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ON THE BINARY SYSTEM OF FACTORS OF FORMAL MATRIX RINGS

Weining Chen, Guixin Deng, Huadong Su, Nanning

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Abstract. We investigate the formal matrix ring over R defined by a certain system of factors. We give a method for constructing formal matrix rings from non-negative integer matrices. We also show that the principal factor matrix of a binary system of factors determine the structure of the system.

Keywords: formal matrix ring; bimodule; system of factors; Wedderburn-Artin theorem

MSC 2020: 16S50, 15B99

1. Introduction

Throughout, rings are associative rings with nonzero identity, and modules and bimodules are unitary. The Jacobson radical, the center, the set of zero-divisors and the group of units of a ring R are denoted by $J(R)$, $C(R), Z(R)$ and $U(R)$, respectively. Let $M_n(R)$ be the $n \times n$ matrix ring over a ring R.

Let R, S be two rings, M be an R-S-bimodule and N be an S-R-bimodule. Suppose that there are two bimodule homomorphisms $\varphi\colon M \otimes N \to R$ and $\psi\colon N \otimes M \to S$. We write $mn = \varphi(m \otimes n)$ and $nm = \psi(n \otimes m)$ for all $m \in M$, $n \in N$. Let K denote the set of all matrices of the form

$$\begin{pmatrix} r & m \\ n & s \end{pmatrix}, \quad r \in R,\ s \in S,\ m \in M,\ n \in N.$$

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Then K is an abelian group under usual matrix addition. We define the multiplication of two matrices in K as

$$\begin{pmatrix} r_1 & m_1 \\ n_1 & s_1 \end{pmatrix}\begin{pmatrix} r_2 & m_2 \\ n_2 & s_2 \end{pmatrix} = \begin{pmatrix} r_1r_2 + m_1n_1 & r_1m_2 + m_1s_2 \\ n_1r_2 + s_1n_2 & n_1m_2 + s_1s_2 \end{pmatrix}.$$

It is routine to verify that K is a ring if and only if the associative law holds, that is, $(m_1n_1)m_2 = m_1(n_1m_2)$ and $(n_1m_1)n_2 = n_1(m_1n_2)$ for any $m_i \in M$ and $n_i \in N$. The ring K is called *generalized matrix ring* in [12] and [13], and is also called formal matrix ring in [8], [16]. If $M = 0$ or $N = 0$, then K is called triangular formal matrix ring. An important class of these rings consists of Morita context rings, see [10], [11] and [14]. Another important classes of these rings are triangular formal matrix rings which appeared in the representation theory of artinian algebra, see [3]. Many people studied the modules over these rings, see [4], [5], [7] and [8].

The ring K is called *formal matrix ring* over R when $R = S$ and $M = N = R$ with the usual R-R-bimodule structure. In this case, Krylov in [6] observed that K is determined by a central element s of R and he denoted this ring by $K_s(R)$. To be more precise, $K_s(R)$ consists of all 2×2 matrices over R with ordinary addition and the multiplication is defined by

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & b_1b_2 + sy_1x_2 \end{pmatrix}.$$

If R is commutative and $Z(R) \subseteq J(R)$, then $K_s(R) \simeq K_t(R)$ if and only if there exists an automorphism ϕ of R and $u \in U(R)$ such that $s = u\phi(t)$, see [6]. In particular, $K_s(R)$ is isomorphic to the usual matrix ring $M_2(R)$ if and only if $s \in U(R)$. Tang and Zhou in [15] obtained necessary and sufficient conditions for $K_s(R)$ to be a strongly clean ring.

The *formal matrix ring* over R was generalized to the formal matrix ring of order n over R by Tang and Zhou, see [16]. For a collection of central elements $\Sigma = \{s_{ijk}\}$ of R such that $s_{iij} = s_{ijj} = 1$ and $s_{ijk}s_{jkl} = s_{ijl}s_{jkl}$ for all $i,\ j,\ k,\ l$, one can define a formal matrix ring $K_n(R,\Sigma)$ of order n over R. Such a collection Σ is called a system of factors over R in [1], [2] and [9].

Let $s \in C(R)$ and $\Sigma = \{s_{ijk}\}$, where

$$s_{ijk} = \begin{cases} 1, & i = j \text{ or } j = k, \\ s, & i,\ j,\ k \text{ are pairwise distinct}, \\ s^2, & k = i \neq j. \end{cases}$$

Then Σ is a system of factors and the formal matrix ring defined by Σ is called a *Tang-Zhou ring* by Krylov and Tuganbaev, see [9]. Many properties of usual matrix rings are extended to the Tang-Zhou rings. In particular, the determinant is defined and the Cayley-Hamilton Theorem is proved for matrices in a Tang-Zhou ring. Abyzov and Tapkin in [1], [2], constructed another class of formal matrix rings which contained the class of the Tang-Zhou rings. Krylov and Tuganbaev in [9] obtained many basic properties of system of factors. They generalized the determinant and proved the Cayley-Hamilton Theorem for matrices in formal matrix rings defined by arbitrary systems of factors. As we can see the importance of systems of factors, to study formal matrix rings actually means to study the systems of factors in some ways.

This paper is organized as follows. In Section 2, we introduce definitions and notations of formal matrix rings over a ring R. We also list some basic properties and examples of the system of factors. In Section 3, we show how to construct system of factors from non-negative integer matrices. In Section 4, we mainly study binary system of factors. We show that the principal factor matrix of a binary system of factors determine the structure of the system.

2. Preliminaries

Let $R_1, R_2, \ldots, R_n$ be rings and M_{ij} be R_i-R_j-bimodule with $M_{ii} = R_i$. Suppose that there is bimodule homomorphism $\varphi_{ijk}\colon M_{ij} \otimes_{R_j} M_{jk} \to M_{ik}$ such that $\varphi_{iij}\colon R_i \otimes_{R_i} M_{ij} \to M_{ij}$ and $\varphi_{ijj}\colon M_{ij} \otimes_{R_j} R_j \to M_{ij}$ are the canonical isomorphisms for any i, j, k. Write $a \circ b = \varphi_{ijk}(m_{ij} \otimes m_{jk})$ for any i, j, k. Let $K = K(\{M_{ij}\}; \{\varphi_{ijk}\})$ be the set of all $n \times n$ matrices (m_{ij}) such that $m_{ij} \in M_{ij}$, with the following operations:

$$(m_{ij}) + (n_{ij}) := (m_{ij} + n_{ij}), \quad (m_{ij})(n_{ij}) := \left(\sum_{k=1}^{n} m_{ik} \circ n_{kj}\right).$$

It is routine to verify that K is a ring if and only if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in K$.

Now suppose that $R_1 = R_2 = \ldots = R_n = R$ and $M_{ij} = R$ with the usual R-R-bimodule structure. If K is a ring, then $\varphi_{ijk}(a \otimes b) = ab\varphi_{ijk}(1 \otimes 1) = \varphi_{ijk}(1 \otimes 1)ab$. Putting $b = 1$, we obtain $s_{ijk} = \varphi_{ijk}(1 \otimes 1) \in C(R)$. We claim that for any i, j, k, l

(i) $s_{iij} = s_{ijj} = 1$,

(ii) $s_{ijk}s_{ikl} = s_{ijl}s_{jkl}$.

In fact, the first equality follows immediately from that φ_{iij} and φ_{ijj} are canonical isomorphisms. By the associativity of $(a \circ b) \circ c = a \circ (b \circ c)$, we have $s_{ijk}s_{ikl}abc = s_{ijl}s_{jkl}abc$. Putting $a = b = c = 1$, we obtain equality (ii).

Conversely, for a given collection $\{s_{ijk}\colon 1 \leqslant i, j, k \leqslant n\}$ satisfying equalities (i) and (ii), we may set $\varphi_{ijk}(a \otimes b) = s_{ijk}ab$ for any $a, b \in R$. This defines a formal matrix ring $K(R, \{\varphi_{ijk}\})$ over R of order n. This class of formal matrix rings was first introduced in [9] and $\{s_{ijk}\}$ is called a system of factors in [2] and [9]. For completeness, we summarize it in the following definition.

Definition 2.1. A system of factors over a ring R of order n is a collection $\Sigma = \{s_{ijk} \colon\ s_{ijk} \in C(R),\ 1 \leqslant i, j, k \leqslant n\}$ such that for any i, j, k, l,

$$s_{iij} = s_{ijj} = 1, \quad s_{ijk}s_{ikl} = s_{ijl}s_{jkl}. \tag{2.1}$$

The elements s_{ijk} are called factors and the matrix $S = (s_{iji})_{1 \leqslant i,j \leqslant n}$ is called the *principal factor matrix* of Σ. The formal matrix ring $\mathbb{M}(R, \Sigma)$ defined by Σ is consists of all $n \times n$ matrices over R with ordinary matrix addition and the multiplication is given by

$$(a_{ij})(b_{ij}) := \Bigl(\sum_{k=1}^{n} s_{ikj}a_{ik}b_{kj}\Bigr).$$

Remark 2.2. Let $i = k$ in (2.1) and exchange i and j. We obtain

$$s_{iji} = s_{jij} = s_{ijl} \cdot s_{jil} = s_{lij} \cdot s_{lji}.$$

Therefore, the principal factor matrix of a system of factors is symmetric.

Now, we present some examples of system of factors.

Example 2.3 ([16])**.** Let $s \in C(R)$ and

$$s_{ijk} = \begin{cases} 1, & i = j \text{ or } j = k, \\ s, & i,\ j,\ k \text{ are pairwise distinct}, \\ s^2, & k = i \neq j. \end{cases}$$

Then $\Sigma = \{s_{ijk}\}$ is a system of factors and the formal matrix ring $\mathbb{M}(R, \Sigma)$ is called Tang-Zhou ring in [2] and [9].

Example 2.4 ([9])**.** Let $s \in C(R)$ and

$$s_{ijk} = \begin{cases} s, & i < k < j \text{ or } j < i < k \text{ or } k < j < i, \\ 1, & \text{otherwise}. \end{cases}$$

Then $\Sigma = \{s_{ijk}\}$ is a system of factors and the principal factor matrix of Σ is

$$\begin{pmatrix} 1 & s & \dots & s \\ s & 1 & \dots & s \\ \vdots & \vdots & \ddots & \vdots \\ s & s & \dots & 1 \end{pmatrix}.$$

Example 2.5 ([1]). Let $\beta_1, \beta_2, \ldots, \beta_n \in C(R)$ and

$$s_{ijk} = \begin{cases} 1, & i = j \text{ or } j = k, \\ \beta_j, & i,\ j,\ k \text{ are pairwise distinct}, \\ \beta_i\beta_j, & k = i \neq j. \end{cases}$$

Then $\Sigma = \{s_{ijk}\}$ is a system of factors of order n and the formal matrix ring $\mathbb{M}(R, \Sigma)$ is denoted by $\mathbb{M}_{\beta_1,\beta_2,\ldots,\beta_n}(R)$ in [1]. This class of systems of factors contains the class of systems of factors in Example 2.3.

Next, we present some basic properties of system of factors, which are used in the sequel. Let $S = (s_{ij})$ be an $n \times n$ matrix and σ be a permutation on $\{1, 2, \ldots, n\}$. Let $\sigma(S) = (s_{\sigma(i)\sigma(j)})$ and S^t denote the transpose of S.

Lemma 2.6 ([9]). *Let $\Sigma = \{s_{ijk}\colon 1 \leqslant i, j, k \leqslant n\}$ be a system of factors with principal factor matrix S, and σ be a permutation on $\{1, \ldots, n\}$. Then $\sigma(\Sigma) = \{s_{\sigma(i)\sigma(j)\sigma(k)}\colon 1 \leqslant i, j, k \leqslant n\}$ is a system of factors with principal matrix $\sigma(S)$ and $\mathbb{M}(R, \Sigma) \simeq \mathbb{M}(R, \sigma(\Sigma))$.*

Lemma 2.7 ([9]). *Let $\Sigma_l = \{s_{ijk}^{(l)}\colon 1 \leqslant i, j, k \leqslant n\}$ be a system of factors for any $l \in \{1, \ldots, m\}$. Then*

$$\prod_{l=1}^{m} \Sigma_l := \left\{\prod_{l=1}^{m} s_{ijk}^{(l)}\colon 1 \leqslant i, j, k \leqslant n\right\}$$

is a system of factors.

Lemma 2.8 ([9]). *Let $\Sigma_1 = \{s_{ijk}\colon 1 \leqslant i, j, k \leqslant n\}$ and $\Sigma_2 = \{t_{ijk}\colon 1 \leqslant i, j, k \leqslant n\}$ be two systems of factors over R. Then for any given $l \in \{1, \ldots, n\}$,*

$$\phi_l\colon\ \mathbb{M}(R, \Sigma_1\Sigma_2) \to \mathbb{M}(R, \Sigma_1), \quad (a_{ij}) \mapsto (t_{ijl}a_{ij})$$

is a ring homomorphism. Moreover, if $t_{ijk} \in U(R)$ for any i, j, k, then ϕ_l is an isomorphism.

3. Construction of system of factors

In this section, we mainly give a method for constructing the system of factors. We begin with a lemma.

Lemma 3.1. *Suppose that $a, b \in C(R)$ and $a \notin Z(R)$. Then the equation $ax = b$ has at most one solution. Moreover, if $ac = b$ and $b \notin Z(R)$, then $c \in C(R)$ and $c \notin Z(R)$.*

P r o o f. Assume $ax_1 = b$ and $ax_2 = b$. Then $a(x_1 - x_2) = 0$. As a is not a zero-divisor, we have $x_1 - x_2 = 0$ and $x_1 = x_2$. So the equation $ax = b$ has at most one solution. Let $d \in R$. We have $acd = bd = db = dac = adc$, and $a(cd - dc) = 0$. Since $a \notin Z(R)$, one has $cd = dc$ and thus $c \in C(R)$. If $dc = 0$, then $acd = bd = 0$ and $d = 0$. Hence, $c \notin Z(R)$. $\square$

Theorem 3.2. *Let $T = (t_{ij})$ with $t_{ij} \in C(R)$ and $t_{ii} = 1$. Suppose that $t_{ij} \notin Z(R)$ and $t_{ik} \mid t_{ij}t_{jk}$ for any i, j, k. Let s_{ijk} be a solution of $x \cdot t_{ik} = t_{ij} \cdot t_{jk}$. Then $\Sigma = \{s_{ijk}\}$ is a system of factors. Moreover, if $t_{il} = t_{jl} = 1$ or $t_{li} = t_{lj} = 1$, then $t_{ij} = s_{ijl}$ or $t_{ij} = s_{lij}$.*

P r o o f. By Lemma 3.1, $s_{ijk} \in C(R)$ and $s_{ijk} \notin Z(R)$. It is clear that $s_{ijk} = 1$ if $i = j$ or $j = k$. We have

$$\begin{aligned}(s_{ijk} \cdot s_{ikr}) \cdot t_{ik} \cdot t_{ir} \cdot t_{jr} &= (s_{ijk} \cdot t_{ik}) \cdot (s_{ikr} \cdot t_{ir}) \cdot t_{jr} = (t_{ij} \cdot t_{jk}) \cdot (t_{ik} \cdot t_{kr}) \cdot t_{jr} \\ &= (t_{ij} \cdot t_{jr}) \cdot (t_{jk} \cdot t_{kr}) \cdot t_{ik} = (s_{ijr} \cdot t_{ir}) \cdot (s_{jkr} \cdot t_{jr}) \cdot t_{ik} \\ &= (s_{ijr} \cdot s_{jkr}) \cdot t_{ik} \cdot t_{ir} \cdot t_{jr}.\end{aligned}$$

Since $t_{ik} \cdot t_{ir} \cdot t_{jr}$ is not a zero-divisor, one has $s_{ijk} \cdot s_{ikr} = s_{ijr} \cdot s_{jkr}$ and thus Σ is a system of factors. $\square$

By Theorem 3.2, we see that if each factor in a system of factors $\Sigma = \{s_{ijk}\}$ is not a zero-divisor, then Σ is determined by a matrix (s_{ijl}) or (s_{lij}) of factors. A class of systems of factors whose factors are powers of the same central element s is investigated by Krylov and Tuganbaev in [9]. Motivated by their work and Theorem 3.2, we introduce the following definition.

Claim 3.3. Let $\mathcal{M}_n$ be the set of all $n \times n$ non-negative integer matrices (m_{ij}) such that $m_{ii} = 0$ and $m_{ij} + m_{jk} - m_{ik} \geqslant 0$ for any $1 \leqslant i, j, k \leqslant n$. Then $\mathcal{M}_n$ is a semigroup with ordinary matrix addition.

Lemma 3.4. *Let $T = (m_{ij}) \in \mathcal{M}_n$ and $s \in C(R)$. Then*

$$\Sigma_{T,s} := \{s_{ijk} = s^{m_{ij}+m_{jk}-m_{ik}} \colon 1 \leqslant i, j, k \leqslant n\}$$

is a system of factors over R with the principal factor matrix $S = (s^{m_{ij}+m_{ji}})$.

P r o o f. It is straight to verify that (2.1) holds for $\Sigma_{T,s}$ and the principal factor matrix of $\Sigma_{T,s}$ is $(s_{iji}) = (s_{ijk} \cdot s_{jik}) = (s^{m_{ij}+m_{jk}-m_{ik}} \cdot s^{m_{ji}+m_{ik}-m_{jk}}) = (s^{m_{ij}+m_{ji}})$. $\square$

Remark 3.5. Let $s, \beta_1, \beta_2, \ldots, \beta_n \in C(R)$ and

$$T = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}.$$

It is easy to verify that $T, R \in \mathcal{M}_n$, $\Sigma_{T,s}$ and $\Sigma_{R,s}$ are the systems of factors in Example 2.3 and Example 2.4, respectively.

Let $T_l = (m_{ij}^{(l)})$, where

$$m_{ij}^{(l)} = \begin{cases} 1, & j = l \neq i, \\ 0, & \text{otherwise.} \end{cases}$$

That is, the lth column of T_l is equal to the lth column of T, and the other columns are all zero.

If i, j, k are pairwise distinct, then

$$m_{ij}^{(l)} + m_{jk}^{(l)} - m_{ik}^{(l)} = \begin{cases} 1, & j = l, \\ 0, & \text{otherwise.} \end{cases}$$

If $i = k \neq j$, then

$$m_{ij}^{(l)} + m_{jk}^{(l)} - m_{ik}^{(l)} = m_{ij}^{(l)} + m_{ji}^{(l)} = \begin{cases} 1, & l = i \text{ or } l = j, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $\Sigma = \Big\{ s_{ijk} = \prod\limits_{l=1}^{n} \beta_l^{m_{ij}^{(l)} + m_{jk}^{(l)} - m_{ik}^{(l)}} \Big\} = \prod\limits_{l=1}^{n} \Sigma_{T_l, \beta_l}$ is the system of factors in Example 2.5.

Theorem 3.6. *Let Σ be a system of factors of order n over R. Suppose that R is a unique factorization domain and $0 \notin \Sigma$. Then there exist non-negative integer matrices $T_i \in \mathcal{M}_n$ and irreducible elements $t_i \in R$ such that*

$$\mathbb{M}(R, \Sigma) \simeq \mathbb{M}\Big(R, \prod_{i=1}^{v} \Sigma_{T_i, t_i}\Big).$$

P r o o f. Let $\{t_r\}_{r \in J}$ be a representative class of irreducible elements in R. Since R is a unique factorization domain, we can write $s_{ijk} = u_{ijk} \prod\limits_{r=1}^{v} t_r^{m_{ijk}^{(r)}}$ for any i, j, k, where $u_{ijk} \in U(R)$ and $m_{ijk}^{(r)} \geqslant 0$. Since $s_{ijk} s_{ikl} = s_{ijl} s_{jkl}$, one has $u_{ijk} u_{ikl} = u_{ijl} u_{jkl}$

and $m_{ijk}^{(r)} + m_{ikl}^{(r)} = m_{ijl}^{(r)} + m_{jkl}^{(r)}$. Putting $\Sigma_0 = \{u_{ijk}\}$ and $T_r = (m_{ijl}^{(r)})_{1\leqslant i,j\leqslant n}$ for a fixed $l \in \{1,\ldots,n\}$. By Lemma 3.4, $\Sigma_{T_r,t_r} = \{t_r^{m_{ijk}^{(r)}}\colon 1 \leqslant i,j,k \leqslant n\}$ is a system of factors. It follows that $\Sigma = \Sigma_0 \cdot \prod\limits_{r=1}^{v} \Sigma_{T_r,t_r}$. Since each factor in Σ_0 is a unit, by Lemma 2.8 we have

$$\mathbb{M}(R,\Sigma) \simeq \mathbb{M}\biggl(R, \prod_{i=1}^{v} \Sigma_{T_i,t_i}\biggr).$$

This completes our proof. $\square$

A partition of a matrix is called a *square partition* if the number of blocks at each row is equal to the number of blocks at each column, and the blocks at the main diagonal are squares.

Definition 3.7. Let $S = (s_{ij})$ be a symmetric matrix such that $s_{ij} \in \{1, s\}$. We say that S has *canonical form* if S has a square partition such that all elements standing at the diagonal block are 1, and the other elements standing on all remaining positions are s. If there exists a permutation σ such that $\sigma(S)$ has the canonical form, then we say that S can be reduced to the canonical form.

Remark 3.8. Suppose that S has a square partition. If S_1 is obtained by exchanging the ith block row and the jth block row from S, and S_2 is obtained by exchanging the ith block column and the jth block column from S_1, then there exists a permutation σ such that $\sigma(S) = S_2$.

Let us look at the following simple example.

Example 3.9. Let

$$S = \begin{pmatrix} 1 & 1 & s \\ 1 & 1 & s \\ s & s & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & s & 1 \\ s & 1 & s \\ 1 & s & 1 \end{pmatrix}.$$

Then S has the canonical form and T can be reduced to canonical form, since $\sigma(T) = S$ for $\sigma = (23)$.

A *system of factors* $\{s_{ijk}\}$ is called *binary* if $s_{ijk} \in \{1, s\}$ for some $s \in C(R)$. It was showed in [9] that if Σ is a system of factors with principal factor matrix S such that each factor belongs to $\{1, s\}$ and $s^2 \neq 1, s$, then S can be reduced to the canonical form. In [9], the authors also showed that there exists a system of factors with principal factor matrix S for any given S with the canonical form. Here, we give a more straightforward proof of this fact based on Lemma 3.4.

Theorem 3.10. *Suppose that $S = (s_{ij})$ has the canonical form such that $s_{ij} \in \{1, s\}$ and $s \in C(R)$. Then there exists a system of factors $\Sigma = \{s_{ijk}\}$ over R with principal factor matrix S such that $s_{ijk} \in \{1, s\}$ for any i, j, k.*

P r o o f. Let $T = (m_{ij})$ with the same partition of the canonical form of S such that $m_{ij} = 1$ if the (i,j)-position stands above the diagonal block, and $m_{ij} = 0$ for any remaining positions. It is easy to verify that $0 \leqslant m_{ij} + m_{jk} - m_{ik} \leqslant 1$. Therefore, by Lemma 3.4, $\Sigma_{T,s} = \{s^{m_{ij}+m_{jk}-m_{ik}} \colon 1 \leqslant i,j,k \leqslant n\}$ is a binary system of factors such that each factor is contained in $\{1, s\}$. By the construction of T we have $m_{ij} + m_{ji} = 0$ if the (i,j)-position is in the diagonal block of T, and $m_{ij} + m_{ji} = 1$ for the remaining blocks. By Lemma 3.4 again, the principal factor matrix of $\Sigma_{T,s}$ is $(s^{m_{ij}+m_{ji}}) = S$. □

4. Structure of binary system of factors

In this section, we study the structure of binary system of factors over R. We use $\mathbf{1}_{m\times n}$ or $\mathbf{s}_{m\times n}$ to denote the $m \times n$ matrices such that all elements are 1 or s, respectively. The first lemma is a modification of Lemma 7.1 in [9].

Lemma 4.1. *Let $S = (s_{iji})$ be the principal factor matrix of a system of factors $\{s_{ijk}\}$ over a ring R. Then there exists a permutation σ such that $\sigma(S)$ has a square partition such that all elements standing at the diagonal block are units, and the other elements standing on all remaining positions are not unit.*

P r o o f. We first show that if $S_0 = (s_{ij})_{n\times n}$ is a matrix over R which satisfies the following two conditions, then S has the desired property of the lemma.

(1) $s_{ij}, s_{jk} \in U(R)$ implies that $s_{ik} \in U(R)$,

(2) S_0 is symmetric and $s_{ii} \in U(R)$.

Let $\mathbf{U}_{p\times q}$ be a set of $p \times q$ matrices with elements in $U(R)$ and let $\mathbf{V}_{p\times q}$ be a set of $p \times q$ matrices with elements in $R \setminus U(R)$. We induct on n. The case $n = 1, 2$ is trivial. Since $(s_{ij})_{1\leqslant i,j\leqslant n-1}$ satisfies the conditions (1) and (2), by induction there exists a permutation σ such that $\sigma(n) = n$ and

$$S_1 = \sigma(S_0) = \begin{pmatrix} T & A^t \\ A & s_{nn} \end{pmatrix} = (t_{ij}),$$

where $A \in R^{1\times(n-1)}$ and T has a square partition such that all elements standing at the diagonal block of are units, and the other elements standing on all remaining positions are not unit. We divide the remaining part of the proof into two cases.

Case 1: Suppose that $T \in \mathbf{U}_{(n-1)\times(n-1)}$. That is, $t_{ij} \in U(R)$ for $1 \leqslant i,j \leqslant n-1$. If $t_{ni} \in U(R)$ for some $i \in \{1,\ldots,n-1\}$, then we have $t_{nj} \in U(R)$ by condition (1) for every j. It follows that $A \in \mathbf{U}_{1\times(n-1)}$ or $A \in \mathbf{V}_{1\times(n-1)}$.

Case 2: Suppose that the number of blocks at the diagonal of T is $r \geqslant 2$. We write

$$S_1 = \begin{pmatrix} \mathbf{u}_1 & & \dots & & A_1^t \\ & \mathbf{u}_2 & \dots & & A_2^t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & \dots & \mathbf{u}_r & A_r^t \\ A_1 & A_2 & \dots & A_r & u \end{pmatrix},$$

where $\mathbf{u}_i \in \mathbf{U}_{n_i \times n_i}$, and $A_i \in \mathbf{U}_{1 \times n_i}$ or $A_i \in \mathbf{V}_{1 \times n_i}$ for any $i = 1, \dots, r$. If $A_i \in \mathbf{U}_{1 \times n_i}$ for any $i \in \{1, \dots, r\}$, then $t_{ij} \in U(R)$ for any i, j by condition (1). It follows that $S_1 \in \mathbf{U}_{n \times n}$, a contradiction. Hence, $A_i \in \mathbf{V}_{1 \times n_i}$ for some i. Now we exchange the ith row block and the last row block of S_1, and then exchange the ith column block and the last column block. This reduces S_1 to

$$S_2 = \begin{pmatrix} B & \mathbf{v}^T \\ \mathbf{v} & \mathbf{u}_i \end{pmatrix},$$

where $\mathbf{v} \in \mathbf{V}_{n_i \times (n - n_i)}$. By induction, B has the desired property of the lemma and is also S_2.

It remains to show that the principal matrix $S = (s_{iji})$ satisfies conditions (1) and (2). By Definition 2.1, we have $s_{iii} = 1$. By Remark 2.2, S is symmetric and

$$s_{iji} = s_{ijk} s_{jik} = s_{kij} s_{kji}, \quad s_{jkj} = s_{jki} s_{kji} = s_{ijk} s_{ikj}, \quad s_{iki} = s_{ikj} s_{kij} = s_{jik} s_{jki}.$$

If $s_{iji}, s_{jkj} \in U(R)$, then $s_{ijk}, s_{jik}, s_{jki}, s_{kji} \in U(R)$ and $s_{iki} \in U(R)$. The proof is complete. $\square$

Corollary 4.2. *Let $\Sigma = \{s_{ijk}\}$ be a system of factors over R with $s_{ijk} \in \{1, s\}$ and $S = (s_{iji})$ be the principal factor matrix of Σ. Suppose that $s \notin U(R)$. Then S can be reduced to the canonical form.*

Proof. It follows immediately from Lemma 4.1 and the fact that $s \notin U(R)$. $\square$

Definition 4.3. Let $S = (s_{ij})_{1 \leqslant i,j \leqslant n}$ such that $s_{ij} \in \{1, s\}$. We say that S has the triangular canonical form if S has a square partition such that all elements standing above (or below) the diagonal block are s, and the other elements standing on the diagonal blocks and below (or above) the diagonal blocks are 1. If there exists a permutation σ such that $\sigma(S)$ has the triangular canonical form, then we say that S can be reduced to the triangular canonical form.

Example 4.4. Let

$$S = \begin{pmatrix} 1 & s & s & s \\ 1 & 1 & s & s \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & s & s & s \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & s & s & 1 \end{pmatrix}.$$

Then $\sigma(T) = S$ for $\sigma = (24)$. So S has the triangular canonical form and T can be reduced to the triangular canonical form.

Lemma 4.5. *Let $S = (s_{ij})_{1 \leqslant i,j \leqslant n}$ such that $s_{ii} = 1$ and $s_{ij} \in \{1, s\}$. Suppose that*

(1) *$s_{ij} = s_{jk} = 1$ implies that $s_{ik} = 1$,*

(2) *S has a square partition such that all elements in the diagonal blocks are 1, and $(s_{ij}, s_{ji}) = (1, s)$ or $(s, 1)$ for any (i, j)-position not contained in the diagonal blocks.*

Then S can be reduced to the triangular canonical form. Moreover, if S has the triangular canonical form, then there exists a permutation σ such that $\sigma(S) = S^t$.

Proof. We prove the statement by induction on n. The case $n = 1$ is trivial. If $n = 2$, then S is one of the following:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ s & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & s \\ 1 & 1 \end{pmatrix}.$$

Each of them has the triangular canonical form. Suppose the statement is true for any $k < n$. By induction, S can be reduced to

$$S_1 = \begin{pmatrix} C & B \\ A & 1 \end{pmatrix},$$

where C is $(n-1) \times (n-1)$ with the triangular canonical form. We divide the remaining part of the proof into cases.

Case 1: If $C = \mathbf{1}_{(n-1)\times(n-1)}$, then $A = \mathbf{1}_{1\times(n-1)}$ or $\mathbf{s}_{1\times(n-1)}$ by similar discussions as in Case 1 of the proof of Lemma 4.1. It follows that S_1 has the triangular canonical form.

Case 2: If the number of blocks at the diagonal of C is $r \geqslant 2$, then we can partition

$$S_1 = \begin{pmatrix} \mathbf{1} & \mathbf{s} & \dots & \mathbf{s} & B_1^t \\ \mathbf{1} & \mathbf{1} & \dots & \mathbf{s} & B_2^t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{1} & \mathbf{1} & \dots & \mathbf{1} & B_r^t \\ A_1 & A_2 & \dots & A_r & 1 \end{pmatrix},$$

where both A_i and B_i are of $1 \times n_i$, and all elements in A_i or B_i are equal. Since $s_{ij} = 1$ for any $j \leqslant i < n$, if $s_{kn} = 1$, then $s_{ln} = 1$ for any $l < k$.

Subcase 2.1: If $(A_1, \dots, A_r) = \mathbf{1}_{1\times(n-1)}$, then $(B_1, \dots, B_r) = \mathbf{s}_{1\times(n-1)}$ and S has the triangular canonical form.

Subcase 2.2: If $(A_1,\dots,A_r)=\mathbf{s}_{1\times(n-1)}$, then $(B_1,\dots,B_r)=\mathbf{1}_{1\times(n-1)}$. By induction, there exists a permutation σ_1 such that $\sigma_1(C)=C^t$. Thus, S can be reduced to the triangular canonical form.

Subcase 2.3: If there exists j such that $A_i=\mathbf{1}_{1\times n_i}$ for some $i\leqslant j$ and $A_i=\mathbf{s}_{1\times n_i}$ for $i>j$, then we consider another partition of

$$S_1=\begin{pmatrix} J & \mathbf{s}_{m\times(n-m)} \\ \mathbf{1}_{(n-m)\times m} & H \end{pmatrix},$$

where $m=\sum\limits_{i=1}^{j} n_i$. By induction, there exists a permutation matrix P such that PHP^{-1} has the triangular canonical form with s standing above the diagonal blocks. Let $U=\left(\begin{smallmatrix} E_m & O \\ O & P \end{smallmatrix}\right)$. Then U is also a permutation matrix and

$$\begin{aligned} US_1U^{-1} &= \begin{pmatrix} E_m & O \\ O & P \end{pmatrix}\begin{pmatrix} J & \mathbf{s}_{m\times(n-m)} \\ \mathbf{1}_{(n-m)\times m} & H \end{pmatrix}\begin{pmatrix} E_m & O \\ O & P^{-1} \end{pmatrix} \\ &= \begin{pmatrix} J & \mathbf{s}_{m\times(n-m)} \\ \mathbf{1}_{(n-m)\times m} & PH \end{pmatrix}\begin{pmatrix} E_m & O \\ O & P^{-1} \end{pmatrix} = \begin{pmatrix} J & \mathbf{s}_{m\times(n-m)} \\ \mathbf{1}_{(n-m)\times m} & PHP^{-1} \end{pmatrix}. \end{aligned}$$

Since both J and PHP^{-1} have the triangular canonical form with s standing above the diagonal blocks, so does US_1U^{-1}. The proof is complete. □

Corollary 4.6. *Let $\Sigma=\{s_{ijk}\colon \{1\leqslant i,j,k\leqslant n\}$ be a system of factors over R with $s_{ijk}\in\{1,s\}$. Let $T_l=(s_{ijl})_{1\leqslant i,j\leqslant n}$ or $(s_{lij})_{1\leqslant i,j\leqslant n}$ for any given $l\in\{1,\dots,n\}$. Suppose that $s^2\neq 1,s$. Then T_l can be reduced to the triangular canonical form.*

P r o o f. We only deal with the case when $T_l=(s_{ijl})$ since the proof of $T_l=(s_{lij})$ is similar. By Corollary 4.2, $S=(s_{iji})$ can be reduced to the canonical form. After a permutation if necessary, we may assume that S has the canonical form. Then S has a square partition such that 1 stands at the diagonal blocks and s stands at the remaining positions. We partition $T_l=(s_{ijl})$ as the canonical form of S. Since $s_{ijl}\cdot s_{jil}=s_{iji}$ and $s^2\neq s,1$, we have $s_{ijl}=s_{jil}=1$ if (i,j)-position is in the diagonal blocks, and $\{s_{ijl},s_{jil}\}=\{1,s\}$ if (i,j)-position is not in the diagonal blocks. If $s_{ijl}=s_{jkl}=1$, then $s_{ikl}\cdot s_{jkl}=s_{ijl}\cdot s_{jkl}=1$, and thus $s_{ikl}=1$. By Lemma 4.5, T_l can be reduced to the triangular canonical form. □

Lemma 4.7. *Let Σ_1,Σ_2 be two systems of factors with principal factor matrices S_1, S_2, respectively. Suppose that each factor of Σ_i is 1 or s, and $s^2\neq s,1$. Then $\sigma(\Sigma_1)=\Sigma_2$ for some permutation σ if and only if S_1 and S_2 can be reduced to the same canonical form.*

P r o o f. The necessity follows from Lemma 2.6. Now we prove the sufficiency. Suppose that S_1 and S_2 can be reduced to the same canonical form. By Corollary 4.6, after permutations we may assume that $\Sigma_1 = \{s_{ijk}\}$, $\Sigma_2 = \{t_{ijk}\}$ such that $(s_{ijl})_{1\leqslant i,j\leqslant n} = (t_{ijl})_{1\leqslant i,j\leqslant n}$ has the triangular canonical form. Since $s_{ijl}\cdot s_{jkl} = s_{ikl}\cdot s_{ijk}$ and $t_{ijl}\cdot t_{jkl} = t_{ikl}\cdot t_{ijk}$, combining with that $s^2 \neq 1, s$, we obtain $s_{ijk} = t_{ijk}$ for any i, j, k. $\square$

Lemma 4.8 ([9], Proposition 6.3)**.** *Let I be an ideal of the formal matrix ring $\mathbb{M}(R,\Sigma)$. Then*

$$I = \begin{pmatrix} I_{11} & I_{12} & \dots & I_{1n} \\ I_{21} & I_{22} & \dots & I_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n1} & I_{n2} & \dots & I_{nn} \end{pmatrix},$$

where all I_{ij} are ideals of R and the following hold:

$$I_{ii} \subseteq \bigcap_{l=1}^{n} (I_{il} \cap I_{li}), \quad s_{iji}I_{ij} \subseteq I_{ii} \cap I_{jj}$$

for all i, j, and

$$s_{ikj}I_{kj} \subseteq I_{ij}, \quad s_{jki}I_{jk} \subset I_{ji}$$

for all pairwise distinct i, j, k.

Lemma 4.9 ([9], Proposition 6.4)**.** *Let $I = (I_{ij})$ be an ideal of a formal matrix ring $\mathbb{M}(R,\Sigma)$, where $\Sigma = \{s_{ijk}\}$ is a system of factors. Then:*

(1) *The set of matrices*

$$\overline{K} = \begin{pmatrix} \dfrac{R}{I_{11}} & \dfrac{R}{I_{12}} & \dots & \dfrac{R}{I_{1n}} \\ \dfrac{R}{I_{21}} & \dfrac{R}{I_{22}} & \dots & \dfrac{R}{I_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{R}{I_{n1}} & \dfrac{R}{I_{n2}} & \dots & \dfrac{R}{I_{nn}} \end{pmatrix}$$

is a formal matrix ring with bimodule homomorphisms

$$\varphi_{ijk}\colon \frac{R}{I_{ij}} \otimes_{R/I_{jj}} \frac{R}{I_{jk}} \to \frac{R}{I_{ik}}, \quad \varphi_{ijk}(\bar{x}\otimes\bar{y}) = s_{ijk}xy + I_{ik}.$$

(2) *There is an isomorphism*

$$\sigma\colon \frac{K}{I} \to \overline{K}, \quad (x_{ij}) + I \mapsto (x_{ij} + I_{ij}).$$

Lemma 4.10 ([9]). *Let $\Sigma = \{s_{ijk}\colon 1 \leqslant i, j, k \leqslant n\}$ be a system of factors over R. Then the Jacobson radical of $\mathbb{M}(R, \Sigma)$ is*

$$\begin{pmatrix} J(R) & J_{12}(R) & \dots & J_{1n}(R) \\ J_{21}(R) & J(R) & \dots & J_{2n}(R) \\ \vdots & \vdots & \ddots & \vdots \\ J_{n1}(R) & J_{n2}(R) & \dots & J(R) \end{pmatrix},$$

where $J_{ij}(R) = \{x\colon\ s_{iji}x \in J(R)\} = J_{ji}(R)$. In particular, $J_{ij}(R) = R$ if $s_{iji} \in J(R)$.

Corollary 4.11. *Let $\Sigma = \{s_{ijk}\}$ be a system of factors over a ring R and $K = M(R, \Sigma)$ be the formal matrix ring defined by Σ. Let $\overline{R} = R/J(R)$. Suppose that $s_{iji} \in J(R) \cup U(R)$ for any i, j. Then there exist $n_1, n_2, \dots, n_r$ such that $\sum\limits_{i=1}^{r} n_i = n$ and*

$$\frac{K}{J(K)} \simeq M_{n_1}(\overline{R}) \times M_{n_2}(\overline{R}) \times \ldots \times M_{n_r}(\overline{R}).$$

Proof. By Lemma 4.1 and the hypothesis that $s_{iji} \in J(R) \cup U(R)$, after a permutation if necessary, we may assume that the principal factor matrix $S = (s_{iji})$ has a square partition such that all elements standing at the diagonal blocks are units and the other elements are contained in $J(R)$. By Lemma 4.10, we have $J(K) = (J_{ij}(R))$, where $J_{ij}(R) = J(R)$ if (i, j)-position is at the diagonal blocks of S and $J_{ij}(R) = R$ if (i, j)-position is not at the diagonal blocks of S.

Assume that the ith block at the diagonal of S is $n_i \times n_i$. Let $N_i = \sum\limits_{l=1}^{i} n_l$. By Lemma 4.9,

$$\frac{K}{J(K)} \simeq \mathbb{M}_{n_1}(\overline{R}, \Sigma_1) \times \mathbb{M}_{n_2}(\overline{R}, \Sigma_2) \times \ldots \times \mathbb{M}_{n_r}(\overline{R}, \Sigma_r),$$

where $\Sigma_l = \{s_{ijk} + J(R)\colon\ N_l + 1 \leqslant i, j, k \leqslant N_{l+1}\}$. However, by hypothesis, s_{ijk} is a unit when s_{iji} is a unit. It follows that $\mathbb{M}_{n_l}(\overline{R}, \Sigma_l) \simeq M_{n_l}(\overline{R})$. This completes the proof. $\square$

Now, we state and prove one of the main results of this paper.

Theorem 4.12. *Let $s \in C(R) \cap J(R)$. Let Σ_1, Σ_2 be two systems of factors over R such that each factor of them belongs to $\{1, s\}$. Suppose that R is left artinian. Then $\mathbb{M}(R, \Sigma_1) \simeq \mathbb{M}(R, \Sigma_2)$ if and only if the principal factor matrices of Σ_1 and Σ_2 can be reduced to the same canonical form.*

P r o o f. The sufficiency follows immediately from Lemma 4.7 and Lemma 2.6. Now suppose that $\mathbb{M}(R,\Sigma_1)\simeq\mathbb{M}(R,\Sigma_2)$. Let $K_i=\mathbb{M}(R,\Sigma_i)$ and S_i be the principal matrices of Σ_i. After permutations if necessary, we may assume that both S_1 and S_2 have canonical form, and the sizes of diagonal blocks of S_1 are $n_1,\dots,n_r$, the sizes of diagonal blocks of S_2 are $m_1,\dots,m_{r'}$ with $n_1\leqslant n_2\leqslant\dots\leqslant n_r$ and $m_1\leqslant m_2\leqslant\dots\leqslant m_{r'}$.

Since R is left artinian, by the Wedderburn-Artin theorem, there exist division rings $D_1,\dots,D_s$ and positive integers $k_1,\dots,k_s$ such that $\overline{R}=R/J(R)\simeq M_{k_1}(D_1)\times M_{k_2}(D_2)\times\dots\times M_{k_s}(D_s)$. By Corollary 4.11,

$$\begin{aligned}\frac{K_1}{J(K_1)}&\simeq M_{n_1}(\overline{R})\times M_{n_2}(\overline{R})\times\dots\times M_{n_r}(\overline{R})\\&\simeq\prod_{i=1}^{s}M_{n_1k_i}(D_i)\times\prod_{i=1}^{s}M_{n_2k_i}(D_i)\times\dots\times\prod_{i=1}^{s}M_{n_rk_i}(D_i)\simeq\prod_{i=1}^{s}\prod_{j=1}^{r}M_{n_jk_i}(D_i).\end{aligned}$$

Similarly, we have

$$\frac{K_2}{J(K_2)}\simeq\prod_{i=1}^{s}\prod_{j=1}^{r'}M_{m_jk_i}(D_i).$$

By the Wedderburn-Artin theorem again, we obtain that $r=r'$ and $n_i=m_i'$ for $i=1,\dots,r$. Thus, $S_1=S_2$ and the proof is completed by Lemma 4.7. $\square$

Example 4.13. Let $R=F$ be a field and let $\Sigma_1=\{s_{ijk}\colon 1\leqslant i,j,k\leqslant 3\}$, $\Sigma_2=\{t_{ijk}\colon 1\leqslant i,j,k\leqslant 3\}$ be two systems of factors, where

$$s_{ijk}=\begin{cases}1, & i=j\text{ or }j=k,\\ 1, & ijk=123,231,312,\\ 0, & \text{otherwise.}\end{cases}$$

and

$$t_{ijk}=\begin{cases}1, & i=j\text{ or }j=k,\\ 0, & \text{otherwise.}\end{cases}$$

Then both Σ_1 and Σ_2 are systems of factors with the same principal factor matrix E_3. Since Σ_1 and Σ_2 have distinct number of factors 1, $\sigma(\Sigma_1)\neq\Sigma_2$ for any permutation σ. We also show that $K_1=\mathbb{M}(F,\Sigma_1)$ is not isomorphic to $K_2=\mathbb{M}(F,\Sigma_2)$.

Suppose to the contrary that there exists a ring isomorphism $\varphi\colon K_2\to K_1$. Let E_{ij} be the standard matrix bases. Let $\mathrm{Ann}_r C=\{X\colon CX=0\}$ be the right annihilator of C in a ring. By the construction of $\{t_{ijk}\}$, we have $E_{ij}E_{kl}=\delta_{jk}t_{ijl}E_{il}$ in K_2. If $i\neq j$ and $X=(x_{ij})$ in K_2, then $E_{ij}X=0$ if and only if $x_{jj}=0$. In K_2, we have

$$\dim_F\mathrm{Ann}_r(E_{ij})=8,\quad i\neq j.$$

Let $A = (a_{ij})$ and $Y = (y_{ij})$. By the construction of $\{s_{ijk}\}$, in K_1 we have

$$AY = \begin{pmatrix} a_{11}y_{11} & a_{11}y_{12} + a_{12}y_{22} & a_{11}y_{13} + a_{12}y_{23} + a_{13}y_{33} \\ a_{21}y_{11} + a_{22}y_{21} + a_{23}y_{31} & a_{22}y_{22} & a_{22}y_{23} + a_{23}y_{33} \\ a_{31}y_{11} + a_{33}y_{31} & a_{31}y_{12} + a_{32}y_{22} + a_{33}y_{32} & a_{33}y_{33} \end{pmatrix}.$$

Thus, $Y \in \mathrm{Ann}_r(A)$ if and only if Y is a solution of the system of homogeneous linear equations with nine indeterminates y_{11}, y_{21}, y_{31}, y_{12}, y_{22}, y_{32}, y_{13}, y_{23}, y_{33} and coefficient matrix $C_0 = \mathrm{diag}(C_1, C_2, C_3)$, where

$$C_1 = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}, \quad C_2 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad C_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}.$$

By the standard result of linear algebra, $\dim_F \mathrm{Ann}_r(A) = 8$ if and only if

$$\mathrm{rank}(C_0) = \sum_{i=1}^{3} \mathrm{rank}(C_i) = 1.$$

By a simple computation, we see that in K_1 $\dim_F \mathrm{Ann}_r(A) = 8$ if and only if $A = cE_{13}$, cE_{32} or cE_{21}, $c \neq 0$.

Since $\varphi\colon K_2 \to K_1$ is a ring isomorphism, we have

$$\dim_F \mathrm{Ann}_r(A) = \dim_F \mathrm{Ann}_r(\varphi(A))$$

for any $A \in K_2$. It follows that $\varphi(E_{ij})$ is contained in the subspace spanned by E_{13}, E_{32} and E_{21} for any $i \neq j$. That is, $\{E_{ij}\colon 1 \leqslant i,j \leqslant 3,\ i \neq j\}$ is linear independent, but $\{\varphi(E_{ij})\colon 1 \leqslant i,j \leqslant 3,\ i \neq j\}$ is linear dependent. This contradicts the fact that φ is an isomorphism.

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Author's address: Weining Chen, School of Mathematics and Statistics, Nanning Normal University, 175 Mingxiu E Rd, Xixiangtang, Nanning, Guangxi, 530023, P. R. China, e-mail: `chenwn2013@163.com`; Guixin Deng, School of Mathematics and Statistics, Nanning Normal University, 175 Mingxiu E Rd, Xixiangtang, Nanning, Guangxi, 530023, P. R. China and Guangxi College and Universities Key Laboratory of Data Science, Nanning, Guangxi, 530023, P. R. China, e-mail: `dengguixin@live.com`; Huadong Su (corresponding author), School of Mathematics and Statistics, Nanning Normal University, 175 Mingxiu E Rd, Xixiangtang, Nanning, Guangxi, 530023, P. R. China, e-mail: `huadongsu@sohu.com`.