} \end{pmatrix},$$

the maximum is attained at least twice. This means, by Theorem 2.1, $\tilde{A}(\lambda)$ is singular. Hence, λ is an algebraic eigenvalue of A by Proposition 3.3.

"Only if" part: Suppose λ is an algebraic eigenvalue of A. First, we consider the case $\lambda \neq \varepsilon$. From Proposition 3.3 and Theorem 2.1 there exists a nontrivial vector $\widetilde{\boldsymbol{u}} = \binom{\boldsymbol{u}}{\boldsymbol{u}} \in \mathbb{R}^{2n}_{\max}$ such that the maximum of each row of

$$\begin{pmatrix} A & \lambda \otimes E \\ E & E \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{u} \end{pmatrix}$$

is attained at least twice. Let $\mathcal{C}$ be a λ-maximal multi-circuit in $G(A)$. We define matrices P and Q and a vector $\boldsymbol{b}$ by

$$P = \begin{pmatrix} A_{\mathcal{C}} & \lambda \otimes E_{\setminus\mathcal{C}} \\ E_{\setminus\mathcal{C}} & E_{\mathcal{C}} \end{pmatrix}, \quad Q = \begin{pmatrix} A_{\setminus\mathcal{C}} & \lambda \otimes E_{\mathcal{C}} \\ E_{\mathcal{C}} & E_{\setminus\mathcal{C}} \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} (A \oplus \lambda \otimes E) \otimes \boldsymbol{u} \\ \boldsymbol{u} \end{pmatrix}.$$

Since the (i, j) entry of P is finite if and only if $j = \pi^{\mathcal{C}}(i)$, P has its inverse P^{-1}. We consider the equation $P \otimes \widetilde{\boldsymbol{x}} = Q \otimes \widetilde{\boldsymbol{x}} \oplus \boldsymbol{b}$ and its solution of the form

$$\widetilde{\boldsymbol{x}} = (P^{-1} \otimes Q)^* \otimes (P^{-1} \otimes \boldsymbol{b}).$$

Then, the vector consisting of the first n entries of $\widetilde{\boldsymbol{x}}$ is the desired algebraic eigenvector. Indeed, since we compute

$$\widetilde{\boldsymbol{x}} = (P^{-1} \otimes Q)^* \otimes P^{-1} \otimes (P \oplus Q) \otimes \widetilde{\boldsymbol{u}} = (P^{-1} \otimes Q)^* \otimes \widetilde{\boldsymbol{u}},$$

we have $\widetilde{\boldsymbol{x}} \geqslant \widetilde{\boldsymbol{u}}$. From our choice of $\widetilde{\boldsymbol{u}}$ we have

$$Q \otimes \widetilde{\boldsymbol{x}} \geqslant Q \otimes \widetilde{\boldsymbol{u}} = (P \oplus Q) \otimes \widetilde{\boldsymbol{u}} = \boldsymbol{b}.$$

Thus, we obtain $P \otimes \widetilde{\boldsymbol{x}} = Q \otimes \widetilde{\boldsymbol{x}}$. By the last n rows of this equation, $\widetilde{\boldsymbol{x}}$ is of the form $\binom{\boldsymbol{x}}{\boldsymbol{x}}$. Checking the first n rows, we have

$$(A_{\setminus\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{x} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus\mathcal{C}}) \otimes \boldsymbol{x}.$$

Next, we consider the case $\lambda = \varepsilon$. For sufficiently small number t we define

$$P_t = \begin{pmatrix} A_{\mathcal{C}} & t \otimes E_{\backslash \mathcal{C}} \\ E_{\backslash \mathcal{C}} & E_{\mathcal{C}} \end{pmatrix}, \quad Q_t = \begin{pmatrix} A_{\backslash \mathcal{C}} & t \otimes E_{\mathcal{C}} \\ E_{\mathcal{C}} & E_{\backslash \mathcal{C}} \end{pmatrix}, \quad \boldsymbol{b}_t = (t, t, \ldots, t)^{\top}.$$

The vector $\widetilde{\boldsymbol{x}}_t = (P_t^{-1} \otimes Q_t)^* \otimes (P_t^{-1} \otimes \boldsymbol{b}_t)$ satisfies $P_t \otimes \widetilde{\boldsymbol{x}}_t = Q_t \otimes \widetilde{\boldsymbol{x}}_t \oplus \boldsymbol{b}_t$. Taking the limit $t \to -\infty$, we obtain the desired vector as the first n entries of $\widetilde{\boldsymbol{x}} = \lim_{t \to -\infty} \widetilde{\boldsymbol{x}}_t$. The fact that $\widetilde{\boldsymbol{x}} \in \mathbb{R}_{\max}^{2n} \backslash \{(\varepsilon, \varepsilon, \ldots, \varepsilon)^{\top}\}$ can be proved as follows. We first verify that all entries of $\widetilde{\boldsymbol{x}}_t$ are of the form $c + dt$, $d \geqslant 0$, by easy computations, which implies $\widetilde{\boldsymbol{x}} \in \mathbb{R}_{\max}^{2n}$. We next see that $\widetilde{\boldsymbol{x}}$ is nontrivial. Since $\lambda = \varepsilon$ is an algebraic eigenvalue, $V(\mathcal{C})$ must not be $\{1, 2, \ldots, n\}$. Take $k \notin V(\mathcal{C})$. From the kth and $(k+n)$th rows of $P_t \otimes \widetilde{\boldsymbol{x}}_t = Q_t \otimes \widetilde{\boldsymbol{x}}_t \oplus \boldsymbol{b}_t$ we have

$$\begin{aligned} &[P_t]_{k,k+n} \otimes [\widetilde{\boldsymbol{x}}_t]_{k+n} \geqslant [\boldsymbol{b}_t]_k, \\ &[P_t]_{k+n,k} \otimes [\widetilde{\boldsymbol{x}}_t]_k \geqslant [Q_t]_{k+n,k+n} \otimes [\widetilde{\boldsymbol{x}}_t]_{k+n}. \end{aligned}$$

Since $[P_t]_{k,k+n} = [\boldsymbol{b}_t]_k = t$ and $[P_t]_{k+n,k} = [Q_t]_{k+n,k+n} = 0$, we have

$$[\widetilde{\boldsymbol{x}}_t]_k \geqslant [\widetilde{\boldsymbol{x}}_t]_{k+n} \geqslant 0.$$

As this holds for arbitrarily small value t, $[\widetilde{\boldsymbol{x}}_t]_k$ is a finite constant independent of t. Thus, $[\widetilde{\boldsymbol{x}}]_k \neq \varepsilon$. $\square$

Example 3.6. Let us consider the max-plus matrix

$$A = \begin{pmatrix} \varepsilon & 9 & 8 & \varepsilon & 0 & \varepsilon \\ 7 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon \\ 6 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

The characteristic polynomial of A is

$$\varphi_A(t) = (t \oplus 8)^{\otimes 2} \otimes (t \oplus 2)^{\otimes 2} \otimes t^{\otimes 2}.$$

Take an algebraic eigenvalue 2 of A and a 2-maximal multi-circuit $\mathcal{C} = \{(1, 2, 1)\}$. Then, the defining equation of the algebraic eigenvectors in Theorem 3.4 becomes

$$\left(\begin{pmatrix} \varepsilon & \varepsilon & 8 & \varepsilon & 0 & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon \\ 6 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \oplus 2 \otimes \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}\right) \otimes \boldsymbol{x}$$

$$= \left(\begin{pmatrix} \varepsilon & 9 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 7 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \oplus 2 \otimes \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}\right) \otimes \boldsymbol{x}.$$

It can be easily verified that $\boldsymbol{x} = (0,5,6,4,4,5)^\top$ is an algebraic eigenvector of A with respect to the algebraic eigenvalue 2. We will show later in Example 3.11 how to compute this algebraic eigenvector.

3.3. Algebraic eigenspaces. Next we describe the set of all algebraic eigenvectors. Let $A \in \mathbb{R}_{\max}^{n\times n}$. For an algebraic eigenvalue λ of A and a multi-circuit $\mathcal{C}$ in $G(A)$, we define

$$W(\lambda,\mathcal{C}) = \{\boldsymbol{x} \in \mathbb{R}_{\max}^{n} \mid (A_{\setminus\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{x} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\setminus\mathcal{C}}) \otimes \boldsymbol{x}\}.$$

Lemma 3.7. *Let $\lambda \neq \varepsilon$ be an algebraic eigenvalue of $A \in \mathbb{R}_{\max}^{n\times n}$ and $\mathcal{C}$ be a λ-maximal multi-circuit in $G(A)$. Then for all multi-circuits $\mathcal{C}'$ in $G(A)$ we have $W(\lambda,\mathcal{C}') \subset W(\lambda,\mathcal{C})$.*

Proof. Let $\boldsymbol{x} = (x_1,x_2,\ldots,x_n)^\top \in W(\lambda,\mathcal{C}')$. We set $x_{j+n} = x_j$ for $j = 1,2,\ldots,n$. Then we have

$$\tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes x_{\pi^{\mathcal{C}}(i)} \leqslant \bigoplus_{j=1}^{2n} \tilde{a}_{ij} \otimes x_j = \tilde{a}_{i\pi^{\mathcal{C}'}(i)} \otimes x_{\pi^{\mathcal{C}'}(i)}$$

for $i = 1,2,\ldots,2n$, where $\tilde{A}(\lambda) = (\tilde{a}_{ij})$. We first assume that all entries of $\boldsymbol{x}$ are finite. Since we have

$$\bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes x_{\pi^{\mathcal{C}}(i)} \leqslant \bigotimes_{i=1}^{2n}\bigoplus_{j=1}^{2n} \tilde{a}_{ij} \otimes x_j = \bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}'}(i)} \otimes x_{\pi^{\mathcal{C}'}(i)} = \bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}'}(i)} \otimes \bigotimes_{i=1}^{2n} x_{\pi^{\mathcal{C}'}(i)}$$
$$\leqslant \bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes \bigotimes_{i=1}^{2n} x_{\pi^{\mathcal{C}}(i)} = \bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes x_{\pi^{\mathcal{C}}(i)},$$

we see that

$$\bigotimes_{i=1}^{2n} \tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes x_{\pi^{\mathcal{C}}(i)} = \bigotimes_{i=1}^{2n} \bigoplus_{j=1}^{2n} \tilde{a}_{ij} \otimes x_j.$$

As this value is finite, we have

$$\tilde{a}_{i\pi^{\mathcal{C}}(i)} \otimes x_{\pi^{\mathcal{C}}(i)} = \bigoplus_{j=1}^{2n} \tilde{a}_{ij} \otimes x_j, \quad i = 1, 2, \ldots, 2n.$$

In particular, we get that $\boldsymbol{x} \in W(\lambda, \mathcal{C})$ from the equalities for $i = 1, 2, \ldots, n$.

Next, we assume that some but not all entries of $\boldsymbol{x} \in W(\lambda, \mathcal{C}')$ are ε. Let $K = \{j \mid x_j \neq \varepsilon\}$ and $L = \{j \mid x_j = \varepsilon\}$. For $i \in K$, since we have

$$\tilde{a}_{i\pi^{\mathcal{C}'}(i)} \otimes x_{\pi^{\mathcal{C}'}(i)} = \bigoplus_{j=1}^{2n} \tilde{a}_{ij} \otimes x_j \geqslant \lambda \otimes x_{i+n} = \lambda \otimes x_i \neq \varepsilon,$$

we obtain $x_{\pi^{\mathcal{C}'}(i)} \neq \varepsilon$. This implies $\pi^{\mathcal{C}'}(K) = K$ and hence $\pi^{\mathcal{C}'}(L) = L$. For $i \in L$ and $k \in K$ we have

$$\tilde{a}_{ik} \otimes x_k \leqslant \bigotimes_{j=1}^{2n} \tilde{a}_{ij} \otimes v_j = \tilde{a}_{i\pi^{\mathcal{C}'}(i)} \otimes x_{\pi^{\mathcal{C}'}(i)} = \varepsilon.$$

Thus $\tilde{a}_{ik}$ must be ε. Since $\det \tilde{A}(\lambda) \neq \varepsilon$ for any finite value λ, $\pi^{\mathcal{C}}$ satisfies $\pi^{\mathcal{C}}(K) = K$ and $\pi^{\mathcal{C}}(L) = L$. Restricting calculations to only the rows and columns indexed by K and making the same argument as above, we obtain $\boldsymbol{x} \in W(\lambda, \mathcal{C})$. □

Let λ be a finite algebraic eigenvalue of A. For λ-maximal multi-circuits $\mathcal{C}_1$ and $\mathcal{C}_2$ in $G(A)$, we have both $W(\lambda, \mathcal{C}_1) \subset W(\lambda, \mathcal{C}_2)$ and $W(\lambda, \mathcal{C}_2) \subset W(\lambda, \mathcal{C}_1)$, which implies that the set $W(\lambda, \mathcal{C})$ does not depend on the choice of λ-maximal multi-circuit $\mathcal{C}$. On the other hand, if $\lambda = \varepsilon$, the λ-maximal multi-circuit is unique under Assumption 3.2. Thus, we write $W(\lambda) := W(\lambda, \mathcal{C})$, where $W(\lambda)$ is the set of all algebraic eigenvectors of A with respect to λ. Since $W(\lambda)$ is the set of solutions of a homogeneous linear system, $W(\lambda)$ is a max-plus subspace of $\mathbb{R}^n_{\max}$. Hence, it is called the *algebraic eigenspace of A with respect to λ*. We also see that the (geometric) eigenspace $U(\lambda)$ is contained in the algebraic eigenspace $W(\lambda)$ by setting $\mathcal{C}' = \emptyset$ in Lemma 3.7.

3.4. Dimensions and multiplicities. In this subsection, we give an upper bound for the dimension of the algebraic eigenspace by the multiplicity of the algebraic eigenvalues.

Theorem 3.8. *Let λ be an algebraic eigenvalue of $A \in \mathbb{R}^{n \times n}_{\max}$. Then, the dimension of the algebraic eigenspace $W(\lambda)$ does not exceed the multiplicity of the root λ in the characteristic polynomial $\varphi_A(t)$.*

To prove this theorem, we distinguish the case where λ is finite from the case where $\lambda = \varepsilon$. We first consider the case, where $\lambda \neq \varepsilon$. Let $\mathcal{C}$ be a λ-maximal multi-circuit in $G(A)$. Then an algebraic eigenvector $\boldsymbol{x} \in W(\lambda)$ satisfies

$$(A_{\backslash\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}}) \otimes \boldsymbol{x} = (A_{\mathcal{C}} \oplus \lambda \otimes E_{\backslash\mathcal{C}}) \otimes \boldsymbol{x}.$$

Since $A_{\mathcal{C}} \oplus \lambda \otimes E_{\backslash\mathcal{C}}$ is invertible, we have

$$((A_{\mathcal{C}} \oplus \lambda \otimes E_{\backslash\mathcal{C}})^{-1} \otimes (A_{\backslash\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}})) \otimes \boldsymbol{x} = \boldsymbol{x}.$$

This means that $\boldsymbol{x}$ is an eigenvector of $B_{\mathcal{C}} := (A_{\mathcal{C}} \oplus \lambda \otimes E_{\backslash\mathcal{C}})^{-1} \otimes (A_{\backslash\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}})$ with respect to the eigenvalue 0 of $B_{\mathcal{C}}$ in the usual sense. Thus, from Theorem 2.5, we see that the dimension of the algebraic eigenspace $W(\lambda)$ is the number of the connected components of $G^c(B_{\mathcal{C}})$.

Lemma 3.9. *Let $\lambda \neq \varepsilon$ be an algebraic eigenvalue of $A \in \mathbb{R}_{\max}^{n\times n}$ and $\mathcal{C}$ be a λ-maximal multi-circuit in $G(A)$. Then, from any multi-circuit $\mathcal{D}$ with weight 0 in $G(B_{\mathcal{C}})$, we can find a multi-circuit $\mathcal{C}'$ satisfying*

$$(l(\mathcal{C}') - l(\mathcal{C}))\lambda = w(\mathcal{C}') - w(\mathcal{C})$$

and

$$(V(\mathcal{C}') \setminus V(\mathcal{C})) \cup (V(\mathcal{C}) \setminus V(\mathcal{C}')) \subset V(\mathcal{D}).$$

It follows from the first equality that $\mathcal{C}'$ is also a λ-maximal multi-circuit in $G(A)$.

Since the proof of this lemma is quite complicated and rather technical, we detail it in the Appendix.

Proof of Theorem 3.8 for the case $\lambda \neq \varepsilon$. Let $\mathcal{C}$ be a λ-maximal multi-circuit with the minimum length in $G(A)$ and m be the dimension of $W(\lambda)$. Then there are m disjoint circuits $D_1, D_2, \ldots, D_m$ with (average) weights 0 in $G(B_{\mathcal{C}})$. Then, it follows from Lemma 3.9 that we find λ-maximal circuits $\mathcal{C}_i, i = 1, 2, \ldots, m$, corresponding to $D_i, i = 1, 2, \ldots, m$. Let $\mathcal{C}'$ be the λ-maximal multi-circuit obtained from $\mathcal{D} := \{D_1, D_2, \ldots, D_m\}$. Since we have

$$\begin{aligned}((V(\mathcal{C}_i) \setminus V(\mathcal{C})) \cup (V(\mathcal{C}) \setminus V(\mathcal{C}_i))) \cap ((V(\mathcal{C}_j) \setminus V(\mathcal{C})) \cup (V(\mathcal{C}) \setminus V(\mathcal{C}_j)))\\ \subset V(D_i) \cap V(D_j) = \emptyset\end{aligned}$$

for $i \neq j$, we see that the construction of the λ-maximal multi-circuit $\mathcal{C}_i$ from circuits D_i in $B_{\mathcal{C}}$ does not interfere with each other. Hence, we have

$$V(\mathcal{C}') \setminus V(\mathcal{C}) = \bigcup_{i=1}^{m} (V(\mathcal{C}_i) \setminus V(\mathcal{C})), \quad V(\mathcal{C}) \setminus V(\mathcal{C}') = \bigcup_{i=1}^{m} (V(\mathcal{C}) \setminus V(\mathcal{C}_i)).$$

From Assumption 3.2 and the minimality of the length of $\mathcal{C}$, we see $l(\mathcal{C}_i) - l(\mathcal{C}) \geqslant 1$. Further, we note

$$l(\mathcal{C}_i) - l(\mathcal{C}) = |V(\mathcal{C}_i)| - |V(\mathcal{C})| = |V(\mathcal{C}_i) \setminus V(\mathcal{C})| - |V(\mathcal{C}) \setminus V(\mathcal{C}_i)|$$

for $i = 1, 2, \ldots, m$. Thus, we have

$$\begin{aligned} l(\mathcal{C}') - l(\mathcal{C}) &= |V(\mathcal{C}') \setminus V(\mathcal{C})| - |V(\mathcal{C}) \setminus V(\mathcal{C}')| \\ &= \sum_{i=1}^{m} (|V(\mathcal{C}_i) \setminus V(\mathcal{C})| - |V(\mathcal{C}) \setminus V(\mathcal{C}_i)|) \geqslant m. \end{aligned}$$

This means there exists a multi-circuit $\mathcal{C}'$ in $G(A)$ satisfying $l(\mathcal{C}') \geqslant l(\mathcal{C}) + m$ and $\mathrm{r.ave}(\mathcal{C}, \mathcal{C}') = \lambda$. Algorithm 2.6 implies that m cannot exceed the multiplicity of λ. $\square$

To prove the case $\lambda = \varepsilon$, we use the following result.

Lemma 3.10 (see, e.g. [8], Theorem 2.10). *The equation $\boldsymbol{x} = A \otimes \boldsymbol{x} \oplus \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^* \otimes \boldsymbol{b}$ if every circuit in $G(A)$ has negative weight.*

P r o o f of Theorem 3.8 for the case $\lambda = \varepsilon$. Let $\mathcal{C}$ be the ε-maximal multi-circuit of length l in $G(A)$. We assume without loss of generality that $V(\mathcal{C}) = \{1, 2, \ldots, l\}$. Let $\boldsymbol{x} \in W(\varepsilon)$ and $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ be the first l rows and the last $(n-l)$ rows of $\boldsymbol{x}$. Then we have

$$\begin{pmatrix} A^1_{\setminus\mathcal{C}} & A^2 \\ A^3 & A^4 \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{x}^1 \\ \boldsymbol{x}^2 \end{pmatrix} = \begin{pmatrix} A^1_{\mathcal{C}} & \mathcal{E} \\ \mathcal{E} & \mathcal{E} \end{pmatrix} \otimes \begin{pmatrix} \boldsymbol{x}^1 \\ \boldsymbol{x}^2 \end{pmatrix}$$

with $A = \begin{pmatrix} A^1 & A^2 \\ A^3 & A^4 \end{pmatrix}$, $A^1 \in \mathbb{R}_{\max}^{l \times l}$, $A^2 \in \mathbb{R}_{\max}^{l \times (n-l)}$, $A^3 \in \mathbb{R}_{\max}^{(n-l) \times l}$, $A^4 \in \mathbb{R}_{\max}^{(n-l) \times (n-l)}$, yielding two equations:

$$A^1_{\setminus\mathcal{C}} \otimes \boldsymbol{x}^1 \oplus A^2 \otimes \boldsymbol{x}^2 = A^1_{\mathcal{C}} \otimes \boldsymbol{x}^1, \quad A^3 \otimes \boldsymbol{x}^1 \oplus A^4 \otimes \boldsymbol{x}^2 = (\varepsilon, \varepsilon, \ldots, \varepsilon)^\top.$$

When we fix a vector $\boldsymbol{x}^2 \in \mathbb{R}_{\max}^{n-l}$, the first equation has the unique solution

$$\boldsymbol{x}^1 = \xi(\boldsymbol{x}^2) := ((A^1_{\mathcal{C}})^{-1} \otimes A^1_{\setminus\mathcal{C}})^* \otimes ((A^1_{\mathcal{C}})^{-1} \otimes A^2 \otimes \boldsymbol{x}^2),$$

because every circuit in $G((A^1_{\mathcal{C}})^{-1} \otimes A^1_{\setminus\mathcal{C}})$ has negative weight by Assumption 3.2. Combining this solution with the second equation, we have

$$W(\varepsilon) = \left\{ \boldsymbol{x} = \begin{pmatrix} \xi(\boldsymbol{x}^2) \\ \boldsymbol{x}^2 \end{pmatrix} \;\middle|\; [\boldsymbol{x}]_j = \varepsilon \text{ if } a_{ij} \neq \varepsilon \text{ for some } i = l+1, \ldots, n \right\}.$$

In particular, the basis of $W(\varepsilon)$ is the set

$$\left\{ \begin{pmatrix} \xi(\tilde{\boldsymbol{e}}_k) \\ \tilde{\boldsymbol{e}}_k \end{pmatrix} \;\middle|\; \begin{array}{l} \text{the } k\text{th column of } A^4 \text{ is } (\varepsilon, \varepsilon, \ldots, \varepsilon)^\top, \\ [\xi(\tilde{\boldsymbol{e}}_k)]_j = \varepsilon \text{ if } a_{ij} \neq \varepsilon \text{ for some } i = l+1, \ldots, n \end{array} \right\},$$

where $\tilde{\boldsymbol{e}}_k, 1 \leqslant k \leqslant n-l$, are the standard basis vectors of $\mathbb{R}^{n-l}_{\min}$. Hence, the dimension of $W(\varepsilon)$ does not exceed $n-l$, which is the multiplicity of the root ε in $\varphi_A(t)$. □

Example 3.11. We again consider

$$A = \begin{pmatrix} \varepsilon & 9 & 8 & \varepsilon & 0 & \varepsilon \\ 7 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon & \varepsilon \\ 6 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

The algebraic eigenspace $W(8)$ is the same as the (geometric) eigenspace of A. Hence, computing

$$((-8) \otimes A)^* = \begin{pmatrix} 0 & 1 & 0 & -4 & -8 & -15 \\ -1 & 0 & -1 & -5 & -9 & -16 \\ -6 & -5 & 0 & -4 & -8 & -15 \\ -2 & -1 & -2 & 0 & -10 & -17 \\ -14 & -13 & -14 & -18 & 0 & -7 \\ -7 & -6 & -7 & -11 & -15 & 0 \end{pmatrix},$$

we identify the basis $(0, -1, -6, -2, -14, -7)^\top$ of $W(8)$.

From the discussion after the statement of Theorem 3.8, the algebraic eigenspace $W(2)$ is the same as the eigenspace of

$$\begin{aligned} B_{\{(1,2,1)\}} &= (A_{\{(1,2,1)\}} \oplus 2 \otimes E_{\setminus\{(1,2,1)\}})^{-1} \otimes (A_{\setminus\{(1,2,1)\}} \oplus 2 \otimes E_{\{(1,2,1)\}}) \\ &= \begin{pmatrix} \varepsilon & -5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ -7 & \varepsilon & -1 & \varepsilon & -9 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & \varepsilon \\ 4 & -2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -1 \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}. \end{aligned}$$

We see that $G(B_{\{(1,2,1)\}})$ has exactly one circuit $(1, 2, 3, 4, 1)$ with average weight 0. Computing $(B_{\{(1,2,1)\}})^*$, we have the basis $(0, 5, 6, 4, 4, 5)^\top$ of $W(2)$.

For the algebraic eigenvalue ε, the ε-maximal multi-circuit in $G(A)$ is $\mathcal{C} = \{(1,3,4,1),(2,2)\}$. The map $\xi\colon\ \mathbb{R}^2_{\max} \to \mathbb{R}^4_{\max}$ in the above proof is given by

$$\xi(\boldsymbol{x}^2) = \left(\begin{pmatrix} \varepsilon & \varepsilon & 8 & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 \\ 6 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}^{-1} \otimes \begin{pmatrix} \varepsilon & 9 & \varepsilon & \varepsilon \\ 7 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \end{pmatrix}\right)^{*} \otimes \left(\begin{pmatrix} \varepsilon & \varepsilon & 8 & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 \\ 6 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}^{-1} \otimes \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}\right) \otimes \boldsymbol{x}^2.$$

Since the first column of the right bottom 2×2 block of A is $(\varepsilon, \varepsilon)^\top$, the vector $((\xi(\tilde{\boldsymbol{e}}_1))^\top, \tilde{\boldsymbol{e}}_1^\top) = (\varepsilon, \varepsilon, -8, \varepsilon, 0, \varepsilon)^\top$ is the basis of $W(\varepsilon)$.

Thus, we have computed the basis of all algebraic eigenspaces of A and have found that the dimensions of all algebraic eigenspaces are 1, which is less than their multiplicities in $\varphi_A(t) = (t \oplus 8)^{\otimes 2} \otimes (t \oplus 2)^{\otimes 2} \otimes t^{\otimes 2}$.

4. Concluding remarks

In this paper, we introduced algebraic eigenvectors with respect to the roots of max-plus characteristic polynomials. We restricted our argument to a matrix such that every essential term of the characteristic polynomial is attained with a single permutation. Without the assumption, the "if part" of Theorem 3.4 does not hold. Hence, it may happen that there is a vector satisfying equation (1.1) for λ that is not a root of the characteristic polynomial. Moreover, the dimension of the algebraic eigenspace cannot be evaluated for the case where our assumption is not satisfied. Thus, some additional conditions will be needed if we make the definition of algebraic eigenvectors in the general case.

Comparison of the dimension of algebraic eigenspaces and the multiplicities of the roots of characteristic polynomials reminds us the diagonalization of matrices. In the max-plus algebra, we conjectured that algebraic eigenvectors with respect to different roots are independent; we have no proof. Hence, we have not derived conditions under which the given matrix can be diagonalized by the transformation matrix consisting of the algebraic eigenvectors. Some of the authors have also tried to define Jordan canonical forms [13], but it succeeded for a very restricted class of matrices. It is a future work to establish a max-plus analogue of the conventional diagonalization theory.

5. Appendix A. Proof of Lemma 3.9

Let $\lambda \neq \varepsilon$ be an algebraic eigenvalue of $A \in \mathbb{R}_{\min}^{n\times n}$ and $\mathcal{C}$ be a λ-maximal multi-circuit in $G(A)$. For any vertex $i \in V(\mathcal{C})$ we denote by $\sigma(i)$ the succeeding vertex of i in the circuit in $\mathcal{C}$; $\sigma^{-1}(i)$ is the preceding vertex of i in $\mathcal{C}$. Recalling that $B_{\mathcal{C}} := (A_{\mathcal{C}} \oplus \lambda \otimes E_{\backslash\mathcal{C}})^{-1} \otimes (A_{\backslash\mathcal{C}} \oplus \lambda \otimes E_{\mathcal{C}})$ and computing the entires of $B_{\mathcal{C}}$, we obtain the correspondence in edges between $G(A)$ and $G(B_{\mathcal{C}})$ shown in Table 1. We see that $G(B_{\mathcal{C}})$ has a multi-circuit $\overleftarrow{\mathcal{C}}$ consisting of the edges $(\sigma(i), i)$, $i \in V(\mathcal{C})$.

$G(A)$		$G(B_c)$	
edge	weight	edge	weight
(i, i'), $i \notin V(\mathcal{C})$	$a_{ii'}$	(i, i')	$-\lambda + a_{ii'}$
$(i, \sigma(i))$, $i \in V(\mathcal{C})$	$a_{i\sigma(i)}$	$(\sigma(i), i)$	$\max\{\lambda, a_{ii}\} - a_{i\sigma(i)}$
(i, i'), $i \in V(\mathcal{C})$, $i' \neq \sigma(i)$	$a_{ii'}$	$(\sigma(i), i')$	$a_{ii'} - a_{i\sigma(i)}$

Table 1. Correspondence between $G(A)$ and $G(B_{\mathcal{C}})$.

Let $\mathcal{D}$ be a multi-circuit in $G(B_{\mathcal{C}})$ with (average) weight 0. We construct a multi-circuit $\mathcal{C}'$ in $G(A)$ by the following steps:

(1) Set $\mathcal{C}' := \emptyset$.

(2) Choose any edge of $\mathcal{D}$ that is not in $E(\overleftarrow{\mathcal{C}})$ and denote the terminal vertex of that edge by i. We define the initial sequence of vertices by $\widehat{C} := (i)$.

(3) The succeeding vertex of i in $\widehat{C}$ is determined by the following rules.

   (a) If $i \notin V(\mathcal{C})$, let i' be the succeeding vertex of i in $\mathcal{D}$. Append i' to $\widehat{C}$ and set $i := i'$.

   (b) If $i \in V(\mathcal{C})$ and $\sigma(i) \notin V(\mathcal{D})$, append $\sigma(i)$ to $\widehat{C}$ and set $i := \sigma(i)$.

   (c) If $i \in V(\mathcal{C})$ and $\sigma(i) \in V(\mathcal{D})$, let i' be the succeeding vertex of $\sigma(i)$ in $\mathcal{D}$. Append i' to $\widehat{C}$ and set $i := i'$.

(4) Repeat (3) until the original vertex i selected in (2) appears again. If we return to i, append the circuit $\widehat{C}$ to $\mathcal{C}'$.

(5) Repeat (2)–(4) while there exist edges (or corresponding terminal vertices) satisfying (2).

(6) Append all circuits in $\mathcal{C}$ that have no common vertices with $\mathcal{D}$ to $\mathcal{C}'$.

(7) Find all loops on $V(\mathcal{D}) \setminus V(\mathcal{C}')$ whose weights are greater than λ. Append them to $\mathcal{C}'$.

An example of these steps is illustrated in Figure 1. The steps (2)–(5) give a union of disjoint circuits because this vertex search is uniquely traced back as follows:

▷ If $j \in V(\mathcal{D})$, let j' be the preceding vertex of j in $\mathcal{D}$. The preceding vertex of j in $\widehat{C}$ is $\sigma^{-1}(j')$ if $j' \in \mathcal{C}$; otherwise it is j'.
▷ If $j \notin V(\mathcal{D})$, the preceding vertex of j in $\widehat{C}$ is $\sigma^{-1}(j)$.

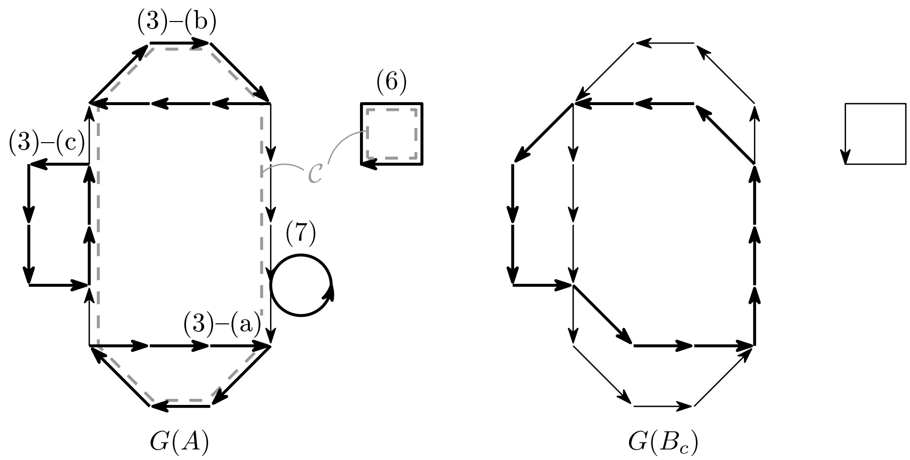

Figure 1. Graph $G(A)$ and circuit $\mathcal{C}$ (left) and graph $G(B_{\mathcal{C}})$ (right). Bold arrows represent the multi-circuit $\mathcal{D}$ (right) and the corresponding multi-circuit $\mathcal{C}'$ (left). Examples of five types of edges are also illustrated.

Lemma A.1. *Let $\mathcal{C}'$ be the multi-circuit in $G(A)$ constructed as above. We have $V(\mathcal{C}) \setminus V(\mathcal{C}') \subset V(\mathcal{D})$ and $(\sigma(i), i) \in E(\mathcal{D})$ for any $i \in V(\mathcal{C}) \setminus V(\mathcal{C}')$.*

Proof. We assume the contrary. Suppose there is a vertex $j \in (V(\mathcal{C}) \setminus V(\mathcal{C}')) \setminus V(\mathcal{D})$. In that case, we show that, without loss of generality, we may assume $\sigma^{-1}(j) \in V(\mathcal{D})$. In order to show the assumption is proper, first we prove $\sigma^{-1}(j) \notin V(\mathcal{C}')$ if we have $\sigma^{-1}(j) \notin V(\mathcal{D})$: If we have $\sigma^{-1}(j) \in V(\mathcal{C}') \setminus V(\mathcal{D})$, it occurs after step (3) was executed for $i := \sigma^{-1}(j)$. Since $\sigma^{-1}(j) \in V(\mathcal{C})$ and $j = \sigma(\sigma^{-1}(j)) \notin V(\mathcal{D})$, case (b) occurs and we have $j \in V(\mathcal{C}')$, leading to a contradiction. Thus, we can continue replacing j with $\sigma^{-1}(j)$ until $\sigma^{-1}(j)$ is contained in $V(\mathcal{D})$. The edge in $\mathcal{D}$ whose terminal vertex is $\sigma^{-1}(j)$ exists but it is not $(j, \sigma^{-1}(j))$ since $j \notin V(\mathcal{D})$. Thus, $\sigma^{-1}(j)$ must be in $V(\mathcal{C}')$ and case (3)(b) occurs for $i := \sigma^{-1}(j)$, which implies $j \in V(\mathcal{C}')$, leading to a contradiction. Hence, we conclude $V(\mathcal{C}) \setminus V(\mathcal{C}') \subset V(\mathcal{D})$, which is the first assertion of the lemma. In particular, if $(\sigma(i), i)$ were not an edge of $\mathcal{D}$ for some $i \in V(\mathcal{C}) \setminus V(\mathcal{C}')$, there would be another edge in $\mathcal{D}$ whose terminal vertex is i. By step (2), this means $i \in V(\mathcal{C}')$, leading to a contradiction. $\square$

Proof of Lemma 3.9. The inclusion $V(\mathcal{C}) \setminus V(\mathcal{C}') \subset V(\mathcal{D})$ is proved in Lemma A.1. On the other hand, from the above procedure, each vertex in $V(\mathcal{C}')$ is contained in $V(\mathcal{C})$ or $V(\mathcal{D})$, which shows $V(\mathcal{C}') \setminus V(\mathcal{C}) \subset V(\mathcal{D})$.

We next prove the equality for weights. Let $\{(\alpha_k, \alpha'_k)\}$ be the set of the edges in $\mathcal{C}'$ constructed by (3)(a), $\{(\beta_k, \sigma(\beta_k))\}$ by (3)(b), $\{(\gamma_k, \gamma'_k)\}$ by (3)(c), $\{(\delta_k, \sigma(\delta_k))\}$ by (6), $\{(\varepsilon_k, \varepsilon_k)\}$ by (7). We denote by $l_\alpha, l_\beta, l_\gamma, l_\delta$ and l_ε the numbers of those edges, respectively. Then we have

$$\begin{aligned}
&w(\mathcal{C}') - w(\mathcal{C}) \\
&\quad = \sum_{k=1}^{l_\alpha} a_{\alpha_k \alpha'_k} + \sum_{k=1}^{l_\beta} a_{\beta_k \sigma(\beta_k)} + \sum_{k=1}^{l_\gamma} a_{\gamma_k \gamma'_k} + \sum_{k=1}^{l_\delta} a_{\delta_k \sigma(\delta_k)} + \sum_{k=1}^{l_\varepsilon} a_{\varepsilon_k \varepsilon_k} - \sum_{i \in V(\mathcal{C})} a_{i\sigma(i)} \\
&\quad = \sum_{k=1}^{l_\alpha} a_{\alpha_k \alpha'_k} + \sum_{k=1}^{l_\gamma} a_{\gamma_k \gamma'_k} + \sum_{k=1}^{l_\varepsilon} a_{\varepsilon_k \varepsilon_k} - \sum_{i \in V(\mathcal{C}) \setminus \{\beta_k, \delta_k\}} a_{i\sigma(i)}.
\end{aligned}$$

Let b_{ij} be the weight of the edge (i, j) in $G(B_{\mathcal{C}})$. From Table 1 we have

$$\begin{aligned}
\sum_{k=1}^{l_\alpha} a_{\alpha_k \alpha'_k} &= l_\alpha \lambda + \sum_{k=1}^{l_\alpha} b_{\alpha_k \alpha'_k}, \\
\sum_{k=1}^{l_\gamma} a_{\gamma_k \gamma'_k} &= \sum_{k=1}^{l_\gamma} a_{\gamma_k \sigma(\gamma_k)} + \sum_{k=1}^{l_\gamma} b_{\sigma(\gamma_k) \gamma'_k}, \\
\sum_{k=1}^{l_\varepsilon} a_{\varepsilon_k \varepsilon_k} &= \sum_{k=1}^{l_\varepsilon} (a_{\varepsilon_k \sigma(\varepsilon_k)} + b_{\sigma(\varepsilon_k) \varepsilon_k}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&w(\mathcal{C}') - w(\mathcal{C}) \\
&\quad = l_\alpha \lambda + \sum_{k=1}^{l_\alpha} b_{\alpha_k \alpha'_k} + \sum_{k=1}^{l_\gamma} b_{\sigma(\gamma^k) \gamma'^k} + \sum_{k=1}^{l_\varepsilon} b_{\sigma(\varepsilon_k) \varepsilon_k} - \sum_{i \in V(\mathcal{C}) \setminus \{\beta_k, \gamma_k, \delta_k, \varepsilon_k\}} a_{i\sigma(i)} \\
&\quad = l_\alpha \lambda + \sum_{k=1}^{l_\alpha} b_{\alpha_k \alpha'_k} + \sum_{k=1}^{l_\gamma} b_{\sigma(\gamma^k) \gamma'^k} + \sum_{k=1}^{l_\varepsilon} b_{\sigma(\varepsilon_k) \varepsilon_k} \\
&\qquad - \left((l(\mathcal{C}) - l_\beta - l_\gamma - l_\delta - l_\varepsilon)\lambda - \sum_{i \in V(\mathcal{C}) \setminus \{\beta_k, \gamma_k, \delta_k, \varepsilon_k\}} b_{\sigma(i) i} \right) \\
&\quad = (l(\mathcal{C}') - l(\mathcal{C}))\lambda + \sum_{k=1}^{l_\alpha} b_{\alpha_k \alpha'_k} + \sum_{k=1}^{l_\gamma} b_{\sigma(\gamma^k) \gamma'^k} \\
&\qquad + \sum_{k=1}^{l_\varepsilon} b_{\sigma(\varepsilon_k) \varepsilon_k} + \sum_{i \in V(\mathcal{C}) \setminus \{\beta_k, \gamma_k, \delta_k, \varepsilon_k\}} b_{\sigma(i) i}.
\end{aligned}$$

From our procedure and Lemma A.1 we have

$$E(\mathcal{D}) = \{(\alpha_k, \alpha'_k) \mid k = 1, 2, \dots, l_\alpha\} \cup \{(\sigma(\gamma_k), \gamma'_k) \mid k = 1, 2, \dots, l_\gamma\} \\ \cup \{(\sigma(\varepsilon_k), \varepsilon_k) \mid k = 1, 2, \dots, l_\varepsilon\} \cup \{(\sigma(i), i) \mid i \in V(\mathcal{C}) \setminus \{\beta_k, \gamma_k, \delta_k, \varepsilon_k\}\}.$$

Thus, we obtain

$$w(\mathcal{C}') - w(\mathcal{C}) = (l(\mathcal{C}') - l(\mathcal{C}))\lambda + w(\mathcal{D}).$$

Since $w(\mathcal{D}) = 0$, we have the desired equality. □

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