Dao Van Duong; Kieu Huu Dung; Nguyen Minh Chuong

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WEIGHTED ESTIMATES FOR COMMUTATORS OF MULTILINEAR HAUSDORFF OPERATORS ON VARIABLE EXPONENT MORREY-HERZ TYPE SPACES

Dao Van Duong, Tuy Hoa,
Kieu Huu Dung, Nguyen Minh Chuong, Hanoi

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Abstract. We establish the boundedness for the commutators of multilinear Hausdorff operators on the product of some weighted Morrey-Herz type spaces with variable exponent with their symbols belonging to both Lipschitz space and central BMO space. By these, we generalize and strengthen some previously known results.

Keywords: multilinear Hausdorff operator; Hardy-Cesàro operator; commutator; Lipschitz space; central BMO space; Morrey-Herz space; A_p weight; variable exponent

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1. Introduction

It is well known that the Hausdorff operator is one of important operators in harmonic analysis, and it is used to solve certain classical problems in analysis. Let Φ be a locally integrable function on $\mathbb{R}^n$. The n-dimensional Hausdorff operator $H_{\Phi,A}$ is defined by Brown and Móricz in [4] as

$$H_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} f(A(t)x)\,\mathrm{d}t, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $A(t)$ is an $n \times n$ invertible matrix for almost every t in the support of Φ. It should be pointed out that if the function Φ and the matrix A are taken appropriately, then the Hausdorff operator $H_{\Phi,A}$ reduces to many classical operators in analysis such as the Hardy operator, the Cesàro operator, the Hardy-Littlewood average operator and the Riemann-Liouville fractional integral operator. Some of their results have been significantly seen in [4], [8], [9], [17], [29], [32] and the references therein.

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In addition, it is natural to extend the study from the theory of linear operators to multilinear operators, which is actually necessary (see [10] and the references therein). Recently, the authors of this paper in [7] have investigated the multilinear operators of Hausdorff type $H_{\Phi,\vec{A}}$ defined as

$$H_{\Phi,\vec{A}}(\vec{f})(x) = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} f_i(A_i(t)x)\,\mathrm{d}t, \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\Phi\colon\ \mathbb{R}^n \to [0,\infty)$ and $A_i(t)$ (for $i = 1,\dots,m$) are $n\times n$ invertible matrices for almost every t in the support of Φ, $f_1, f_2, \dots, f_m\colon\ \mathbb{R}^n \to \mathbb{C}$ are measurable functions, $\vec{f} = (f_1,\dots,f_m)$, $\vec{A} = (A_1,\dots,A_m)$. It is useful to remark that the weighted multilinear Hardy operators (see [16]) and weighted multilinear Hardy-Cesàro operators (see [9]) are two special cases of the multilinear Hausdorff operators $H_{\Phi,\vec{A}}$.

Let b be a measurable function. Let $\mathcal{M}_b$ be the multiplication operator defined by $\mathcal{M}_b f(x) = b(x)f(x)$ for any measurable function f. If $\mathcal{T}$ is a linear operator on some measurable function space, the commutator of Coifman-Rochberg-Weiss type formed by $\mathcal{M}_b$ and $\mathcal{T}$ is defined by $[\mathcal{M}_b,\mathcal{T}]f(x) = (\mathcal{M}_b\mathcal{T} - \mathcal{T}\mathcal{M}_b)f(x)$. In particular, if $\mathcal{T} = H^{\vec{b}}_{\Phi,\vec{A}}$, then we have the commutators of Coifman-Rochberg-Weiss type of the multilinear Hausdorff operator given as follows.

Definition 1.1. Let Φ, $\vec{A}$, $\vec{f}$ be as above. The Coifman-Rochberg-Weiss type commutator of multilinear Hausdorff operator is defined by

$$H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})(x) = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} f_i(A_i(t)x) \prod_{i=1}^{m} (b_i(x) - b_i(A_i(t)x))\,\mathrm{d}t, \quad x \in \mathbb{R}^n, \tag{1.3}$$

where $\vec{b} = (b_1,\dots,b_m)$ and b_i are locally integrable functions on $\mathbb{R}^n$ for all $i = 1,\dots,m$.

Moreover, if we now take $m = n \geqslant 2$, $\Phi(t) = |t|^m\cdot\omega(t)\chi_{[0,1]^m}(t)$ and $A_i(t) = t_i\cdot I_m$ (I_m is an identity matrix), for $t = (t_1,t_2,\dots,t_m)$, where $\omega\colon\ [0,1]^m \to [0,\infty)$ is a measurable function, then $H^{\vec{b}}_{\Phi,\vec{A}}$ reduces to the commutator of weighted multilinear Hardy operator due to Fu et al. (see [16]) defined by

$$H^{\vec{b}}_{\omega}(\vec{f})(x) = \int_{[0,1]^m} \prod_{i=1}^{m} f_i(t_i x) \prod_{i=1}^{m} (b_i(x) - b_i(t_i x))\omega(t)\,\mathrm{d}t, \quad x \in \mathbb{R}^m. \tag{1.4}$$

Also, by $\Phi(t) = |t|^n\cdot\psi(t)\chi_{[0,1]^n}(t)$ and $A_i(t) = s_i(t)\cdot I_n$, where $\psi\colon\ [0,1]^n \to [0,\infty)$, $s_1,\dots,s_m\colon\ [0,1]^n \to \mathbb{R}$ are measurable functions, it is clear to see that $H^{\vec{b}}_{\Phi,A}$ reduces to the commutator of multilinear Hardy-Cesàro operator $U^{m,n,\vec{b}}_{\psi,\vec{s}}$ introduced by Hung and Ky (see [20]) as

$$(1.5)\qquad U^{m,n,\vec{b}}_{\psi,\vec{s}}(x) = \int_{[0,1]^n} \prod_{i=1}^{m} f_i(s_i(t)x) \prod_{i=1}^{m} (b_i(x) - b_i(s_i(t)x))\psi(t)\,\mathrm{d}t, \quad x \in \mathbb{R}^n.$$

In recent years, the theory of function spaces with variable exponents has attracted much more interest of many mathematicians (see, e.g. [1], [3], [7], [13], [19], [23], [31] and others). It is interesting to see that this theory has had some important applications in the electronic fluid mechanics, elasticity, fluid dynamics, recovery of graphics, harmonic analysis and partial differential equations (see [2], [6], [12], [14] and the references therein).

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and T be a Calderón-Zygmund singular integral operator with rough kernels. From classical result of Coifman, Rochberg, and Weiss in [11], Karlovich and Lerner in [25] developed the boundedness of commutator $[b,T]$ to generalized L^p spaces with variable exponent. Also, in order to generalize the result of Chanillo in [5], Izuki in [23] established the boundedness of the higher order commutator on Herz spaces with variable exponent. More recently, Wu in [30] considered the mth-order commutator for the fractional integral as

$$I^m_{\beta,b}(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)(b(x)-b(y))^m}{|x-y|^{n-\beta}}\,\mathrm{d}y,$$

where $\beta \in (0,n)$, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $m \in \mathbb{N}$. Then the author established the boundedness for commutators of fractional integrals on Herz-Morrey spaces with variable exponent.

Motivated by above mentioned results, the goal of this paper is to establish the boundedness for commutators of multilinear Hausdorff operators on the product of weighted Lebesgue, central Morrey, Herz and Morrey-Herz spaces with variable exponent with their symbols belonging to both Lipschitz spaces and central BMO spaces. By these, our results extend and improve some previously known results in the papers [9] and [16].

Our paper is organized as follows. In Section 2, we give necessary preliminaries on weighted Lebesgue spaces, central Morrey spaces, Herz spaces, Morrey-Herz spaces with variable exponent and Lipschitz spaces, central BMO spaces. In Section 3, our main theorems are given. Finally, the results of this paper are proved in Section 4.

2. Preliminaries

In this section, let us recall some basic facts and notations which will be used throughout this paper. The letter C denotes a positive constant which is independent of the main parameters, but may be different from line to line. Given a measurable set Ω, let us denote by χ_Ω its characteristic function, by $|\Omega|$ its Lebesgue measure. For any $a \in \mathbb{R}^n$ and $r > 0$ we denote by $B(a,r)$ the ball centered at a with radius r.

Next, we write $a \lesssim b$ to mean that there is a positive constant C, independent of the main parameters, such that $a \leqslant Cb$. Besides that, we denote $\chi_k = \chi_{C_k}$, $C_k = B_k \setminus B_{k-1}$ and $B_k = \{x \in \mathbb{R}^n \colon\ |x| \leqslant 2^k\}$ for all $k \in \mathbb{Z}$.

Now, we present the definition of the Lebesgue space with variable exponent. For further readings on its deep applications in harmonic analysis, the interested reader may refer to the works [12], [13] and [14].

Definition 2.1. Let $\mathcal{P}(\mathbb{R}^n)$ be the set of all measurable functions $p(\cdot)\colon \mathbb{R}^n \to [1,\infty]$. For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all complex-valued measurable functions f defined on $\mathbb{R}^n$ such that there exists constant $\eta > 0$ satisfying

$$F_p(f/\eta) := \int_{\mathbb{R}^n\setminus\Omega_\infty} \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x + \|f/\eta\|_{L^\infty(\Omega_\infty)} < \infty,$$

where $\Omega_\infty = \{x \in \mathbb{R}^n \colon\ p(x) = \infty\}$. When $|\Omega_\infty| = 0$, it is straightforward that

$$F_p(f/\eta) := \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x < \infty.$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ then becomes a norm space equipped with a norm

$$\|f\|_{L^{p(\cdot)}} = \inf\Big\{\eta > 0\colon\ F_p\Big(\frac{f}{\eta}\Big) \leqslant 1\Big\}.$$

Let us denote by $\mathcal{P}_b(\mathbb{R}^n)$ the class of exponents $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that

$$1 < q_- \leqslant q(x) \leqslant q_+ < \infty \quad \forall\, x \in \mathbb{R}^n,$$

where $q_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} q(x)$ and $q_+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} q(x)$. For $p \in \mathcal{P}_b(\mathbb{R}^n)$, it is useful to remark that we have the following inequalities which are usually used in the sequel.

$$\text{(2.1)}\qquad \begin{aligned} &\text{(i)}\quad \text{If } F_p(f) \leqslant C, \text{ then } \|f\|_{L^{p(\cdot)}} \leqslant \max\{C^{1/q_-}, C^{1/q_+}\} \quad \forall\, f \in L^{p(\cdot)}.\\ &\text{(ii)}\quad \text{If } F_p(f) \geqslant C, \text{ then } \|f\|_{L^{p(\cdot)}} \geqslant \min\{C^{1/q_-}, C^{1/q_+}\} \quad \forall\, f \in L^{p(\cdot)}.\end{aligned}$$

The space $\mathcal{P}_\infty(\mathbb{R}^n)$ is defined by the set of all measurable functions $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and there exists a constant q_∞ such that

$$q_\infty = \lim_{|x|\to\infty} q(x).$$

For $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the weighted variable exponent Lebesgue space $L^{p(\cdot)}_\omega(\mathbb{R}^n)$ is the set of all complex-valued measurable functions f such that $f\omega$ belongs to the $L^{p(\cdot)}(\mathbb{R}^n)$ space and f has norm

$$\|f\|_{L^{p(\cdot)}_\omega} = \|f\omega\|_{L^{p(\cdot)}}.$$

Let $\mathbf{C}^{\log}_0(\mathbb{R}^n)$ denote the set of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at the origin

$$|\alpha(x) - \alpha(0)| \leqslant \frac{C^\alpha_0}{\log(\mathrm{e} + |x|^{-1})} \quad \forall\, x \in \mathbb{R}^n.$$

Denote by $\mathbf{C}^{\log}_\infty(\mathbb{R}^n)$ the set of all log-Hölder continuous functions $\alpha(\cdot)$ satisfying at infinity

$$|\alpha(x) - \alpha_\infty| \leqslant \frac{C^\alpha_\infty}{\log(\mathrm{e} + |x|)} \quad \forall\, x \in \mathbb{R}^n.$$

Next, we give the definition of variable exponent weighted Herz spaces $\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}$ and variable exponent weighted Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}$ (see [26], [31] for more details).

Definition 2.2. Let $0 < p < \infty$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha(\cdot)\colon \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The variable exponent weighted Herz space $\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}$ is defined by

$$\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega} = \{f \in L^{q(\cdot)}_{\rm loc}(\mathbb{R}^n \setminus \{0\})\colon \|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha(\cdot),p}_{q(\cdot),\omega}} = \biggl(\sum_{k=-\infty}^{\infty} \|2^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}_\omega}\biggr)^{1/p}.$$

Definition 2.3. Assume that $0 \leqslant \lambda < \infty$, $0 < p < \infty$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$ and $\alpha(\cdot)\colon \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The variable exponent weighted Morrey-Herz space $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}$ is defined by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega} = \{f \in L^{q(\cdot)}_{\rm loc}(\mathbb{R}^n \setminus \{0\})\colon \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\biggl(\sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^p_{L^{q(\cdot)}_\omega}\biggr)^{1/p}.$$

It is easy to see that $M\dot{K}^{\alpha(\cdot),0}_{p,q(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot),p}_{q(\cdot)}(\mathbb{R}^n)$. Consequently, the Herz space with variable exponent is a special case of Morrey-Herz space with variable exponent.

Let us next state the following corollary which is used in the sequel. The proof is trivial and may be found in [31].

Lemma 2.4. *Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$, $p \in (0,\infty)$ and $\lambda \in [0,\infty)$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then*

$$\|f\chi_j\|_{L^{q(\cdot)}_\omega} \leqslant C \cdot 2^{j(\lambda-\alpha(0))} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} \quad \forall j \in \mathbb{Z}^-,$$
$$\|f\chi_j\|_{L^{q(\cdot)}_\omega} \leqslant C \cdot 2^{j(\lambda-\alpha_\infty)} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot),\omega}} \quad \forall j \in \mathbb{N}.$$

We recall the definition of two-weight λ-central Morrey spaces with variable exponent (see [7]).

Definition 2.5. For $\lambda \in \mathbb{R}$ and $p \in \mathcal{P}_\infty(\mathbb{R}^n)$ we denote by $\dot{B}^{p(\cdot),\lambda}_{\omega_1,\omega_2}$ the class of locally integrable functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{\dot{B}^{p(\cdot),\lambda}_{\omega_1,\omega_2}} = \sup_{R>0} \frac{1}{\omega_1(B(0,R))^{\lambda+1/p_\infty}} \|f\|_{L^{p(\cdot)}_{\omega_2}(B(0,R))} < \infty,$$

where $\|f\|_{L^{p(\cdot)}_{\omega_2}(B(0,R))} = \|f\chi_{B(0,R)}\|_{L^{p(\cdot)}_{\omega_2}}$ and ω_1, ω_2 are non-negative and local integrable functions. Moreover, as $p(\cdot)$ is constant and $\omega_1 = \omega$ and $\omega_2 = \omega^{1/p}$, we define $\dot{B}^{p,\lambda}(\omega) := \dot{B}^{p(\cdot),\lambda}_{\omega,\omega^{1/p}}$.

Next, the following theorem is stated as an embedding result for the Lebesgue spaces with variable exponent (see, for example, [3], Theorem 2, [12], Theorem 2.45, [13], Lemma 3.3.1).

Theorem 2.6. *Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $q(x) \leqslant p(x)$ for almost every $x \in \mathbb{R}^n$, and*

$$\frac{1}{r(\cdot)} := \frac{1}{q(\cdot)} - \frac{1}{p(\cdot)} \quad \textit{and} \quad \|1\|_{L^{r(\cdot)}} < \infty.$$

Then there exists a constant K such that

$$\|f\|_{L^{q(\cdot)}_\omega} \leqslant K\|1\|_{L^{r(\cdot)}} \|f\|_{L^{p(\cdot)}_\omega}.$$

Let us recall the definition of the Lipschitz space and BMO space of John-Nirenberg (see, for example, [24], [28] for more details).

Definition 2.7. Let $0 < \beta \leqslant 1$. The Lipschitz space $\mathrm{Lip}^{\beta}(\mathbb{R}^n)$ is defined as the set of all functions $f\colon \mathbb{R}^n \to \mathbb{C}$ satisfying $\|f\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)} := \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\beta}}.$$

Definition 2.8. The space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions $f\colon \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{Q} \frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes, and f_Q stands for the mean of f on Q, namely, $f_Q = |Q|^{-1}\int_Q f(x)\,\mathrm{d}x$.

Definition 2.9. Let $1 \leqslant q < \infty$ and ω be a weight function. The central bounded mean oscillation space $\dot{\mathrm{CMO}}^q(\omega)$ is defined as the set of all functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{\mathrm{CMO}}^q(\omega)} = \sup_{R>0}\biggl(\frac{1}{\omega(B(0,R))}\int_{B(0,R)} |f(x)-f_{\omega,B(0,R)}|^q \omega(x)\,\mathrm{d}x\biggr)^{1/q},$$

where

$$\omega(B(0,R)) = \int_{B(0,R)} \omega(x)\,\mathrm{d}x \quad \text{and} \quad f_{\omega,B(0,R)} = \frac{1}{\omega(B(0,R))}\int_{B(0,R)} f(x)\omega(x)\,\mathrm{d}x.$$

Remark that Fefferman in [15] discovered the famous result that the space $\mathrm{BMO}(\mathbb{R}^n)$ is the dual space of Hardy space $H^1(\mathbb{R}^n)$. When $\omega = 1$, we write simply $\mathrm{CMO}^q(\mathbb{R}^n) := \mathrm{CMO}^q(\omega)$. The space $\mathrm{CMO}(\mathbb{R}^n)$ can be seen as a local version of $\mathrm{BMO}(\mathbb{R}^n)$ at the origin. Moreover, $\mathrm{BMO}(\mathbb{R}^n) \subsetneqq \mathrm{CMO}^q(\mathbb{R}^n)$, where $1 \leqslant q < \infty$, and the John-Nirenberg inequality is not true in $\mathrm{CMO}^q(\mathbb{R}^n)$.

Next, we recall the A_p weights due to Benjamin Muckenhoupt. One may find more details in the work [27].

Definition 2.10. Let $1 < l < \infty$. We say that a weight $\omega \in A_l$ if there exists a constant C such that for all balls B,

$$\biggl(\frac{1}{|B|}\int_B \omega(x)\,\mathrm{d}x\biggr)\biggl(\frac{1}{|B|}\int_B \omega(x)^{-1/(l-1)}\,\mathrm{d}x\biggr)^{l-1} \leqslant C.$$

We say that a weight $\omega \in A_1$ if there is a constant C such that for all balls B,

$$\frac{1}{|B|}\int_B \omega(x)\,\mathrm{d}x \leqslant C \operatorname*{ess\,inf}_{x\in B} \omega(x).$$

We denote $A_\infty = \bigcup\limits_{1\leqslant l<\infty} A_l$.

We give the following standard result related to the Muckenhoupt weights.

Proposition 2.11.
(i) $A_l \subsetneq A_q$ *for* $1 \leqslant l < q < \infty$.
(ii) *If* $\omega \in A_l$ *for* $1 < l < \infty$, *then there is an* $\varepsilon > 0$ *such that* $l - \varepsilon > 1$ *and* $\omega \in A_{l-\varepsilon}$.

A closing relation to A_∞ is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant C such that

$$\left(\frac{1}{|B|}\int_B \omega(x)^r \,\mathrm{d}x\right)^{1/r} \leqslant \frac{C}{|B|}\int_B \omega(x)\,\mathrm{d}x$$

for all balls $B \subset \mathbb{R}^n$, we then say that ω *satisfies the reverse Hölder condition of order* r and write $\omega \in \mathrm{RH}_r$. According to [22], Theorem 19 and Corollary 21, $\omega \in A_\infty$ if and only if there exists some $r > 1$ such that $\omega \in \mathrm{RH}_r$. Moreover, if $\omega \in \mathrm{RH}_r$, $r > 1$, then $\omega \in \mathrm{RH}_{r+\varepsilon}$ for some $\varepsilon > 0$. We thus write $r_\omega = \sup\{r > 1\colon \omega \in \mathrm{RH}_r\}$ to denote the critical index of ω for the reverse Hölder condition. For the reverse Hölder property on spaces of homogeneous type, see [21].

Let us give the following standard characterization of A_l weights (see [18], [28] for more details).

Proposition 2.12. *Let* $\omega \in A_l \cap \mathrm{RH}_r$, $l \geqslant 1$ *and* $r > 1$. *Then there exist constants* C_1, $C_2 > 0$ *such that*

$$C_1\left(\frac{|E|}{|B|}\right)^l \leqslant \frac{\omega(E)}{\omega(B)} \leqslant C_2\left(\frac{|E|}{|B|}\right)^{(r-1)/r}$$

for any measurable subset E *of the ball* B.

Proposition 2.13. *If* $\omega \in A_l$, $1 \leqslant l < \infty$, *then for any* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and any ball* $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\int_B |f(x)|\,\mathrm{d}x \leqslant C\left(\frac{1}{\omega(B)}\int_B |f(x)|^l \omega(x)\,\mathrm{d}x\right)^{1/l}.$$

3. Statement of the results

Before stating our main results, let us introduce some notations which will be used throughout this section. Let $\gamma_1, \ldots, \gamma_m \in \mathbb{R}$, $\lambda_1, \ldots, \lambda_m \geqslant 0$, $p_1, \ldots, p_m$, $p \in (0, \infty)$, $0 < \beta_1, \ldots, \beta_m \leqslant 1$, $q_i \in \mathcal{P}_b(\mathbb{R}^n)$, $r_i \in \mathcal{P}_\infty(\mathbb{R}^n)$ for $i = 1, \ldots, m$ and $\alpha_1, \ldots, \alpha_m \in L^\infty(\mathbb{R}^n) \cap \mathbf{C}_0^{\log}(\mathbb{R}^n) \cap \mathbf{C}_\infty^{\log}(\mathbb{R}^n)$. The functions $\alpha^*(\cdot)$, $q(\cdot)$, $\gamma(\cdot)$ and numbers β, λ are defined as follows:

$$\beta = \beta_1 + \ldots + \beta_m,$$
$$\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_m,$$

$$\gamma(\cdot)=\gamma_1+\ldots+\gamma_m+\frac{\gamma_1}{r_1(\cdot)}+\ldots+\frac{\gamma_m}{r_m(\cdot)},$$
$$\frac{1}{q(\cdot)}=\frac{1}{q_1(\cdot)}+\ldots+\frac{1}{q_m(\cdot)}+\frac{1}{r_1(\cdot)}+\ldots+\frac{1}{r_m(\cdot)},$$
$$\alpha^*(\cdot)=\alpha_1(\cdot)+\ldots+\alpha_m(\cdot)-\beta_1-\ldots-\beta_m-\frac{\gamma_1+n}{r_1(\cdot)}-\ldots-\frac{\gamma_m+n}{r_m(\cdot)}.$$

For a matrix $A=(a_{ij})_{n\times n}$ we define the norm of A as

$$\|A\|=\Big(\sum_{i,j=1}^{n}|a_{ij}|^2\Big)^{1/2}. \tag{3.1}$$

As above, we conclude that $|Ax|\leqslant\|A\||x|$ for any vector $x\in\mathbb{R}^n$. In particular, if A is invertible, then we have

$$\|A\|^{-n}\leqslant|\det(A^{-1})|\leqslant\|A^{-1}\|^n, \tag{3.2}$$
$$|x|^\alpha\min\{\|A\|^\alpha,\|A^{-1}\|^{-\alpha}\}\leqslant|Ax|^\alpha\leqslant|x|^\alpha\max\{\|A\|^\alpha,\|A^{-1}\|^{-\alpha}\} \tag{3.3}$$

for any $\alpha\in\mathbb{R}$, $x\in\mathbb{R}^n\setminus\{0\}$.

Now, we are ready to state the main results of this paper.

Theorem 3.1. *Let* $\zeta>0$, $\omega_1(x)=|x|^{\gamma_1},\ldots,\omega_m(x)=|x|^{\gamma_m}$, $\omega(x)=|x|^{\gamma(x)}$, $q(\cdot)\in\mathcal{P}_b(\mathbb{R}^n)$, $\alpha^*\in L^\infty(\mathbb{R}^n)\cap\mathbf{C}_0^{\log}(\mathbb{R}^n)\cap\mathbf{C}_\infty^{\log}(\mathbb{R}^n)$, $b_i\in\mathrm{Lip}^{\beta_i}$, $\lambda_1,\ldots,\lambda_m>0$ *and the following conditions be true:*

(3.4) $q_i(A_i^{-1}(t)\cdot)\leqslant\zeta\cdot q_i(\cdot)$ *and* $\|1\|_{L^{\vartheta_i(t,\cdot)}}<\infty$, *a.e.* $t\in\operatorname{supp}(\Phi)$ $\forall\, i=1,\ldots,m$,

(3.5) $\alpha_i(0)-\alpha_{i\infty}\geqslant 0$ $\forall\, i=1,\ldots,m$,

(3.6) either $\gamma_1,\ldots,\gamma_m>-n$, $r_1(0)=r_{1+}, r_{1\infty}=r_{1-},\ldots,r_m(0)=r_{m+}, r_{m\infty}=r_{m-}$,
or $\gamma_1,\ldots,\gamma_m<-n$, $r_1(0)=r_{1-}$, $r_{1\infty}=r_{1+}$, $\ldots$, $r_m(0)=r_{m-}$, $r_{m\infty}=r_{m+}$,
or $\gamma_1=\ldots=\gamma_m=-n$.

Then if

$$\mathcal{C}_1=\int_{\mathbb{R}^n}\frac{\Phi(t)}{|t|^n}\prod_{i=1}^{m}c_{A_i,q_i,\gamma_i}(t)\|I_n-A_i(t)\|^{\beta_i}\|1\|_{L^{\vartheta_i(t,\cdot)}}\phi_{A_i,\lambda_i}(t)\,\mathrm{d}t<\infty, \tag{3.7}$$

where

$$\phi_{A_i,\lambda_i}(t)=\max\{\|A_i(t)\|^{\lambda_i-\alpha_i(0)},\|A_i(t)\|^{\lambda_i-\alpha_{i\infty}}\} \times\max\Big\{\sum_{r=\Theta_n^*-1}^{0}2^{r(\lambda_i-\alpha_i(0))},\sum_{r=\Theta_n^*-1}^{0}2^{r(\lambda_i-\alpha_{i\infty})}\Big\} \tag{3.8}$$

with $\Theta_n^* = \Theta_n^*(t)$ *being the greatest integer number satisfying*

$$\max_{i=1,\dots,m}\{\|A_i(t)\|\cdot\|A_i^{-1}(t)\|\} < 2^{-\Theta_n^*} \quad \text{for a.e. } t \in \mathbb{R}^n,$$

$$c_{A_i,q_i,\gamma_i}(t) = \max\{\|A_i(t)\|^{-\gamma_i}, \|A_i^{-1}(t)\|^{\gamma_i}\}\max\{|\det A_i^{-1}(t)|^{1/q_{i+}},\ |\det A_i^{-1}(t)|^{1/q_{i-}}\},$$

$$\frac{1}{\vartheta_i(t,\cdot)} = \frac{1}{q_i(A_i^{-1}(t)\cdot)} - \frac{1}{\zeta q_i(\cdot)} \quad \forall\, i = 1,\dots,m,$$

we have that $H^{\vec{b}}_{\Phi,\vec{A}}$ *is a bounded operator from* $M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1,\zeta q_1(\cdot),\omega_1} \times \ldots \times M\dot{K}^{\alpha_m(\cdot),\lambda_m}_{p_m,\zeta q_m(\cdot),\omega_m}$ *to* $M\dot{K}^{\alpha^*(\cdot),\lambda}_{p,q(\cdot),\omega}$.

Theorem 3.2. *Suppose that we have the given supposition of Theorem* 3.1. *Let* $1 \leqslant p,\ p_1,\dots,p_m < \infty$, $\lambda_i = 0$ *and* $\alpha_i(0) = \alpha_{i\infty}$ *for all* $i = 1,\dots,m$. *At the same time, let*

$$\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}, \tag{3.9}$$

$$\mathcal{C}_2 = \int_{\mathbb{R}^n} (2-\Theta_n^*)^{m-1/p}\frac{\Phi(t)}{|t|^n} \times \prod_{i=1}^m c_{A_i,q_i,\gamma_i}(t)\|I_n - A_i(t)\|^{\beta_i}\|1\|_{L^{\vartheta_i(t,\cdot)}}\phi_{A_i,0}(t)\,\mathrm{d}t < \infty. \tag{3.10}$$

Then $H^{\vec{b}}_{\Phi,\vec{A}}$ *is a bounded operator from* $\dot{K}^{\alpha_1(\cdot),p_1}_{\zeta q_1(\cdot),\omega_1} \times \ldots \times \dot{K}^{\alpha_m(\cdot),p_m}_{\zeta q_m(\cdot),\omega_m}$ *to* $\dot{K}^{\alpha^*(\cdot),p}_{q(\cdot),\omega}$.

In particular, let us consider the case when $q_1,\dots,q_m$, $r_1,\dots,r_m$, $\alpha_1,\dots,\alpha_m$ are constant. Thus, because of the relation between the commutators of multilinear Hausdorff operator and the commutators of multilinear Hardy-Cesàro operator as mentioned in Section 1, we immediately show that the two theorems above may extend and strengthen some results of Theorem 4.1 in [9] with power weights.

By using the ideas in the proof of Theorem 3.1, we also give an analogous result for the Lebesgue spaces with variable exponent as follows.

Theorem 3.3. *Let* $\zeta > 0$, $\gamma_1,\dots,\gamma_m < 0$, $\omega_1(x) = |x|^{\gamma_1},\dots,\omega_m(x) = |x|^{\gamma_m}$, $\omega(x) = |x|^{\gamma(x)}$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$, $b_1,\dots,b_m \in \mathrm{Lip}^{\beta_i}$, *and let the hypothesis* (3.4) *in Theorem* 3.1 *hold. Thus, if the conditions*

$$\||\cdot|^{\beta_i+\gamma_i/r_i(\cdot)}\|_{L^{r_i(\cdot)}} < \infty \quad \forall\, i = 1,\dots,m, \tag{3.11}$$

$$\mathcal{C}_3 = \int_{\mathbb{R}^n}\frac{\Phi(t)}{|t|^n}\prod_{i=1}^m c_{A_i,q_i,\gamma_i}(t)\cdot\|I_n - A_i(t)\|^{\beta_i}\|1\|_{L^{\vartheta_i(t,\cdot)}}\,\mathrm{d}t < \infty, \tag{3.12}$$

are true, then we have

$$\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\|_{L^{q(\cdot)}_{\omega}} \lesssim \mathcal{C}_3 \cdot \mathcal{B}_{\rm Lip} \cdot \prod_{i=1}^{m} \|f_i\|_{L^{\zeta q_i(\cdot)}_{\omega_i}}.$$

Next, we consider the special case when all of $r_1(\cdot),\dots,r_m(\cdot)$ are constant and the following condition holds:

$$(\mathrm{H}_1)\qquad \alpha_1(\cdot)+\ldots+\alpha_m(\cdot)-\frac{\gamma_1+n}{r_1}-\ldots-\frac{\gamma_m+n}{r_m}=\alpha^{**}(\cdot).$$

Then we obtain some interesting results as well:

Theorem 3.4. *Let $\zeta>0$, $\lambda_1,\dots,\lambda_m>0$, $\gamma_1,\dots,\gamma_m>-n$, $\omega_1(x)=|x|^{\gamma_1}$, …, $\omega_m(x)=|x|^{\gamma_m}$, $\omega(x)=|x|^{\gamma(x)}$, $q(\cdot)\in\mathcal{P}_b(\mathbb{R}^n)$, $b_1\in \dot{\mathrm{CMO}}^{r_1}(\omega_1),\dots,b_m\in\dot{\mathrm{CMO}}^{r_m}(\omega_m)$, the hypotheses* (3.4) *and* (3.5) *in Theorem* 3.1 *hold. Then if*

$$(3.13)\qquad \mathcal{C}_4=\int_{\mathbb{R}^n}\frac{\Phi(t)}{|t|^n}\prod_{i=1}^{m} c_{A_i,q_i,\gamma_i}(t)\cdot\|1\|_{L^{\vartheta_i(t,\cdot)}}\phi_{A_i,\lambda_i}(t) \times(1+\psi_{A_i,\gamma_i,r_i}(t)+2\eta_{A_i,\gamma_i}(t)+\varphi_{A_i}(t))\,\mathrm{d}t<\infty,$$

where

$$\psi_{A_i,\gamma_i,r_i}(t)=(|\det A_i^{-1}(t)|\max\{\|A_i^{-1}(t)\|^{\gamma_i},\|A_i(t)\|^{-\gamma_i}\})^{1/r_i}\|A_i(t)\|^{(\gamma_i+n)/r_i},$$
$$\eta_{A_i,\gamma_i}(t)=\frac{\|A_i(t)\|^{\gamma_i+n}}{\min\{\|A_i(t)\|^{\gamma_i},\|A_i^{-1}(t)\|^{-\gamma_i}\}|\det A_i(t)|},$$
$$\varphi_{A_i}(t)=\max\Big\{\log(2\|A_i(t)\|),\log\frac{1}{\|A_i(t)\|}\Big\},$$

we have that $H^{\vec{b}}_{\Phi,\vec{A}}$ is a bounded operator from $M\dot{K}^{\alpha_1(\cdot),\lambda_1}_{p_1,\zeta q_1(\cdot),\omega_1}\times\ldots\times M\dot{K}^{\alpha_m(\cdot),\lambda_m}_{p_m,\zeta q_m(\cdot),\omega_m}$ to $M\dot{K}^{\alpha^{**}(\cdot),\lambda}_{p,q(\cdot),\omega}$.

Theorem 3.5. *Let $1\leqslant p,\ p_1,\dots,p_m<\infty$, $\lambda_i=0$, $\alpha_i(0)=\alpha_{i\infty}$ for all $i=1,\dots,m$. Also, both the assumptions of Theorem* 3.4 *and the hypothesis* (3.9) *in Theorem* 3.2 *are true. In addition, the following condition holds:*

$$(3.14)\qquad \mathcal{C}_5=\int_{\mathbb{R}^n}(2-\Theta_n^*)^{m-1/p}\frac{\Phi(t)}{|t|^n}\prod_{i=1}^{m} c_{A_i,q_i,\gamma_i}(t)\|1\|_{L^{\vartheta_i(t,\cdot)}}\phi_{A_i,0}(t) \times(1+\psi_{A_i,\gamma_i,r_i}(t)+2\eta_{A_i,\gamma_i}(t)+\varphi_{A_i}(t))\,\mathrm{d}t<\infty.$$

Then we have

$$\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\|_{\dot{K}^{\alpha^{**}(\cdot),p}_{q(\cdot),\omega}} \lesssim \mathcal{C}_5\prod_{i=1}^{m}\|f_i\|_{\dot{K}^{\alpha_i(\cdot),p_i}_{\zeta q_i(\cdot),\omega_i}}.$$

Let us now assume that $q(\cdot)$ and $q_i(\cdot) \in \mathcal{P}_\infty(\mathbb{R}^n)$, λ, α, γ, β, r_i, λ_i, α_i, γ_i, β_i are real numbers such that $\lambda \in (-1/q_\infty, 0)$, $r_i \in (1, \infty)$, $\lambda_i \in (-1/q_{i\infty}, 0)$, $\alpha_i, \gamma_i \in (-n, \infty)$, $\beta_i \in (0, 1]$, $i = 1, 2, \ldots, m$ and denote

$$\begin{aligned}\beta &= \beta_1 + \ldots + \beta_m,\\ \alpha &= \alpha_1 + \ldots + \alpha_m + \frac{\alpha_1}{r_1} + \ldots + \frac{\alpha_m}{r_m},\\ \frac{1}{q(\cdot)} &= \frac{1}{q_1(\cdot)} + \ldots + \frac{1}{q_m(\cdot)} + \frac{1}{r_1} + \ldots + \frac{1}{r_m}.\end{aligned}$$

We are also interested in the commutators of multilinear Hausdorff operators on the product of weighted λ-central Morrey spaces with variable exponent. More precisely, we have the following interesting result.

Theorem 3.6. *Let $\omega_i(x) = |x|^{\gamma_i}$, $v_i(x) = |x|^{\alpha_i}$, $b_i \in \mathrm{Lip}^{\beta_i}$ for all $i = 1, \ldots, m$, $\omega(x) = |x|^\gamma$, $v(x) = |x|^\alpha$ and the following conditions be true:*

$$q_i(A_i^{-1}(t)\cdot) \leqslant q_i(\cdot) \ \text{ and } \ \|1\|_{L^{\vartheta_{1i}(t,\cdot)}} < \infty, \quad \text{a.e. } t \in \operatorname{supp}(\Phi) \ \ \forall\, i = 1, \ldots, m, \tag{3.15}$$

$$\beta + \alpha - \frac{\gamma}{q_\infty} + \sum_{i=1}^{m} (\gamma_i + n)\lambda_i - \alpha_i + \frac{\gamma_i}{q_{i\infty}} = (\gamma + n)\lambda, \tag{3.16}$$

$$\begin{aligned}\mathcal{C}_6 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \|A_i(t)\|^{(\gamma_i+n)(1/q_{i\infty}+\lambda_i)} c_{A_i,q_i,\alpha_i}(t)&\\ \times \|1\|_{L^{\vartheta_{1i}(t,\cdot)}} \|I_n - A_i(t)\|^{\beta_i}\, \mathrm{d}t < \infty,&\end{aligned} \tag{3.17}$$

where

$$\frac{1}{\vartheta_{1i}(t,\cdot)} = \frac{1}{q_i(A_i^{-1}(t)\cdot)} - \frac{1}{q_i(\cdot)} \quad \forall\, i = 1, \ldots, m.$$

Then we have that $H^{\vec{b}}_{\Phi,\vec{A}}$ is a bounded operator from $\dot{B}^{q_1(\cdot),\lambda_1}_{\omega_1,v_1} \times \ldots \times \dot{B}^{q_m(\cdot),\lambda_m}_{\omega_m,v_m}$ to $\dot{B}^{q(\cdot),\lambda}_{\omega,v}$.

Theorem 3.7. *Given $\omega_i(x) = |x|^{\gamma_i}$, $v_i(x) = |x|^{\alpha_i}$, $b_i \in \dot{\mathrm{CMO}}^{r_i}(v_i)$ for all $i = 1, \ldots, m$, $\omega(x) = |x|^\gamma$, $v(x) = |x|^\alpha$, the hypothesis* (3.15) *in Theorem* 3.6*. In addition, the following statements are true:*

$$(\gamma + n)\lambda = \alpha - \frac{\gamma}{q_\infty} + \sum_{i=1}^{m} (\gamma_i + n)\lambda_i - \alpha_i + \frac{\gamma_i}{q_{i\infty}}, \tag{3.18}$$

$$\begin{aligned}\mathcal{C}_7 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \|A_i(t)\|^{(\gamma_i+n)(1/q_{i\infty}+\lambda_i)} c_{A_i,q_i,\alpha_i}(t) \cdot \|1\|_{L^{\vartheta_{1i}(t,\cdot)}}&\\ \times (1 + \psi_{A_i,\gamma_i,r_i}(t) + 2\eta_{A_i,\gamma_i}(t) + \varphi_{A_i}(t))\, \mathrm{d}t < \infty.&\end{aligned} \tag{3.19}$$

Then we conclude that $H^{\vec{b}}_{\Phi,\vec{A}}$ *is a bounded operator from* $\dot{B}^{q_1(\cdot),\lambda_1}_{\omega_1,v_1}\times\ldots\times\dot{B}^{q_m(\cdot),\lambda_m}_{\omega_m,v_m}$ *to* $\dot{B}^{q(\cdot),\lambda}_{\omega,v}$.

When all of $q_1(\cdot),\ldots,q_m(\cdot)$ are constant and the weighted function belongs to the class of Muckenhoupt weights, we also get the following result.

Theorem 3.8. *Let* $1\leqslant q,\ q_1,\ldots,q_m,\ r_1,\ldots,r_m,\ p<\infty$, $\omega\in A_p$ *with the finite critical index* r_ω *for the reverse Hölder condition,* $\delta\in(1,r_\omega)$, $\lambda_i\in(-1/q_i,0)$ *and* $b_i\in\dot{\mathrm{CMO}}^{r_i}(\omega)$ *for all* $i=1,\ldots,m$. *Assume that the following conditions hold:*

$$\frac{1}{q}>\Big(\frac{1}{r_1}+\ldots+\frac{1}{r_m}+\frac{1}{q_1}+\ldots+\frac{1}{q_m}\Big)p\frac{r_\omega}{r_\omega-1},\tag{3.20}$$

$$\lambda=\lambda_1+\ldots+\lambda_m,\tag{3.21}$$

$$\begin{aligned}\mathcal{C}_8=\int_{\mathbb{R}^n}\frac{\Phi(t)}{|t|^n}\prod_{i=1}^m&\mu_{A_i,p,r_i}(t)\cdot|\det A_i^{-1}(t)|^{p/q_i}\|A_i(t)\|^{np/q_i}\\&\times(\|A_i(t)\|^{np\lambda_i}\chi_{\{\|A_i(t)\|\leqslant1\}}(t)\\&\quad+\|A_i(t)\|^{n\lambda_i(\delta-1)/\delta}\chi_{\{\|A_i(t)\|>1\}}(t))\,\mathrm{d}t<\infty,\end{aligned}\tag{3.22}$$

where

$$\mu_{A_i,p,r_i}(t)=1+2\frac{\|A_i(t)\|^{np}}{|\det A_i(t)|^p}+|\det A_i^{-1}(t)|^{p/r_i}\|A_i(t)\|^{np/r_i}+\varphi_{A_i}(t).$$

Then $H^{\vec{b}}_{\Phi,\vec{A}}$ *is a bounded operator from* $\dot{B}^{q_1,\lambda_1}(\omega)\times\ldots\times\dot{B}^{q_m,\lambda_m}(\omega)$ *to* $\dot{B}^{q,\lambda}(\omega)$.

4. Proofs of the theorems

Firstly, for simplicity of notation, we denote

$$\mathcal{B}_{\mathrm{Lip}}=\prod_{i=1}^m\|b_i\|_{\mathrm{Lip}^{\beta_i}},\quad \mathcal{B}_{\mathrm{CMO},\vec{\omega}}=\prod_{i=1}^m\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}\quad\text{and}\quad \mathcal{F}=\prod_{i=1}^m\|f_i\|_{M\dot{K}^{\alpha_i(\cdot),\lambda_i}_{p_i,\varsigma q_i(\cdot),\omega_i}}.$$

4.1. Proof of Theorem 3.1 and Theorem 3.2. By using the versions of the Minkowski inequality for variable Lebesgue spaces from Corollary 2.38 in [12], we have

$$\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|_{L^{q(\cdot)}_\omega}\lesssim\int_{\mathbb{R}^n}\frac{\Phi(t)}{|t|^n}\Big\|\prod_{i=1}^m f(A_i(t)\cdot)(b_i(\cdot)-b_i(A_i(t)\cdot))\chi_k\Big\|_{L^{q(\cdot)}_\omega}\,\mathrm{d}t.\tag{4.1}$$

On the other hand, since $b_i \in \mathrm{Lip}^{\beta_i}$, we get

$$|f(A_i(t)x)(b_i(x) - b_i(A_i(t)x))\chi_k(x)| \leqslant |f(A_i(t)x)|.\|b_i\|_{\mathrm{Lip}^{\beta_i}}\|I_n - A_i(t)\|^{\beta_i}2^{\beta_i k}\chi_k(x).$$

Thus, by applying the Hölder inequality for variable Lebesgue spaces (see also [12], Corollary 2.30) we find

$$\Big\|\prod_{i=1}^{m} f(A_i(t)\cdot)(b_i(\cdot) - b_i(A_i(t)\cdot))\chi_k\Big\|_{L^{q(\cdot)}_{\omega}} \leqslant 2^{k\beta}\mathcal{B}_{\mathrm{Lip}}\prod_{i=1}^{m}\|I_n - A_i(t)\|^{\beta_i}.\prod_{i=1}^{m}\|f_i(A_i(t)\cdot)\chi_k\|_{L^{q_i(\cdot)}_{\omega_i}}\||\cdot|^{\gamma_i/r_i(\cdot)}\chi_k\|_{L^{r_i(\cdot)}}. \tag{4.2}$$

We observe that

$$F_{r_i}(|\cdot|^{\gamma_i/r_i(\cdot)}\chi_k) = \int_{C_k}|x|^{\gamma_i}\,\mathrm{d}x = \int_{2^{k-1}}^{2^k}\int_{S^{n-1}} r^{\gamma_i+n-1}\,\mathrm{d}\sigma(x')\,\mathrm{d}r \lesssim 2^{k(\gamma_i+n)}.$$

Case 1: $k < 0$. Denote

$$\sigma_i = \begin{cases} \dfrac{1}{r_{i+}} & \text{if } (\gamma_i + n) > 0,\\ \dfrac{1}{r_{i-}} & \text{otherwise.}\end{cases}$$

Case 2: $k \geqslant 0$. Denote

$$\sigma_i = \begin{cases} \dfrac{1}{r_{i-}} & \text{if } (\gamma_i + n) > 0,\\ \dfrac{1}{r_{i+}} & \text{otherwise.}\end{cases}$$

From this, by (2.1) we have

$$\||\cdot|^{\gamma_i/r_i(\cdot)}\chi_k\|_{L^{r_i(\cdot)}} \lesssim 2^{k(\gamma_i+n)\sigma_i}. \tag{4.3}$$

Therefore from (4.1)–(4.3) we see that

$$\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|_{L^{q(\cdot)}_{\omega}} \lesssim 2^{k(\beta+\sum_{i=1}^{m}(\gamma_i+n)\sigma_i)}\mathcal{B}_{\mathrm{Lip}}\int_{\mathbb{R}^n}\frac{\Phi(t)}{|t|^n}\prod_{i=1}^{m}\|I_n - A_i(t)\|^{\beta_i}\cdot\|f_i(A_i(t)\cdot)\chi_k\|_{L^{q_i(\cdot)}_{\omega_i}}\,\mathrm{d}t. \tag{4.4}$$

Let us now fix $i \in \{1, 2, \dots, m\}$. Since $\|A_i(t)\| \neq 0$, there exists an integer number $l_i = l_i(t)$ such that $2^{l_i - 1} < \|A_i(t)\| \leqslant 2^{l_i}$. We write $\varrho^*_{\vec{A}}(t)$ by

$$\varrho^*_{\vec{A}}(t) = \max_{i=1,\dots,m} \{\|A_i(t)\| \cdot \|A_i^{-1}(t)\|\}.$$

Hence, by letting $y = A_i(t) \cdot z$ with $z \in C_k$, we arrive at

$$|y| \geqslant \|A_i^{-1}(t)\|^{-1}|z| \geqslant \frac{2^{l_i+k-2}}{\varrho^*_{\vec{A}}} > 2^{k+l_i-2+\Theta^*_n}$$
$$|y| \leqslant \|A_i(t)\| \cdot |z| \leqslant 2^{l_i+k}.$$

These estimations can be used to imply that

$$A_i(t) \cdot C_k \subset \{z \in \mathbb{R}^n \colon\ 2^{k+l_i-2+\Theta^*_n} < |z| \leqslant 2^{k+l_i}\}. \tag{4.5}$$

Now, we will prove the following inequality:

$$\|f_i(A_i(t)\cdot)\chi_k\|_{L^{q_i(\cdot)}_{\omega_i}} \lesssim c_{A_i,q_i,\gamma_i}(t) \cdot \|1\|_{L^{\vartheta_i(t,\cdot)}} \cdot \sum_{r=\Theta^*_n-1}^{0} \|f_i\chi_{k+l_i+r}\|_{L^{\zeta q_i(\cdot)}_{\omega_i}}. \tag{4.6}$$

Indeed, for $\eta > 0$ by (4.5), we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big(\frac{|f_i(A_i(t)x)\chi_k(x)\omega_i(x)|}{\eta}\Big)^{q_i(x)} \mathrm{d}x \\
&\leqslant \int_{A_i(t)C_k} \Big(\frac{|f_i(z)| \max\{\|A_i^{-1}(t)\|^{\gamma_i}, \|A_i(t)\|^{-\gamma_i}\}\omega_i(z)}{\eta}\Big)^{q_i(A_i^{-1}(t)z)} |\det A_i^{-1}(t)| \,\mathrm{d}z \\
&\leqslant \int_{\mathbb{R}^n} \Big(\frac{c_{A_i,q_i,\gamma_i}(t) |\sum_{r=\Theta^*_n-1}^{0} f_i(z)\chi_{k+l_i+r}(z)|\omega_i(z)}{\eta}\Big)^{q_i(A_i^{-1}(t)\cdot z)} \mathrm{d}z.
\end{aligned}$$

From this, by the definition of Lebesgue space with variable exponent, we find

$$\|f_i(A_i(t)\cdot)\chi_k\|_{L^{q_i(\cdot)}_{\omega_i}} \leqslant c_{A_i,q_i,\gamma_i}(t) \cdot \sum_{r=\Theta^*_n-1}^{0} \|f_i\chi_{k+l_i+r}\|_{L^{q_i(A_i^{-1}(t)\cdot)}_{\omega_i}}.$$

In view of (3.4) and Theorem 2.6, we deduce

$$\|f\|_{L^{q_i(A_i^{-1}(t)\cdot)}_{\omega_i}} \lesssim \|1\|_{L^{\vartheta_i(t,\cdot)}} \cdot \|f\|_{L^{\zeta q_i(\cdot)}_{\omega_i}}.$$

This completes the proof of inequality (4.6). Now, combining (4.4) and (4.6), it is easy to see that

$$\begin{aligned}(4.7)\quad \|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|_{L^{q(\cdot)}_{\omega}} \lesssim 2^{k(\beta+\sum_{i=1}^m(\gamma_i+n)\sigma_i)}\mathcal{B}_{\rm Lip}\bigg(\int_{\mathbb{R}^n}\frac{\Phi(t)}{|t|^n}\prod_{i=1}^m c_{A_i,q_i,\gamma_i}(t)\|1\|_{L^{\vartheta_i(t,\cdot)}}\\ \times\|I_n-A_i(t)\|^{\beta_i}\prod_{i=1}^m\sum_{r=\Theta_n^*-1}^{0}\|f_i\chi_{k+l_i+r}\|_{L^{\zeta q_i(\cdot)}_{\omega_i}}\,{\rm d}t\bigg).\end{aligned}$$

Thus, by applying Lemma 2.4 in Section 2, we have

$$\begin{aligned}(4.8)\quad &\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|_{L^{q(\cdot)}_{\omega}}\\ &\lesssim \mathcal{B}_{\rm Lip}\cdot\mathcal{F}\bigg(\int_{\mathbb{R}^n}\frac{\Phi(t)}{|t|^n}\mathcal{U}(t)\prod_{i=1}^m c_{A_i,q_i,\gamma_i}(t)\|1\|_{L^{\vartheta_i(t,\cdot)}}\|I_n-A_i(t)\|^{\beta_i}\,{\rm d}t\bigg).\end{aligned}$$

Here

$$\begin{aligned}\mathcal{U}(t)=2^{k(\beta+\sum_{i=1}^m(\gamma_i+n)\sigma_i)}\prod_{i=1}^m\bigg(&2^{(k+l_i)(\lambda_i-\alpha_i(0))}\sum_{r=\Theta_n^*-1}^{0}2^{r(\lambda_i-\alpha_i(0))}\\ &+2^{(k+l_i)(\lambda_i-\alpha_{i\infty})}\sum_{r=\Theta_n^*-1}^{0}2^{r(\lambda_i-\alpha_{i\infty})}\bigg).\end{aligned}$$

Since $2^{l_i-1}<\|A_i(t)\|\leqslant 2^{l_i}$ for all $i=1,\dots,m$, it implies that

$$2^{l_i(\lambda_i-\alpha_i(0))}+2^{l_i(\lambda_i-\alpha_{i\infty})}\lesssim\max\{\|A_i(t)\|^{\lambda_i-\alpha_i(0)},\|A_i(t)\|^{\lambda_i-\alpha_{i\infty}}\}.$$

From this, we can estimate $\mathcal{U}$ as

$$\begin{aligned}\mathcal{U}(t)&\lesssim 2^{k(\beta+\sum_{i=1}^m(\gamma_i+n)\sigma_i)}\prod_{i=1}^m\max\{\|A_i(t)\|^{\lambda_i-\alpha_i(0)},\|A_i(t)\|^{\lambda_i-\alpha_{i\infty}}\}\\ &\quad\times\bigg\{2^{k(\lambda_i-\alpha_i(0))}\sum_{r=\Theta_n^*-1}^{0}2^{r(\lambda_i-\alpha_i(0))}+2^{k(\lambda_i-\alpha_{i\infty})}\sum_{r=\Theta_n^*-1}^{0}2^{r(\lambda_i-\alpha_{i\infty})}\bigg\}\\ &\lesssim 2^{k(\beta+\sum_{i=1}^m(\gamma_i+n)\sigma_i)}\prod_{i=1}^m\max\{\|A_i(t)\|^{\lambda_i-\alpha_i(0)},\|A_i(t)\|^{\lambda_i-\alpha_{i\infty}}\}\\ &\quad\times\max\bigg\{\sum_{r=\Theta_n^*-1}^{0}2^{r(\lambda_i-\alpha_i(0))},\sum_{r=\Theta_n^*-1}^{0}2^{r(\lambda_i-\alpha_{i\infty})}\bigg\}\bigg\{2^{k(\lambda_i-\alpha_i(0))}+2^{k(\lambda_i-\alpha_{i\infty})}\bigg\}.\end{aligned}$$

This implies that

$$\mathcal{U}(t) \lesssim 2^{k(\beta+\sum_{i=1}^m(\gamma_i+n)\sigma_i)}\prod_{i=1}^m\{2^{k(\lambda_i-\alpha_i(0))}+2^{k(\lambda_i-\alpha_{i\infty})}\}\phi_{A_i,\lambda}(t).$$

Thus, by (4.8) it is not difficult to show that

$$\begin{aligned}(4.9)\quad &\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|_{L^{q(\cdot)}_\omega}\\ &\lesssim \mathcal{C}_1\cdot\mathcal{B}_{\rm Lip}\cdot\mathcal{F}\cdot 2^{k(\beta+\sum_{i=1}^m(\gamma_i+n)\sigma_i)}\prod_{i=1}^m(2^{k(\lambda_i-\alpha_i(0))}+2^{k(\lambda_i-\alpha_{i\infty})}).\end{aligned}$$

Next, using Proposition 2.5 in [26], we have

$$(4.10)\qquad \|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\|_{M\dot{K}^{\alpha^*(\cdot),\lambda}_{p,q(\cdot),\omega}} \lesssim \max\Big\{\sup_{k_0<0,k_0\in\mathbb{Z}}E_1,\ \sup_{k_0\geqslant 0,k_0\in\mathbb{Z}}(E_2+E_3)\Big\},$$

where

$$E_1 = 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha^*(0)p}\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|^p_{L^{q(\cdot)}_\omega}\Big)^{1/p},$$

$$E_2 = 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{-1}2^{k\alpha^*(0)p}\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|^p_{L^{q(\cdot)}_\omega}\Big)^{1/p},$$

$$E_3 = 2^{-k_0\lambda}\Big(\sum_{k=0}^{k_0}2^{k\alpha^*_\infty p}\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|^p_{L^{q(\cdot)}_\omega}\Big)^{1/p}.$$

Now, we need to estimate the upper bounds for E_1, E_2 and E_3. Note that, using (4.9), E_1 is dominated by

$$\begin{aligned}(4.11)\qquad E_1 \lesssim \mathcal{C}_1\cdot\mathcal{B}_{\rm Lip}\cdot\mathcal{F}\cdot 2^{-k_0\lambda}\Big(&\sum_{k=-\infty}^{k_0}2^{k\left(\alpha^*(0)+\beta+\sum_{i=1}^m(\gamma_i+n)\sigma_i\right)p}\\ &\times\prod_{i=1}^m(2^{k(\lambda_i-\alpha_i(0))p}+2^{k(\lambda_i-\alpha_{i\infty})p})\Big)^{1/p} := \mathcal{C}_1\cdot\mathcal{B}_{\rm Lip}\cdot\mathcal{F}\cdot\mathcal{T}_0.\end{aligned}$$

In view of α_* we have

$$\begin{aligned}\mathcal{T}_0 = 2^{-k_0\lambda}\cdot\Big(&\sum_{k=-\infty}^{k_0}2^{k(\sum_{i=1}^m\alpha_i(0)+\sum_{i=1}^m(\gamma_i+n)(\sigma_i-1/r_i(0)))p}\\ &\times\prod_{i=1}^m(2^{k(\lambda_i-\alpha_i(0))p}+2^{k(\lambda_i-\alpha_{i\infty})p})\Big)^{1/p}.\end{aligned}$$

Note that, by defining σ_i and (3.6), it is clear to see that

$$(\gamma_i + n)\Big(\sigma_i - \frac{1}{r_i(0)}\Big) = 0 \quad \forall\, i = 1, \dots, m. \tag{4.12}$$

So, we get

$$\begin{aligned}
\mathcal{T}_0 &= 2^{-k_0\lambda}.\Big(\sum_{k=-\infty}^{k_0} 2^{k\sum_{i=1}^m \alpha_i(0)p} \prod_{i=1}^m \big(2^{k(\lambda_i-\alpha_i(0))p} + 2^{k(\lambda_i-\alpha_{i\infty})p}\big)\Big)^{1/p} \\
&= 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} \prod_{i=1}^m (2^{k\lambda_i p} + 2^{k(\lambda_i-\alpha_{i\infty}+\alpha_i(0))p})\Big)^{1/p} \\
&\lesssim \Big(\prod_{i=1}^m 2^{-k_0\lambda_i p}\Big\{\sum_{k=-\infty}^{k_0} 2^{k\lambda_i p} + \sum_{k=-\infty}^{k_0} 2^{k(\lambda_i-\alpha_{i\infty}+\alpha_i(0))p}\Big\}\Big)^{1/p}.
\end{aligned}$$

From $\lambda_i > 0$ for all $i = 1, \dots, m$ and (3.5) we obtain

$$\begin{aligned}
\mathcal{T}_0 &\lesssim \Big(\prod_{i=1}^m 2^{-k_0\lambda_i p}\Big\{\frac{2^{k_0\lambda_i p}}{1-2^{-\lambda_i p}} + \frac{2^{k_0(\lambda_i-\alpha_{i\infty}+\alpha_i(0))p}}{1-2^{-(\lambda_i-\alpha_{i\infty}+\alpha_i(0))p}}\Big\}\Big)^{1/p} \\
&\lesssim \prod_{i=1}^m \Big\{\frac{1}{1-2^{-\lambda_i p}} + \frac{2^{k_0(-\alpha_{i\infty}+\alpha_i(0))}}{1-2^{-(\lambda_i-\alpha_{i\infty}+\alpha_i(0))p}}\Big\} \lesssim \prod_{i=1}^m (1 + 2^{k_0(\alpha_i(0)-\alpha_{i\infty})}).
\end{aligned}$$

Consequently, from (4.11) we conclude

$$E_1 \lesssim \mathcal{C}_1 \cdot \mathcal{B}_{\rm Lip} \cdot \mathcal{F} \cdot \prod_{i=1}^m (1 + 2^{k_0(\alpha_i(0)-\alpha_{i\infty})}). \tag{4.13}$$

Using a similar argument as E_1, we also get

$$E_2 \lesssim \mathcal{C}_1 \cdot \mathcal{B}_{\rm Lip} \cdot \mathcal{F} \cdot 2^{-k_0\lambda}. \tag{4.14}$$

Next, we see that

$$E_3 \lesssim \mathcal{C}_1 \cdot \mathcal{B}_{\rm Lip} \cdot \mathcal{F} \cdot \mathcal{T}_\infty, \tag{4.15}$$

where

$$\begin{aligned}
\mathcal{T}_\infty = 2^{-k_0\lambda}\Big(&\sum_{k=0}^{k_0} 2^{k(\sum_{i=1}^m \alpha_{i\infty} + \sum_{i=1}^m (\gamma_i+n)(\sigma_i - 1/r_{i\infty})p)} \\
&\times \prod_{i=1}^m (2^{k(\lambda_i-\alpha_i(0))p} + 2^{k(\lambda_i-\alpha_{i\infty})p})\Big)^{1/p}.
\end{aligned}$$

Remark that by defining σ_i and (3.6) we deduce

$$(\gamma_i + n)\Big(\sigma_i - \frac{1}{r_{i\infty}}\Big) = 0 \quad \forall\, i = 1, \ldots, m. \tag{4.16}$$

Thus, by estimating in the same way as $\mathcal{T}_0$, we also have

$$\mathcal{T}_\infty \lesssim \prod_{i=1}^{m} 2^{-k_0\lambda_i}\Big(\sum_{k=0}^{k_0} 2^{k\lambda_i p} + \sum_{k=0}^{k_0} 2^{k(\lambda_i+\alpha_{i\infty}-\alpha_i(0))p}\Big)^{1/p} := \prod_{i=1}^{m} \mathcal{T}_{i,\infty}.$$

In the case when $\lambda_i + \alpha_{i\infty} - \alpha_i(0) = 0$, we have

$$\mathcal{T}_{i,\infty} \leqslant 2^{-k_0\lambda_i}\Big(\frac{2^{k_0\lambda_i p}-1}{2^{\lambda_i\cdot p}-1} + (k_0+1)\Big)^{1/p} \lesssim 2^{-k_0\lambda_i}(k_0+1)^{1/p} + |2^{\lambda_i p}-1|^{-1/p}.$$

Otherwise, we get

$$\begin{aligned}\mathcal{T}_{i,\infty} &\leqslant 2^{-k_0\lambda_i}\Big(\frac{2^{k_0\lambda_i p}-1}{2^{\lambda_i\cdot p}-1} + \frac{2^{k_0(\lambda_i+\alpha_{i\infty}-\alpha_i(0))p}-1}{2^{(\lambda_i+\alpha_{i\infty}-\alpha_i(0))p}-1}\Big)^{1/p}\\ &\lesssim 2^{k_0(\alpha_{i\infty}-\alpha_i(0))} + |2^{\lambda_i p}-1|^{-1/p} + 2^{-k_0\lambda_i}.\end{aligned}$$

This implies $\mathcal{T}_\infty \lesssim \prod_{i=1}^{m} \mathcal{L}_i$, where

$$\mathcal{L}_i = \begin{cases} 2^{k_0(\alpha_{i\infty}-\alpha_i(0))} + |2^{\lambda_i p}-1|^{-1/p} + 2^{-k_0\lambda_i} & \text{if } \lambda_i + \alpha_{i\infty} - \alpha_i(0) \neq 0,\\ 2^{-k_0\lambda_i}(k_0+1)^{1/p} + |2^{\lambda_i p}-1|^{-1/p} & \text{otherwise.}\end{cases}$$

From this, by (4.15) we obtain

$$E_3 \lesssim \mathcal{C}_1 \cdot \mathcal{B}_{\mathrm{Lip}} \cdot \mathcal{F} \cdot \prod_{i=1}^{m} \mathcal{L}_i. \tag{4.17}$$

By (4.10), (4.13), (4.14) and (4.17), the proof of Theorem 3.1 is finished.

Next, let us give the proof for Theorem 3.2. From Proposition 3.8 in [1] it is easy to see that

$$\begin{aligned}\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\|_{\dot{K}^{\alpha^*(\cdot),p}_{q(\cdot),\omega}} &\lesssim \Big(\sum_{k=-\infty}^{-1} 2^{k\alpha^*(0)p}\|H_{\Phi,\vec{A}}(\vec{f})\chi_k\|^p_{L^{q(\cdot)}_\omega}\Big)^{1/p}\\ &\quad + \Big(\sum_{k=0}^{\infty} 2^{k\alpha^*_\infty p}\|H_{\Phi,\vec{A}}(\vec{f})\chi_k\|^p_{L^{q(\cdot)}_\omega}\Big)^{1/p} := \mathcal{H}_0 + \mathcal{H}_1.\end{aligned} \tag{4.18}$$

Next, we need to estimate the upper bound of $\mathcal{H}_0$ and $\mathcal{H}_1$. In view of (4.7) and (4.12), by using the Minkowski inequality, we find

$$\begin{aligned}(4.19)\qquad \mathcal{H}_0 \lesssim \mathcal{B}_{\mathrm{Lip}} \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m c_{A_i,q_i,\gamma_i}(t) \|1\|_{L^{\vartheta_i(t,\cdot)}} \|I_n - A_i(t)\|^{\beta_i} \\ \times \Big\{ \sum_{k=-\infty}^{-1} 2^{k\sum_{i=1}^m \alpha_i(0)p} \prod_{i=1}^m \Big(\sum_{r=\Theta_n^*-1}^{0} \|f_i\chi_{k+l_i+r}\|_{L^{\zeta q_i(\cdot)}_{\omega_i}} \Big)^p \Big\}^{1/p} \,\mathrm{d}t.\end{aligned}$$

Using (3.9) and the Hölder inequality, it follows that

$$\begin{aligned}(4.20)\qquad \Big\{ \sum_{k=-\infty}^{-1} 2^{k\sum_{i=1}^m \alpha_i(0)p} \prod_{i=1}^m \Big(\sum_{r=\Theta_n^*-1}^{0} \|f_i\chi_{k+l_i+r}\|_{L^{\zeta q_i(\cdot)}_{\omega_i}} \Big)^p \Big\}^{1/p} \\ \leqslant \prod_{i=1}^m \Big\{ \sum_{k=-\infty}^{-1} 2^{k\alpha_i(0)p_i} \Big(\sum_{r=\Theta_n^*-1}^{0} \|f_i\chi_{k+l_i+r}\|_{L^{\zeta q_i(\cdot)}_{\omega_i}} \Big)^{p_i} \Big\}^{1/p_i}.\end{aligned}$$

By $p_i \geqslant 1$ for all $i = 1, \dots, m$ we have

$$\Big(\sum_{r=\Theta_n^*-1}^{0} \|f_i\chi_{k+l_i+r}\|_{L^{\zeta q_i(\cdot)}_{\omega_i}} \Big)^{p_i} \leqslant (2-\Theta_n^*)^{p_i-1} \sum_{r=\Theta_n^*-1}^{0} \|f_i\chi_{k+l_i+r}\|^{p_i}_{L^{\zeta q_i(\cdot)}_{\omega_i}}.$$

Thus, combining (4.19) and (4.20), we deduce

$$\begin{aligned}(4.21)\quad \mathcal{H}_0 \lesssim \mathcal{B}_{\mathrm{Lip}} \\ \times \int_{\mathbb{R}^n} (2-\Theta_n^*)^{m-1/p} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m c_{A_i,q_i,\gamma_i}(t) \|1\|_{L^{\vartheta_i(t,\cdot)}} \|I_n - A_i(t)\|^{\beta_i} \mathcal{H}_{0,i} \,\mathrm{d}t.\end{aligned}$$

Here

$$\mathcal{H}_{0,i} = \sum_{r=\Theta_n^*-1}^{0} \Big(\sum_{k=-\infty}^{-1} 2^{k\alpha_i(0)p_i} \|f_i\chi_{k+l_i+r}\|^{p_i}_{L^{\zeta q_i(\cdot)}_{\omega_i}} \Big)^{1/p_i} \qquad \forall\, i = 1, 2, \dots, m.$$

Hence, we estimate

$$\begin{aligned}\mathcal{H}_{0,i} &= \sum_{r=\Theta_n^*-1}^{0} \Big(\sum_{t=-\infty}^{-1+l+r} 2^{(t-l_i-r)\alpha_i(0)p_i} \|f_i\chi_t\|^{p_i}_{L^{\zeta q_i(\cdot)}_{\omega_i}} \Big)^{1/p_i} \\ &\lesssim \sum_{r=\Theta_n^*-1}^{0} 2^{-(l_i+r)\alpha_i(0)} \Big(\sum_{t=-\infty}^{\infty} 2^{t\alpha_i(0)p_i} \|f_i\chi_t\|^{p_i}_{L^{\zeta q_i(\cdot)}_{\omega_i}} \Big)^{1/p_i}.\end{aligned}$$

By $\alpha_i(0) = \alpha_{i\infty}$ and [1], Proposition 3.8 we get

$$\begin{aligned}(4.22)\qquad \mathcal{H}_{0,i} &\lesssim \sum_{r=\Theta_n^*-1}^{0} 2^{-(l_i+r)\alpha_i(0)} \|f_i\|_{\dot{K}^{\alpha_i(\cdot),p_i}_{\zeta q_i(\cdot),\omega_i}} \\ &= 2^{-l_i\alpha_i(0)} \cdot \left(\sum_{r=\Theta_n^*-1}^{0} 2^{-r\alpha_i(0)}\right) \|f_i\|_{\dot{K}^{\alpha_i(\cdot),p_i}_{\zeta q_i(\cdot),\omega_i}}.\end{aligned}$$

Since $2^{l_i-1} < \|A_i(t)\| \leqslant 2^{l_i}$, we deduce that $2^{-l_i\alpha_i(0)} \lesssim \|A_i(t)\|^{-\alpha_i(0)}$. Hence, by (4.22) we have

$$\mathcal{H}_{0,i} \lesssim \phi_{A_i,0}(t) \cdot \|f_i\|_{\dot{K}^{\alpha_i(\cdot),p_i}_{\zeta q_i(\cdot),\omega_i}}.$$

As above, by (4.21) we make

$$\mathcal{H}_0 \lesssim \mathcal{C}_2 \cdot \mathcal{B}_{\mathrm{Lip}} \cdot \prod_{i=1}^{m} \|f_i\|_{\dot{K}^{\alpha_i(\cdot),p_i}_{\zeta q_i(\cdot),\omega_i}}.$$

By estimating as $\mathcal{H}_0$, we also get

$$\mathcal{H}_1 \lesssim \mathcal{C}_2 \cdot \mathcal{B}_{\mathrm{Lip}} \cdot \prod_{i=1}^{m} \|f_i\|_{\dot{K}^{\alpha_i(\cdot),p_i}_{\zeta q_i(\cdot),\omega_i}}.$$

There, by (4.18) we finish the desired conclusion.

4.2. Proof of Theorem 3.4 and Theorem 3.5. Applying the Minkowski inequality and the Hölder inequality for variable Lebesgue spaces, we get

$$\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|_{L^{q(\cdot)}_{\omega}} \lesssim \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \|(b_i(\cdot) - b_i(A_i(t)\cdot))\|_{L^{r_i}(\omega_i,B_k)} \|f_i(A_i(t)\cdot)\chi_k\|_{L^{q_i(\cdot)}_{\omega_i}}\,\mathrm{d}t.$$

By (4.6) we deduce

$$\begin{aligned}(4.23)\qquad \|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|_{L^{q(\cdot)}_{\omega}} &\lesssim \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} c_{A_i,q_i,\gamma_i}(t) \|b_i(\cdot) - b_i(A_i(t)\cdot)\|_{L^{r_i}(\omega_i,B_k)} \\ &\quad \times \|1\|_{L^{\vartheta_i(t,\cdot)}} \prod_{i=1}^{m} \sum_{r=\Theta_n^*-1}^{0} \|f_i\chi_{k+l_i+r}\|_{L^{\zeta q_i(\cdot)}_{\omega_i}}\,\mathrm{d}t.\end{aligned}$$

On the other hand, we need to prove that

$$\begin{aligned}(4.24)\qquad &\|b_i(\cdot) - b_i(A_i(t)\cdot)\|_{L^{r_i}(\omega_i,B_k)} \\ &\quad \lesssim 2^{k(\gamma_i+n)/r_i}(1 + \psi_{A_i,\gamma_i,r_i}(t) + 2\eta_{A_i,\gamma_i}(t) + \varphi_{A_i}(t))\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}.\end{aligned}$$

In fact, we put $a_{1,i}(\cdot) = b_i(\cdot) - b_{i,\omega_i,B_k}$, $a_{2,i}(\cdot) = b_i(A_i(t)\cdot) - b_{i,\omega_i,A_i(t)B_k}$ and $a_{3,i}(\cdot) = b_{i,\omega_i,B_k} - b_{i,\omega_i,A_i(t)B_k}$. Here

$$b_{i,\omega_i,U} = \frac{1}{\omega_i(U)} \int_U b_i(x)\omega_i(x)\,\mathrm{d}x.$$

Then we have

$$\begin{aligned}(4.25)\qquad &\|b_i(\cdot) - b_i(A_i(t)\cdot)\|_{L^{r_i}(\omega_i,B_k)} \\ &\quad \leqslant \|a_{1,i}\|_{L^{r_i}(\omega_i,B_k)} + \|a_{2,i}\|_{L^{r_i}(\omega_i,B_k)} + \|a_{3,i}\|_{L^{r_i}(\omega_i,B_k)}.\end{aligned}$$

From the definition of the space $\dot{\mathrm{CMO}}^{r_i}(\omega_i)$ we immediately have

$$(4.26)\quad \|a_{1,i}\|_{L^{r_i}(\omega_i,B_k)} \leqslant (\omega_i(B_k))^{1/r_i} \cdot \|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)} \lesssim 2^{k(\gamma_i+n)/r_i} \cdot \|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}.$$

To estimate $\|a_{2,i}\|_{L^{r_i}(\omega_i,B_k)}$, we decompose

$$\begin{aligned}(4.27)\qquad \|a_{2,i}\|_{L^{r_i}(\omega_i,B_k)} &= \left(\int_{B_k} |b_i(A_i(t)x) - b_{i,\omega_i,A_i(t)B_k}|^{r_i}\omega_i(x)\,\mathrm{d}x\right)^{1/r_i} \\ &\leqslant \omega_i(B_k)^{1/r_i}|b_{i,\omega_i,A_i(t)B_k} - b_{i,\omega_i,\|A_i(t)\|B_k}| \\ &\quad + \left(\int_{B_k} |b_i(A_i(t)x) - b_{i,\omega_i,\|A_i(t)\|B_k}|^{r_i}\omega_i(x)\,\mathrm{d}x\right)^{1/r_i}.\end{aligned}$$

From the Hölder inequality we have

$$\begin{aligned}(4.28)\quad &|b_{i,\omega_i,A_i(t)B_k} - b_{i,\omega_i,\|A_i(t)\|B_k}| \\ &\leqslant \frac{1}{\omega_i(A_i(t)B_k)} \int_{A_i(t)B_k} |b_i(x) - b_{i,\omega_i,\|A_i(t)\|B_k}|\omega_i(x)\,\mathrm{d}x \\ &\leqslant \frac{1}{\omega_i(A_i(t)B_k)} \left(\int_{\|A_i(t)\|B_k} |b_i(x) - b_{i,\omega_i,\|A_i(t)\|B_k}|^{r_i}\omega_i(x)\,\mathrm{d}x\right)^{1/r_i} \\ &\quad \times \omega_i(\|A_i(t)\|B_k)^{1/r_i'} \\ &\leqslant \frac{\omega_i(\|A_i(t)\|B_k)}{\omega_i(A_i(t)B_k)} \|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}.\end{aligned}$$

Note that by (3.3) we get

$$\begin{aligned}(4.29)\qquad \omega_i(A_i(t)B_k) &= \int_{A_i(t)B_k} |x|^{\gamma_i}\,\mathrm{d}x = \int_{B_k} |A_i(t)z|^{\gamma_i}|\det A_i(t)|\,\mathrm{d}z \\ &\geqslant \min\{\|A_i(t)\|^{\gamma_i}, \|A_i^{-1}(t)\|^{-\gamma_i}\}|\det A_i(t)| \int_{B_k} |z|^{\gamma_i}\,\mathrm{d}z \\ &\simeq \min\{\|A_i(t)\|^{\gamma_i}, \|A_i^{-1}(t)\|^{-\gamma_i}\}|\det A_i(t)|\cdot 2^{k(\gamma_i+n)}.\end{aligned}$$

This gives

$$\frac{\omega_i(\|A_i(t)\|B_k)}{\omega_i(A_i(t)B_k)} \lesssim \frac{(\|A_i(t)\|\cdot 2^k)^{\gamma_i+n}}{\min\{\|A_i(t)\|^{\gamma_i}, \|A_i^{-1}(t)\|^{-\gamma_i}\}|\det A_i(t)|\cdot 2^{k(\gamma_i+n)}} = \eta_{A_i,\gamma_i}(t).$$

Thus, by (4.28) we get

$$|b_{i,\omega_i,A_i(t)B_k} - b_{i,\omega_i,\|A_i(t)\|B_k}| \lesssim \eta_{A_i,\gamma_i}(t)\cdot \|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}. \tag{4.30}$$

On the other hand, by making the formula for change of variables and (3.3), we obtain

$$\begin{aligned}
&\Big(\int_{B_k} |b_i(A_i(t)x) - b_{i,\omega_i,\|A_i(t)\|B_k}|^{r_i}\omega_i(x)\,\mathrm{d}x\Big)^{1/r_i}\\
&\quad= \Big(\int_{A_i(t)B_k} |b_i(z) - b_{i,\omega_i,\|A_i(t)\|B_k}|^{r_i}|A_i^{-1}(t)z|^{\gamma_i}|\det A_i^{-1}(t)|\,\mathrm{d}z\Big)^{1/r_i}\\
&\quad\leqslant (\max\{\|A_i^{-1}(t)\|^{\gamma_i}, \|A_i(t)\|^{-\gamma_i}\}|\det A_i^{-1}(t)|\omega_i(\|A_i(t)\|B_k))^{1/r_i}\\
&\qquad\times\Big(\frac{1}{\omega_i(\|A_i(t)\|B_k)}\int_{\|A_i(t)\|B_k} |b_i(z) - b_{i,\omega_i,\|A_i(t)\|B_k}|^{r_i}\omega_i(z)\,\mathrm{d}z\Big)^{1/r_i}.
\end{aligned}$$

This deduces that

$$\begin{aligned}
&\Big(\int_{B_k} |b_i(A_i(t)x) - b_{i,\omega_i,\|A_i(t)\|B_k}|^{r_i}\omega_i(x)\,\mathrm{d}x\Big)^{1/r_i}\\
&\qquad\qquad\lesssim \psi_{A_i,\gamma_i,r_i}(t)\cdot 2^{k(\gamma_i+n)/r_i}\|b_i\|^{r_i}_{\dot{\mathrm{CMO}}}(\omega_i).
\end{aligned}$$

From this, by (4.27) and (4.30) one has

$$\|a_{2,i}\|_{L^{r_i}(\omega_i,B_k)} \lesssim 2^{k(\gamma_i+n)/r_i}(\eta_{A_i,\gamma_i}(t) + \psi_{A_i,\gamma_i,r_i}(t))\cdot\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}. \tag{4.31}$$

Next, we observe that

$$\|a_{3,i}\|_{L^{r_i}(\omega_i,B_k)} \leqslant (\omega_i(B_k))^{1/r_i}|b_{i,\omega_i,B_k} - b_{i,\omega_i,A_i(t)B_k}|. \tag{4.32}$$

By $\|A_i(t)\| \neq 0$ there exists an integer number $\theta_i = \theta_i(t)$ satisfying $2^{\theta_i-1} < \|A_i(t)\| \leqslant 2^{\theta_i}$. Thus, we define

$$S(\theta_i) = \begin{cases} \{j\in\mathbb{Z}\colon\ 1\leqslant j\leqslant \theta_i\} & \text{if } \theta_i\geqslant 1,\\ \{j\in\mathbb{Z}\colon\ \theta_i+1\leqslant j\leqslant 0\} & \text{otherwise.}\end{cases}$$

At this point, we give the estimation

$$
\begin{aligned}
(4.33)\qquad & |b_{i,\omega_i,B_k} - b_{i,\omega_i,A_i(t)B_k}| \\
&\leqslant \sum_{j\in S(\theta_i)} |b_{i,\omega_i,2^{j-1}B_k} - b_{i,\omega_i,2^j B_k}| + |b_{i,\omega_i,2^{\theta_i}B_k} - b_{i,\omega_i,A_i(t)B_k}|.
\end{aligned}
$$

When $S(\theta_i)$ is an empty set, we should understand that

$$
\sum_{j\in S(\theta_i)} |b_{i,\omega_i,2^{j-1}B_k} - b_{i,\omega_i,2^j B_k}| := 0.
$$

It is not difficult to show that

$$
|b_{i,\omega_i,2^{j-1}B_k} - b_{i,\omega_i,2^j B_k}| \lesssim \|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}.
$$

By the Hölder inequality and (3.3), it follows that

$$
\begin{aligned}
(4.34)\qquad & |b_{i,\omega_i,2^{\theta_i}B_k} - b_{i,\omega_i,A_i(t)B_k}| \\
&\leqslant \frac{1}{\omega_i(A_i(t)B_k)} \int_{A_i(t)B_k} |b_i(x) - b_{i,\omega_i,2^{\theta_i}B_k}|\omega_i(x)\,\mathrm{d}x \\
&\leqslant \frac{1}{\omega_i(A_i(t)B_k)} \left(\int_{2^{\theta_i}B_k} |b_i(x) - b_{i,\omega_i,2^{\theta_i}B_k}|^{r_i}\omega_i(x)\,\mathrm{d}x\right)^{1/r_i} \\
&\quad \times \omega_i(2^{\theta_i}B_k)^{1/r_i'} \\
&\leqslant \frac{\omega_i(2^{\theta_i}B_k)}{\omega_i(A_i(t)B_k)} \|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}.
\end{aligned}
$$

Noting that $2^{\theta_i} \simeq \|A_i(t)\|$ and using (4.29), we compute

$$
\frac{\omega_i(2^{\theta_i}B_k)}{\omega_i(A_i(t)B_k)} \lesssim \frac{(2^{\theta_i}\cdot 2^k)^{n+\gamma_i}}{\min\{\|A_i(t)\|^{\gamma_i}, \|A_i^{-1}(t)\|^{-\gamma_i}\}|\det A_i(t)|\cdot 2^{k(\gamma_i+n)}} = \eta_{A_i,\gamma_i}(t).
$$

Consequently, we have

$$
|b_{i,\omega_i,2^{\theta_i}B_k} - b_{i,\omega_i,A_i(t)B_k}| \lesssim \eta_{A_i,\gamma_i}(t).\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}.
$$

On the other hand, by $2^{\theta_i-1} < \|A_i(t)\| \leqslant 2^{\theta_i}$ we have

$$
(4.35)\qquad |\theta_i| \lesssim \left\{\begin{array}{ll} \log(2\|A_i(t)\|) & \text{if } \theta_i \geqslant 0, \\ \log\|A_i(t)\|^{-1} & \text{otherwise} \end{array}\right\} \lesssim \varphi_{A_i}(t).
$$

Therefore, by (4.33) it follows that

$$\begin{aligned}|b_{i,\omega_i,B_k}-b_{i,\omega_i,A_i(t)B_k}| &\lesssim (|\theta_i|+\eta_{A_i,\gamma_i}(t))\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}\\ &\lesssim (\varphi_{A_i}(t)+\eta_{A_i,\gamma_i}(t))\cdot\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}.\end{aligned}$$

As above, by (4.32) we get

$$\|a_{3,i}\|_{L^{r_i}(\omega_i,B_k)}\lesssim 2^{k(n+\gamma_i)/r_i}(\varphi_{A_i}(t)+\eta_{A_i,\gamma_i}(t))\cdot\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega_i)}.$$

From this, by (4.26), (4.31), we finish the proof of inequality (4.24). Using (4.23) and (4.24), we have

$$\begin{aligned}(4.36)\quad &\|H^{\vec b}_{\Phi,\vec A}(\vec f)\chi_k\|_{L^{q(\cdot)}_\omega}\\ &\lesssim \mathcal{B}_{\mathrm{CMO},\vec\omega}\cdot 2^{k\sum_{i=1}^m(\gamma_i+n)/r_i}\biggl(\int_{\mathbb{R}^n}\frac{\Phi(t)}{|t|^n}\prod_{i=1}^m c_{A_i,q_i,\gamma_i}(t)\|1\|_{L^{\vartheta_i(t,\cdot)}}\\ &\quad\times(1+\psi_{A_i,\gamma_i,r_i}+2\eta_{A_i,\gamma_i}+\varphi_{A_i})\prod_{i=1}^m\sum_{r=\Theta_n^*-1}^{0}\|f_i\chi_{k+l_i+r}\|_{L^{\varsigma q_i(\cdot)}_{\omega_i}}\,\mathrm{d}t\biggr).\end{aligned}$$

At this point, using Lemma 2.4 in Section 2 again, we have results a similar to (4.9) as

$$\|H^{\vec b}_{\Phi,\vec A}(\vec f)\chi_k\|_{L^{q(\cdot)}_\omega}\lesssim \mathcal{C}_4\cdot\mathcal{B}_{\mathrm{CMO},\vec\omega}\cdot\mathcal{F}\cdot 2^{k\sum_{i=1}^m(\gamma_i+n)/r_i}\prod_{i=1}^m(2^{k(\lambda_i-\alpha_i(0))}+2^{k(\lambda_i-\alpha_{i\infty})}).$$

By using [26], Proposition 2.5 again, we get

$$(4.37)\qquad \|H^{\vec b}_{\Phi,\vec A}(\vec f)\|_{M\dot K^{\alpha^{**}(\cdot),\lambda}_{p,q(\cdot),\omega}}\lesssim\max\Bigl\{\sup_{k_0<0,k_0\in\mathbb{Z}}\widetilde E_1,\ \sup_{k_0\geqslant0,k_0\in\mathbb{Z}}(\widetilde E_2+\widetilde E_3)\Bigr\},$$

where

$$\begin{aligned}\widetilde E_1&=2^{-k_0\lambda}\biggl(\sum_{k=-\infty}^{k_0}2^{k\alpha^{**}(0)p}\|H_{\Phi,\vec A}(\vec f)\chi_k\|^p_{L^{q(\cdot)}_\omega}\biggr)^{1/p},\\ \widetilde E_2&=2^{-k_0\lambda}\biggl(\sum_{k=-\infty}^{-1}2^{k\alpha^{**}(0)p}\|H_{\Phi,\vec A}(\vec f)\chi_k\|^p_{L^{q(\cdot)}_\omega}\biggr)^{1/p},\\ \widetilde E_3&=2^{-k_0\lambda}\biggl(\sum_{k=0}^{k_0}2^{k\alpha^{**}_\infty p}\|H_{\Phi,\vec A}(\vec f)\chi_k\|^p_{L^{q(\cdot)}_\omega}\biggr)^{1/p}.\end{aligned}$$

In view of (4.37), by defining α^{**} and estimating as (4.13), (4.14), (4.17), we also have

$$\widetilde{E}_1 \lesssim \mathcal{C}_4 \cdot \mathcal{B}_{\mathrm{CMO},\vec{\omega}} \cdot \mathcal{F} \cdot \prod_{i=1}^{m}(1 + 2^{k_0(\alpha_i(0)-\alpha_{i\infty})}),$$

$$\widetilde{E}_2 \lesssim \mathcal{C}_4 \cdot \mathcal{B}_{\mathrm{CMO},\vec{\omega}} \cdot \mathcal{F} \cdot 2^{-k_0\lambda},$$

$$\widetilde{E}_3 \lesssim \mathcal{C}_4 \cdot \mathcal{B}_{\mathrm{CMO},\vec{\omega}} \cdot \mathcal{F} \cdot \prod_{i=1}^{m} \mathcal{L}_i.$$

Therefore the proof of Theorem 3.4 is completed.

Now, let us give the proof for Theorem 3.5. By using [1], Proposition 3.8 again we obtain

$$\begin{aligned} \|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\|_{\dot{K}^{\alpha^{**}(\cdot),p}_{q(\cdot),\omega}} &\lesssim \Big(\sum_{k=-\infty}^{-1} 2^{k\alpha^{**}(0)p}\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|^p_{L^{q(\cdot)}_{\omega}}\Big)^{1/p} \\ &\quad + \Big(\sum_{k=0}^{\infty} 2^{k\alpha^{**}_{\infty}p}\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\chi_k\|^p_{L^{q(\cdot)}_{\omega}}\Big)^{1/p} := \mathcal{G}_0 + \mathcal{G}_1. \end{aligned} \tag{4.38}$$

Using the Minkowski inequality, by employing (4.36), we find

$$\begin{aligned} \mathcal{G}_0 \lesssim \mathcal{B}_{\mathrm{CMO},\vec{\omega}} \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} &\prod_{i=1}^{m} c_{A_i,q_i,\gamma_i}(t)\|1\|_{L^{\vartheta_i(t,\cdot)}}(1 + \psi_{A_i,\gamma_i,r_i} + 2\eta_{A_i,\gamma_i} + \varphi_{A_i}) \\ &\times \Big\{\sum_{k=-\infty}^{-1} 2^{k\sum_{i=1}^m \alpha_i(0)p} \prod_{i=1}^{m}\Big(\sum_{r=\Theta^*_n-1}^{0} \|f_i\chi_{k+l_i+r}\|_{L^{\zeta q_i(\cdot)}_{\omega_i}}\Big)^p\Big\}^{1/p} \mathrm{d}t, \\ \mathcal{G}_1 \lesssim \mathcal{B}_{\mathrm{CMO},\vec{\omega}} \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} &\prod_{i=1}^{m} c_{A_i,q_i,\gamma_i}(t).\|1\|_{L^{\vartheta_i(t,\cdot)}}(1 + \psi_{A_i,\gamma_i,r_i} + 2\eta_{A_i,\gamma_i} + \varphi_{A_i}) \\ &\times \Big\{\sum_{k=0}^{\infty} 2^{k\sum_{i=1}^m \alpha_i(0)p} \prod_{i=1}^{m}\Big(\sum_{r=\Theta^*_n-1}^{0} \|f_i\chi_{k+l_i+r}\|_{L^{\zeta q_i(\cdot)}_{\omega_i}}\Big)^p\Big\}^{1/p} \mathrm{d}t. \end{aligned}$$

We observe that the other estimations can be done by similar arguments as for Theorem 3.2. Thus, $\mathcal{G}_0$ and $\mathcal{G}_1$ are dominated by $\mathcal{C}\cdot\mathcal{C}_5\cdot\mathcal{B}_{\mathrm{CMO},\vec{\omega}}\prod_{i=1}^{m}\|f_i\|_{\dot{K}^{\alpha_i(\cdot),p_i}_{\zeta q_i(\cdot),\omega_i}}$. This proves the assertion.

4.3. Proof of Theorem 3.6 and Theorem 3.7. For $R > 0$ we write $B := B(0,R)$ and Δ_R as

$$\Delta_R = \frac{1}{\omega(B)^{1/q_\infty+\lambda}}\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\|_{L^{q(\cdot)}_{v}(B)}.$$

By applying the Minkowski inequality for the variable Lebesgue space, we have

$$\Delta_R \lesssim \int_{\mathbb{R}^n} \frac{1}{\omega(B)^{1/q_\infty+\lambda}} \cdot \frac{\Phi(t)}{|t|^n} \Big\| \prod_{i=1}^m f_i(A_i(t)\cdot)(b_i(\cdot) - b_i(A_i(t)\cdot)) \Big\|_{L_v^{q(\cdot)}(B)} \, \mathrm{d}t. \tag{4.39}$$

By estimating as (4.2) above we get

$$\begin{aligned} &\Big\| \prod_{i=1}^m f_i(A_i(t)\cdot)(b_i(\cdot) - b_i(A_i(t)\cdot)) \Big\|_{L_v^{q(\cdot)}(B)} \\ &\quad \lesssim R^\beta \cdot \mathcal{B}_{\rm Lip} \prod_{i=1}^m \|I_n - A_i(t)\|^{\beta_i} \cdot \prod_{i=1}^m \|f_i(A_i(t)\cdot)\|_{L_{v_i}^{q_i(\cdot)}(B)} \||\cdot|^{\alpha_i/r_i}\|_{L^{r_i}(B)} \\ &\quad \lesssim R^{\beta+\sum_{i=1}^m (\alpha_i+n)/r_i} \cdot \mathcal{B}_{\rm Lip} \prod_{i=1}^m \|I_n - A_i(t)\|^{\beta_i} \cdot \prod_{i=1}^m \|f_i(A_i(t)\cdot)\|_{L_{v_i}^{q_i(\cdot)}(B)}. \end{aligned} \tag{4.40}$$

By (3.15) and Theorem 2.6 we find

$$\|f_i(A_i(t)\cdot)\|_{L_{v_i}^{q_i(\cdot)}(B)} \lesssim c_{A_i,q_i,\alpha_i}(t) \cdot \|1\|_{L^{\vartheta_{1i}(t,\cdot)}} \cdot \|f_i\|_{L_{v_i}^{q_i(\cdot)}(B(0,R||A_i(t)||))}. \tag{4.41}$$

By condition (3.16) we estimate

$$\frac{R^{\beta+\sum_{i=1}^m (\alpha_i+n)/r_i}}{\omega(B)^{1/q_\infty+\lambda}} \lesssim \prod_{i=1}^m \frac{\|A_i(t)\|^{(\gamma_i+n)(1/q_{i\infty}+\lambda_i)}}{\omega_i(B(0,R||A_i(t)||))^{1/q_{i\infty}+\lambda_i}}. \tag{4.42}$$

Thus, from (4.39)–(4.42) and the definition of central Morrey spaces with variable exponent, it follows that

$$\Delta_R \lesssim \mathcal{C}_6.\mathcal{B}_{\rm Lip} \cdot \prod_{i=1}^m \|f_i\|_{\dot{B}_{\omega_i,v_i}^{q_i(\cdot),\lambda_i}}.$$

Therefore we conclude that

$$\|H_{\Phi,\vec{A}}^{\vec{b}}(\vec{f})\|_{\dot{B}_{\omega,v}^{q(\cdot),\lambda}} \lesssim \mathcal{C}_6 \cdot \mathcal{B}_{\rm Lip} \cdot \prod_{i=1}^m \|f_i\|_{\dot{B}_{\omega_i,v_i}^{q_i(\cdot),\lambda_i}}.$$

Next, we will prove Theorem 3.7. Indeed, by using the Minkowski inequality and the Hölder inequality for variable Lebesgue spaces again, it is obvious that

$$\Delta_R \lesssim \int_{\mathbb{R}^n} \frac{1}{\omega(B)^{1/q_\infty+\lambda}} \cdot \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m \|b_i(\cdot) - b_i(A_i(t)\cdot)\|_{L^{r_i}(v_i,B)} \|f_i(A_i(t)\cdot)\|_{L_{v_i}^{q_i(\cdot)}(B)} \, \mathrm{d}t.$$

By (4.24) above we deduce

$$\Delta_R \lesssim R^{\sum_{i=1}^m (\alpha_i+n)/r_i} \times \mathcal{B}_{\mathrm{CMO},\vec{v}} \int_{\mathbb{R}^n} \frac{1}{\omega(B)^{1/q_\infty+\lambda}} \cdot \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m (1+\psi_{A_i,\gamma_i,r_i} + 2\eta_{A_i,\gamma_i} + \varphi_{A_i}) \times \prod_{i=1}^m \|f_i(A_i(t)\cdot)\|_{L^{q_i(\cdot)}_{v_i}(B)}\, \mathrm{d}t.$$

For this, by (4.41) we get

$$\Delta_R \lesssim R^{\sum_{i=1}^m (\alpha_i+n)/r_i} \mathcal{B}_{\mathrm{CMO},\vec{v}} \Big(\int_{\mathbb{R}^n} \frac{1}{\omega(B)^{1/q_\infty+\lambda}} \cdot \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m c_{A_i,q_i,\alpha_i}(t) \|1\|_{L^{\vartheta_{1i}(t,\cdot)}} \times (1+\psi_{A_i,\gamma_i,r_i} + 2\eta_{A_i,\gamma_i} + \varphi_{A_i}) \cdot \|f_i\|_{L^{q_i(\cdot)}_{v_i}(B(0,R\|A_i(t)\|))}\, \mathrm{d}t \Big). \tag{4.43}$$

On the other hand, by (3.18) it follows that

$$\frac{R^{\sum_{i=1}^m (\alpha_i+n)/r_i}}{\omega(B)^{1/q_\infty+\lambda}} \lesssim \prod_{i=1}^m \frac{\|A_i(t)\|^{(\gamma_i+n)(1/q_{i\infty}+\lambda_i)}}{\omega_i(B(0,R\|A_i(t)\|))^{1/q_{i\infty}+\lambda_i}}.$$

Consequently, by (4.43) we immediately obtain that

$$\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\|_{\dot{B}^{q(\cdot),\lambda}_{\omega,v}} \lesssim \mathcal{C}_7 \cdot \mathcal{B}_{\mathrm{CMO},\vec{v}} \cdot \prod_{i=1}^m \|f_i\|_{\dot{B}^{q_i(\cdot),\lambda_i}_{\omega_i,v_i}},$$

which completes the proof.

4.4. Proof of Theorem 3.8. For convenience, we denote

$$\mathcal{B}_{\dot{\mathrm{C}\mathrm{MO}},\omega} = \prod_{i=1}^m \|b_i\|_{\dot{\mathrm{C}\mathrm{MO}}^{r_i}(\omega)}$$

and recall $B := B(0,R)$.

In view of inequality (3.20) there exist $r_1^*, \dots, r_m^*$, $q_1^*, \dots, q_m^*$ such that

$$\frac{1}{r_i^*} > \frac{p}{r_i}\frac{r_\omega}{r_\omega-1}, \quad \frac{1}{q_i^*} > \frac{p}{q_i}\frac{r_\omega}{r_\omega-1} \quad \text{and} \quad \sum_{i=1}^m \frac{1}{r_i^*} + \frac{1}{q_i^*} = \frac{1}{q}.$$

Thus, by using the Minkowski inequality and the Hölder inequality, we get

$$(4.44)\quad \|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\|_{L^q(\omega,B)} \leqslant \int_{\mathbb{R}^n}\frac{\Phi(t)}{|t|^n}\prod_{i=1}^m\|b_i(\cdot)-b_i(A_i(t)\cdot)\|_{L^{r_i^*}(\omega,B)}\prod_{i=1}^m\|f_i(A_i(t)\cdot)\|_{L^{q_i^*}(\omega,B)}\,\mathrm{d}t.$$

Now, we also compose

$$(4.45)\quad \begin{aligned}\|b_i(\cdot)-b_i(A_i(t)\cdot)\|_{L^{r_i^*}(\omega,B)} &\leqslant \|b_i(\cdot)-b_{i,\omega,B}\|_{L^{r_i^*}(\omega,B)}+\|b_i(A_i(t)\cdot)-b_{i,\omega,A_i(t)B}\|_{L^{r_i^*}(\omega,B)}\\ &\quad+\|b_{i,\omega,B}-b_{i,\omega,A_i(t)B}\|_{L^{r_i^*}(\omega,B)}:=a^*_{1,i}+a^*_{2,i}+a^*_{3,i}.\end{aligned}$$

Next, it is clear to see that

$$(4.46)\quad a^*_{1,i}\leqslant\omega(B)^{1/r_i^*}\|b_i\|_{\dot{\mathrm{CMO}}^{r_i^*}(\omega)}\leqslant\omega(B)^{1/r_i^*}\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega)}.$$

By estimating as (4.27) and (4.28) we infer

$$(4.47)\quad \begin{aligned}a^*_{2,i}&\leqslant\frac{\omega(\|A_i(t)\|B)}{\omega(A_i(t)B)}\omega(B)^{1/r_i^*}\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega)}\\ &\quad+\Big(\int_B|b_i(A_i(t)x)-b_{i,\omega,\|A_i(t)\|B}|^{r_i^*}\omega(x)\,\mathrm{d}x\Big)^{1/r_i^*}.\end{aligned}$$

From Proposition 2.12 and (4.29) one has

$$(4.48)\quad \frac{\omega(\|A_i(t)\|B)}{\omega(A_i(t)B)}\lesssim\Big(\frac{|\|A_i(t)\|B|}{|A_i(t)B|}\Big)^p\simeq\Big(\frac{|\|A_i(t)\|^nR^n}{|\det A_i(t)|R^n}\Big)^p=\frac{\|A_i(t)\|^{np}}{|\det A_i(t)|^p}.$$

By

$$\frac{1}{r_i^*}>\frac{p}{r_i}\frac{r_\omega}{r_\omega-1}$$

there exists $\beta_{i,0}\in(1,r_\omega)$ such that $r_i=r_i^*p\beta'_{i,0}$. Thus, by the Hölder inequality and the reverse Hölder inequality, we get

$$\begin{aligned}&\Big(\int_B|b_i(A_i(t)x)-b_{i,\omega,\|A_i(t)\|B}|^{r_i^*}\omega(x)\,\mathrm{d}x\Big)^{1/r_i^*}\\ &\quad\leqslant\Big(\int_B|b_i(A_i(t)x)-b_{i,\omega,\|A_i(t)\|B}|^{r_i/p}\,\mathrm{d}x\Big)^{p/r_i}\Big(\int_B\omega(x)^{\beta_{i,0}}\,\mathrm{d}x\Big)^{1/(\beta_{i,0}r_i^*)}\\ &\quad\lesssim|B|^{-p/r_i}\omega(B)^{1/r_i^*}\Big(\int_B|b_i(A_i(t)x)-b_{i,\omega,\|A_i(t)\|B}|^{r_i/p}\,\mathrm{d}x\Big)^{p/r_i}.\end{aligned}$$

By using the formula for change of variable and Proposition 2.13 we have

$$\begin{aligned}
&\left(\int_B |b_i(A_i(t)x) - b_{i,\omega,\|A_i(t)\|B}|^{r_i/p}\,\mathrm{d}x\right)^{p/r_i}\\
&\leqslant |\det A_i^{-1}(t)|^{p/r_i}\left(\int_{\|A_i(t)\|B} |b_i(z) - b_{i,\omega,\|A_i(t)\|B}|^{r_i/p}\,\mathrm{d}z\right)^{p/r_i}\\
&\leqslant |\det A_i^{-1}(t)|^{p/r_i}\frac{|\|A_i(t)\|B|^{p/r_i}}{\omega(\|A_i(t)\|B)^{1/r_i}}\left(\int_{\|A_i(t)\|B} |b_i(z) - b_{i,\omega,\|A_i(t)\|B}|^{r_i}\omega(z)\,\mathrm{d}z\right)^{1/r_i}.
\end{aligned}$$

This deduces that

$$\begin{aligned}
(4.49)\quad &\left(\int_B |b_i(A_i(t)x) - b_{i,\omega,\|A_i(t)\|B}|^{r_i^*}\omega(x)\,\mathrm{d}x\right)^{1/r_i^*}\\
&\lesssim \omega(B)^{1/r_i^*}|\det A_i^{-1}(t)|^{p/r_i}\|A_i(t)\|^{np/r_i}\\
&\quad\times\frac{1}{\omega(\|A_i(t)\|B)^{1/r_i}}\left(\int_{\|A_i(t)\|B} |b_i(z) - b_{i,\omega,\|A_i(t)\|B}|^{r_i}\omega(z)\,\mathrm{d}z\right)^{1/r_i}\\
&\lesssim \omega(B)^{1/r_i^*}|\det A_i^{-1}(t)|^{p/r_i}\|A_i(t)\|^{np/r_i}\cdot\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega)}.
\end{aligned}$$

From this, by (4.47) and (4.48) we have

$$(4.50)\quad a_{2,i}^* \lesssim \omega(B)^{1/r_i^*}\left(\frac{\|A_i(t)\|^{np}}{|\det A_i(t)|^p} + |\det A_i^{-1}(t)|^{p/r_i}\|A_i(t)\|^{np/r_i}\right)\cdot\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega)}.$$

Combining the inequality $2^{\theta_i-1} < \|A_i(t)\| \leqslant 2^{\theta_i}$ and Proposition 2.12, we infer

$$\frac{\omega(2^{\theta_i}B)}{\omega(A_i(t)B)} \lesssim \left(\frac{|2^{\theta_i}B|}{|A_i(t)B|}\right)^p \lesssim \frac{(2^{\theta_i}R)^{np}}{|\det A_i(t)|^p R^{np}} \simeq \frac{\|A_i(t)\|^{np}}{|\det A_i(t)|^p}.$$

Thus, by the same reasons as for (4.32), (4.33), (4.34) and (4.35) above, we estimate

$$\begin{aligned}
a_{3,i}^* &\leqslant \omega(B)^{1/r_i^*}|b_{i,\omega,B} - b_{i,\omega,A_i(t)B}| \lesssim \omega(B)^{1/r_i^*}\left(|\theta_i| + \frac{\omega(2^{\theta_i}B_k)}{\omega(A_i(t)B_k)}\right)\|b_i\|_{\dot{\mathrm{CMO}}^{r_i^*}(\omega)}\\
&\lesssim \omega(B)^{1/r_i^*}\left(\varphi_{A_i}(t) + \frac{\|A_i(t)\|^{np}}{|\det A_i(t)|^p}\right)\cdot\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega)}.
\end{aligned}$$

Hence, by (4.45), (4.46) and (4.50), we have

$$\|b_i(\cdot) - b_i(A_i(t)\cdot)\|_{L^{r_i^*}(\omega,B)} \lesssim \omega(B)^{1/r_i^*}\mu_{A_i,p,r_i}(t)\cdot\|b_i\|_{\dot{\mathrm{CMO}}^{r_i}(\omega)}.$$

Next, from

$$\frac{1}{q_i^*} > \frac{p}{q_i}\frac{r_\omega}{r_\omega - 1}$$

and estimating as (4.49) above, we get

$$\begin{aligned}\left(\int_B |f_i(A_i(t)x)|^{q_i^*}\omega(x)\,\mathrm{d}x\right)^{1/q_i^*} \\ &\lesssim \omega(B)^{1/q_i^*}|\det A_i^{-1}(t)|^{p/q_i}\|A_i(t)\|^{np/q_i}\frac{1}{\omega(\|A_i(t)\|B)^{1/q_i}} \\ &\quad\times\left(\int_{\|A_i(t)\|B}|f_i(z)|^{q_i}\omega(z)\,\mathrm{d}z\right)^{1/q_i} \\ &\leqslant \omega(B)^{1/q_i^*}|\det A_i^{-1}(t)|^{p/q_i}\|A_i(t)\|^{np/q_i}\omega(\|A_i(t)\|B)^{\lambda_i}\|f_i\|_{\dot{B}^{q_i,\lambda_i}(\omega)}.\end{aligned}$$

As a consequence, (3.21), (4.44) and (4.51) imply

$$\begin{aligned}\frac{1}{\omega(B)^{1/q+\lambda}}\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\|_{L^q(\omega,B)} \\ &\lesssim \mathcal{B}_{\dot{\mathrm{CMO}},\omega}\left(\int_{\mathbb{R}^n}\frac{\Phi(t)}{|t|^n}\prod_{i=1}^m \mu_{A_i,p,r_i}(t)|\det A_i^{-1}(t)|^{p/q_i}\|A_i(t)\|^{np/q_i}\right. \\ &\quad\times\left.\left(\frac{\omega(\|A_i(t)\|B)}{\omega(B)}\right)^{\lambda_i}\mathrm{d}t\right)\prod_{i=1}^m\|f_i\|_{\dot{B}^{q_i,\lambda_i}(\omega)}.\end{aligned}$$

For $i=1,\dots,m$, by $\lambda_i<0$, $\omega\in A_p$ and Proposition 2.12, we deduce

$$\left(\frac{\omega(\|A_i(t)\|B)}{\omega(B)}\right)^{\lambda_i}\lesssim\begin{cases}\left(\dfrac{|\|A_i(t)\|B|}{|B|}\right)^{p\lambda_i}\lesssim\|A_i(t)\|^{np\lambda_i} & \text{if } \|A_i(t)\|\leqslant 1,\\ \left(\dfrac{|\|A_i(t)\|B|}{|B|}\right)^{(\delta-1)\lambda_i/\delta}\lesssim\|A_i(t)\|^{n(\delta-1)\lambda_i\delta} & \text{otherwise.}\end{cases}$$

Hence, by the definition of the Morrey space, we have

$$\|H^{\vec{b}}_{\Phi,\vec{A}}(\vec{f})\|_{\dot{B}^{q,\lambda}(\omega)}\lesssim\mathcal{C}_8\cdot\mathcal{B}_{\dot{\mathrm{CMO}},\omega}\cdot\prod_{i=1}^m\|f_i\|_{\dot{B}^{q_i,\lambda_i}(\omega)},$$

which is the desired result.

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References

[1] *A. Almeida, D. Drihem*: Maximal, potential and singular type operators on Herz spaces with variable exponents. J. Math. Anal. Appl. *394* (2012), 781–795. zbl MR doi

[2] *A. Almeida, P. Hästö*: Besov spaces with variable smoothness and integrability. J. Funct. Anal. *258* (2010), 1628–1655. zbl MR doi

[3] *R. A. Bandaliev*: The boundedness of multidimensional Hardy operators in weighted variable Lebesgue spaces. Lith. Math. J. *50* (2010), 249–259. zbl MR doi

[4] *G. Brown, F. Móricz*: Multivariate Hausdorff operators on the spaces $L^p(\mathbb{R}^n)$. J. Math. Anal. Appl. *271* (2002), 443–454. zbl MR doi

[5] *S. Chanillo*: A note on commutators. Indiana Univ. Math. J. *31* (1982), 7–16. zbl MR doi

[6] *N. M. Chuong*: Pseudodifferential Operators and Wavelets over Real and p-adic Fields. Springer, Cham, 2018. zbl MR doi

[7] *N. M. Chuong, D. V. Duong, K. H. Dung*: Multilinear Hausdorff operators on some function spaces with variable exponent. Available at `https://arxiv.org/abs/1709.08185` (2017), 36 pages.

[8] *N. M. Chuong, D. V. Duong, H. D. Hung*: Bounds for the weighted Hardy-Cesàro operator and its commutator on weighted Morrey-Herz type spaces. Z. Anal. Anwend. *35* (2016), 489–504. zbl MR doi

[9] *N. M. Chuong, N. T. Hong, H. D. Hung*: Multilinear Hardy-Cesàro operator and commutator on the product of Morrey-Herz spaces. Anal. Math. *43* (2017), 547–565. zbl MR doi

[10] *R. R. Coifman, Y. Meyer*: On commutators of singular integrals and bilinear singular integrals. Trans. Am. Math. Soc. *212* (1975), 315–331. zbl MR doi

[11] *R. R. Coifman, R. Rochberg, G. Weiss*: Factorization theorems for Hardy spaces in several variables. Ann. Math. *103* (1976), 611–635. zbl MR doi

[12] *D. V. Cruz-Uribe, A. Fiorenza*: Variable Lebesgue Spaces: Foundations and Harmonic Analysis. Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2013. zbl MR doi

[13] *L. Diening, P. Harjulehto, P. Hästö, M. Růžička*: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics 2017, Springer, Berlin, 2011. zbl MR doi

[14] *L. Diening, M Růžička*: Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics. J. Reine Angew. Math. *563* (2003), 197–220. zbl MR doi

[15] *C. Fefferman*: Characterizations of bounded mean oscillation. Bull. Am. Math. Soc. *77* (1971), 587–588. zbl MR doi

[16] *Z. W. Fu, S. L. Gong, S. Z. Lu, W. Yuan*: Weighted multilinear Hardy operators and commutators. Forum Math. *27* (2015), 2825–2851. zbl MR doi

[17] *Z. W. Fu, Z. G. Liu, S. Z. Lu*: Commutators of weighted Hardy operators on $\mathbb{R}^n$. Proc. Am. Math. Soc. *137* (2009), 3319–3328. zbl MR doi

[18] *L. Grafakos*: Modern Fourier Analysis. Graduate Texts in Mathematics 250, Springer, New York, 2009. zbl MR doi

[19] *V. S. Guliyev, J. J. Hasanov, S. G. Samko*: Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces. Math. Scand. *107* (2010), 285–304. zbl MR doi

[20] *H. D. Hung, L. D. Ky*: New weighted multilinear operators and commutators of Hardy-Cesàro type. Acta Math. Sci., Ser. B, Engl. Ed. *35* (2015), 1411–1425. zbl MR doi

[21] *T. Hytönen, C. Pérez, E. Rela*: Sharp reverse Hölder property for A_∞ weights on spaces of homogeneous type. J. Funct. Anal. *263* (2012), 3883–3899. zbl MR doi

[22] *S. Indratno, D. Maldonado, S. Silwal*: A visual formalism for weights satisfying reverse inequalities. Expo. Math. *33* (2015), 1–29. zbl MR doi

[23] *M. Izuki*: Commutators of fractional integrals on Lebesgue and Herz spaces with variable exponent. Rend. Circ. Mat. Palermo (2) *59* (2010), 461–472. zbl MR doi
[24] *F. John, L. Nirenberg*: On functions of bounded mean oscillation. Commun. Pure Appl. Math. *14* (1961), 415–426. zbl MR doi
[25] *A. Y. Karlovich, A. K. Lerner*: Commutators of singular integrals on generalized L^p spaces with variable exponent. Publ. Mat., Barc. *49* (2005), 111–125. zbl MR doi
[26] *Y. Lu, Y. P. Zhu*: Boundedness of multilinear Calderón-Zygmund singular operators on Morrey-Herz spaces with variable exponents. Acta Math. Sin., Engl. Ser. *30* (2014), 1180–1194. zbl MR doi
[27] *B. Muckenhoupt*: Weighted norm inequalities for the Hardy maximal function. Trans. Am. Math. Soc. *165* (1972), 207–226. zbl MR doi
[28] *E. M. Stein*: Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals. Princeton Mathematical Series 43, Princeton University Press, Princeton, 1993. zbl MR doi
[29] *C. Tang, F. Xue, Y. Zhou*: Commutators of weighted Hardy operators on Herz-type spaces. Ann. Polon. Math. *101* (2011), 267–273. zbl MR doi
[30] *J. Wu*: Boundedness for commutators of fractional integrals on Herz-Morrey spaces with variable exponent. Kyoto J. Math. *54* (2014), 483–495. zbl MR doi
[31] *J.-L. Wu, W.-J. Zhao*: Boundedness for fractional Hardy-type operator on variable-exponent Herz-Morrey spaces. Kyoto J. Math. *56* (2016), 831–845. zbl MR doi
[32] *J. Xiao*: L^p and BMO bounds of weighted Hardy-Littlewood averages. J. Math. Anal. Appl. *262* (2001), 660–666. zbl MR doi

Authors' addresses: Dao Van Duong (corresponding author), School of Mathematics, Mientrung University of Civil Engineering, No. 24 Nguyen Du Street, Ward 7, Tuy Hoa City, Phu Yen Province, Vietnam, e-mail: `daovanduong@muce.edu.vn`; Kieu Huu Dung, School of Mathematics, University of Transport and Communications, No. 3 Cau Giay Street, Lang Thuong ward, Dong Da District, Hanoi, Vietnam, e-mail: `khdung@utc2.edu.vn`; Nguyen Minh Chuong, Institute of Mathematics, Vietnamese Academy of Science and Technology, 18 Hoang Quoc Viet Road, Building A5, Cau Giay, Hanoi, Vietnam, e-mail: `nmchuong@math.ac.vn`.