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On the n-fold symmetric product of a space with a σ-(P)-property cn-network (ck-network)

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Abstract. We study the relation between a space X satisfying certain generalized metric properties and its n-fold symmetric product $\mathcal{F}_n(X)$ satisfying the same properties. We prove that X has a σ-(P)-property cn-network if and only if so does $\mathcal{F}_n(X)$. Moreover, if X is regular then X has a σ-(P)-property ck-network if and only if so does $\mathcal{F}_n(X)$. By these results, we obtain that X is strict σ-space (strict $\aleph$-space) if and only if so is $\mathcal{F}_n(X)$.

Keywords: σ-(P)-property; cn-network; ck-network; strict σ-space; strict $\aleph$-space

Classification: 54B20, 54D20

1. Introduction and preliminaries

In 1931, K. Borsuk and S. Ulam introduced the notion of a symmetric product of an arbitrary topological space, see [1]. Moreover, they also show that the n-fold symmetric product $\mathcal{F}_n(X)$ can be obtained as a quotient space of Cartesian product X^n. Recently, C. Good and S. Macías in [3], L.-X. Peng and Y. Sun in [4], Z. Tang, S. Lin and F. Lin in [5], studied the symmetric products of generalized metric spaces. They considered several generalized metric properties and studied the relation between a space X satisfying such property and its n-fold symmetric product satisfying the same property.

In this paper, we also study the relation between a space X satisfying certain generalized metric properties and its n-fold symmetric product satisfying the same properties. We prove that X has a σ-(P)-property cn-network if and only if so does $\mathcal{F}_n(X)$. Moreover, if X is regular then X has a σ-(P)-property ck-network if and only if so does $\mathcal{F}_n(X)$. By these results, we obtain that X is strict σ-space (strict $\aleph$-space) if and only if so is $\mathcal{F}_n(X)$.

Throughout this paper, all spaces are Hausdorff, $\mathbb{N}$ denotes the set of all positive integers.

Given a space X, we define its *hyperspaces* as the following sets:

(1) $CL(X) = \{A \subset X \colon A \text{ is closed and nonempty}\}$;

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(2) $2^X = \{A \in CL(X)\colon A \text{ is compact}\}$;
(3) $\mathcal{F}_n(X) = \big\{A \in 2^X\colon A \text{ has at most } n \text{ points}\big\}$, where $n \in \mathbb{N}$.

Set $CL(X)$ is topologized by the *Vietoris topology* defined as the topology generated by

$$\mathcal{B} = \{\langle U_1, \ldots, U_k\rangle\colon U_1, \ldots, U_k \text{ are open subsets of } X,\ k \in \mathbb{N}\},$$

where

$$\langle U_1, \ldots, U_k\rangle = \Big\{A \in CL(X)\colon A \subset \bigcup_{i \le k} U_i,\ A \cap U_i \ne \emptyset \ \text{ for each } i \le k\Big\}.$$

Note that, by definition, 2^X and $\mathcal{F}_n(X)$ are subspaces of $CL(X)$. Hence, they are topologized with the appropriate restriction of the Vietoris topology. Moreover,

(1) space $CL(X)$ is called the *hyperspace of nonempty closed subsets of* X;
(2) space 2^X is called the *hyperspace of nonempty compact subsets of* X;
(3) space $\mathcal{F}_n(X)$ is called the *n-fold symmetric product of* X.

On the other hand, it is obvious that $\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X)$ for each $n \in \mathbb{N}$.

Remark 1.1 ([3], Remark 2.1)**.** Let X be a space and let $n \ge 2$. Note that $\mathcal{F}_1(X)$ is closed in $\mathcal{F}_n(X)$ and $\xi\colon \mathcal{F}_1(X) \twoheadrightarrow X$ given by $\xi(\{x\}) = x$ is a homeomorphism.

Notation 1.2 ([3], Notation 2.2)**.** Let X be a space and let $n \in \mathbb{N}$. To simplify notation, if $U_1, \ldots, U_s$ are open subsets of X then $\langle U_1, \ldots, U_s\rangle_n$ denotes the intersection of the open set $\langle U_1, \ldots, U_s\rangle$ of the Vietoris topology, with $\mathcal{F}_n(X)$.

Notation 1.3 ([3], Notation 2.3)**.** Let X be a space and let $n \in \mathbb{N}$. If $\{x_1, \ldots, x_r\}$ is a point of $\mathcal{F}_n(X)$ and $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s\rangle_n$, then for each $j \in \{1, \ldots, r\}$, we let $U_{x_j} = \bigcap\{U \in \{U_1, \ldots, U_s\}\colon x_j \in U\}$. Observe that $\langle U_{x_1}, \ldots, U_{x_r}\rangle_n \subset \langle U_1, \ldots, U_s\rangle_n$.

Lemma 1.4 ([4], Lemma 21)**.** *Let X be a space and let $n \in \mathbb{N}$. If $\mathcal{C}$ is a compact subset of $\mathcal{F}_n(X)$, then $\bigcup\mathcal{C}$ is a compact subset of X.*

Definition 1.5. Let $\mathcal{P}$ be a family of subsets of a space X. Then:

(1) Family $\mathcal{P}$ is *point-finite*, if each point $x \in X$ belongs only to finitely many members of $\mathcal{P}$.
(2) Family $\mathcal{P}$ is *point-countable*, if each point $x \in X$ belongs only to countably many members of $\mathcal{P}$.
(3) Family $\mathcal{P}$ is *compact-finite*, if for each compact subset $K \subset X$, the set $\{P \in \mathcal{P}\colon P \cap K \ne \emptyset\}$ is finite.
(4) Family $\mathcal{P}$ is *compact-countable*, if for each compact subset $K \subset X$, the set $\{P \in \mathcal{P}\colon P \cap K \ne \emptyset\}$ is countable.
(5) Family $\mathcal{P}$ is *locally finite*, if for each $x \in X$, there exists a neighborhood V of x such that V meets only finitely many members of $\mathcal{P}$.

(6) Family $\mathcal{P}$ is *locally countable*, if for each $x \in X$, there exists a neighborhood V of x such that V meets only countably many members of $\mathcal{P}$.

Definition 1.6. For a cover $\mathcal{P}$ of a space X, let (P) be one of the following properties: point-finite, compact-finite, locally finite, point-countable, compact-countable, and locally countable. We said that $\mathcal{P}$ has *σ-(P)-property*, if $\mathcal{P}$ can be expressed as $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, where each $\mathcal{P}_n$ has (P)-property, and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n \in \mathbb{N}$.

Definition 1.7 ([2])**.** Let $\mathcal{P}$ be a family of subsets of a space X. Then,

(1) Family $\mathcal{P}$ is a *network at* $x \in X$, if for each neighborhood O_x of x there is a set $P \in \mathcal{P}$ such that $x \in P \subset O_x$; $\mathcal{P}$ is a *network* in X if $\mathcal{P}$ is a network at each point $x \in X$.

(2) Family $\mathcal{P}$ is a *cn-network at* $x \in X$, if for each neighborhood O_x of x, the set $\bigcup\{P \in \mathcal{P} : x \in P \subset O_x\}$ is a neighborhood of x; $\mathcal{P}$ is a *cn-network* in X if $\mathcal{P}$ is a *cn*-network at each point $x \in X$.

(3) Family $\mathcal{P}$ is a *ck-network at* $x \in X$, if for any neighborhood O_x of x, there is a neighborhood $U_x \subset O_x$ of x such that for each compact subset $K \subset U_x$, there exists a finite subfamily $\mathcal{F} \subset \mathcal{P}$ satisfying $x \in \bigcap \mathcal{F}$ and $K \subset \bigcup \mathcal{F} \subset O_x$; $\mathcal{P}$ is a *ck-network* in X if $\mathcal{P}$ is a *ck*-network at each point $x \in X$.

Remark 1.8 ([2])**.** Base (at x) $\Longrightarrow$ *ck*-network (at x) $\Longrightarrow$ *cn*-network (at x) $\Longrightarrow$ network (at x).

Definition 1.9 ([2])**.** Let X be a topological space. Then:

(1) Space X is called a *strict σ-space*, if X has a σ-locally finite *cn*-network.

(2) Space X is called a *strict $\aleph$-space*, if X has a σ-locally finite *ck*-network.

2. Main results

Let $n \in \mathbb{N}$ and $\mathcal{P}$ be a family of subsets of a space X. If we put

$$\mathfrak{P} = \{\langle P_1, \dots, P_s\rangle_n : P_1, \dots, P_s \in \mathcal{P},\ s \leq n\},$$

then observe that $\mathfrak{P}$ is a family of subsets of $\mathcal{F}_n(X)$.

Lemma 2.1. *Let $\langle U_1, \dots, U_s\rangle$, $\langle V_1, \dots, V_r\rangle \subset CL(X)$. If there exists $i_0 \leq s$ such that $U_{i_0} \cap \big(\bigcup_{j\leq r} V_j\big) = \emptyset$, then $\langle U_1, \dots, U_s\rangle \cap \langle V_1, \dots, V_r\rangle = \emptyset$.*

Proof: Assume that there exists $i_0 \leq s$ such that $U_{i_0} \cap \big(\bigcup_{j\leq r} V_j\big) = \emptyset$. Then, we have $\langle U_1, \dots, U_s\rangle \cap \langle V_1, \dots, V_r\rangle = \emptyset$. Otherwise, there exists $F \in \langle U_1, \dots, U_s\rangle \cap \langle V_1, \dots, V_r\rangle$. Hence, $F \cap U_{i_0} \neq \emptyset$, it implies that there exists $x_0 \in F \cap U_{i_0}$. Since $U_{i_0} \cap \big(\bigcup_{j\leq r} V_j\big) = \emptyset$, $x_0 \notin \bigcup_{j\leq r} V_j$. Thus, $F \not\subset \bigcup_{j\leq r} V_j$, this is a contradiction. □

Lemma 2.2. *If $\mathcal{P}$ has (P)-property then so does $\mathfrak{P}$.*

Proof: *Case 1.* (P) is point-finite. Let $F = \{x_1, \dots, x_r\} \in \mathcal{F}_n(X)$. For each $j \leq r$, since $\mathcal{P}$ is point-finite in X, $\mathcal{P}_j = \{P \in \mathcal{P} \colon x_j \in P\}$ is finite. If we put $\mathcal{P}_0 = \bigcup_{j \leq r} \mathcal{P}_j$, then $\mathcal{P}_0$ is finite. Moreover, we have

$$\{\mathcal{W} \in \mathfrak{P} \colon F \in \mathcal{W}\} \subset \{\langle P_1, \dots, P_s \rangle_n \colon P_1, \dots, P_s \in \mathcal{P}_0,\ s \leq n\}.$$

In fact, let $k \leq n$ and $\langle E_1, \dots, E_k \rangle_n \notin \{\langle P_1, \dots, P_s \rangle_n \colon P_1, \dots, P_s \in \mathcal{P}_0,\ s \leq n\}$. Then, there exists $i_0 \leq k$ such that $E_{i_0} \notin \mathcal{P}_0$. It implies that $x_j \notin E_{i_0}$ for every $j \leq r$. Thus, $F \notin \langle E_1, \dots, E_k \rangle_n$. Hence, $\langle E_1, \dots, E_k \rangle_n \notin \{\mathcal{W} \in \mathfrak{P} \colon F \in \mathcal{W}\}$.

Because $\mathcal{P}_0$ is finite, $\{\mathcal{W} \in \mathfrak{P} \colon F \in \mathcal{W}\}$ is finite. Thus, $\mathfrak{P}$ is point-finite in $\mathcal{F}_n(X)$.

Case 2. (P) is point-countable. Similar to the proof of Case 1.

Case 3. (P) is compact-finite. Let $\mathcal{A}$ be a compact subset of $\mathcal{F}_n(X)$. It follows from Lemma 1.4 that $A = \bigcup \mathcal{A}$ is a compact subset of X. Moreover, since $\mathcal{A} \subset \langle A \rangle_n$, we have

$$\{\mathcal{W} \in \mathfrak{P} \colon \mathcal{W} \cap \mathcal{A} \neq \emptyset\} \subset \{\mathcal{W} \in \mathfrak{P} \colon \mathcal{W} \cap \langle A \rangle_n \neq \emptyset\}.$$

Since $\mathcal{P}$ is compact-finite in X, $\mathcal{P}_0 = \{P \in \mathcal{P} \colon P \cap A \neq \emptyset\}$ is finite. On the other hand, we have

$$\{\mathcal{W} \in \mathfrak{P} \colon \mathcal{W} \cap \langle A \rangle_n \neq \emptyset\} \subset \{\langle P_1, \dots, P_s \rangle_n : P_1, \dots, P_s \in \mathcal{P}_0,\ s \leq n\}.$$

In fact, let $k \leq n$ and $\langle E_1, \dots, E_k \rangle_n \notin \{\langle P_1, \dots, P_s \rangle_n \colon P_1, \dots, P_s \in \mathcal{P}_0,\ s \leq n\}$. Then, there exists $i_0 \leq k$ such that $E_{i_0} \notin \mathcal{P}_0$. This implies that $E_{i_0} \cap A = \emptyset$. By Lemma 2.1, $\langle E_1, \dots, E_k \rangle_n \cap \langle A \rangle_n = \emptyset$. Thus, $\langle E_1, \dots, E_k \rangle_n \notin \{\mathcal{W} \in \mathfrak{P} \colon \mathcal{W} \cap \langle A \rangle_n \neq \emptyset\}$.

Since $\mathcal{P}_0$ is finite, $\{\mathcal{W} \in \mathfrak{P} \colon \mathcal{W} \cap \mathcal{A} \neq \emptyset\}$ is finite. Therefore, $\mathfrak{P}$ is compact-finite in $\mathcal{F}_n(X)$.

Case 4. (P) is compact-countable. Similar to the proof of Case 3.

Case 5. (P) is locally finite. Let $F = \{x_1, \dots, x_r\} \in \mathcal{F}_n(X)$. For each $i \leq r$, since $\mathcal{P}$ is locally finite in X, there exists an open neighborhood W_i of x_i intersecting only finitely many elements of $\mathcal{P}$. If we put

$$V_i = W_i \setminus \{x_j \colon j \leq r,\ j \neq i\},$$

then V_i is open in X for every $i \leq r$, and $\langle V_1, \dots, V_r \rangle_n$ is an open neighborhood of F in $\mathcal{F}_n(X)$. On the other hand, $\langle V_1, \dots, V_r \rangle_n$ intersects only finitely many elements of $\mathfrak{P}$. In fact, for each $i \leq r$, since $\mathcal{P}$ is locally finite in X, $\mathcal{P}_i = \{P \in \mathcal{P} \colon P \cap V_i \neq \emptyset\}$ is finite. If we put $\mathcal{P}_0 = \bigcup_{i \leq r} \mathcal{P}_i$, then $\mathcal{P}_0$ is finite. Now, let $k \leq n$ and $\langle E_1, \dots, E_k \rangle_n \notin \{\langle P_1, \dots, P_s \rangle_n \colon P_1, \dots, P_s \in \mathcal{P}_0,\ s \leq n\}$.

Then, there exists $i_0 \leq k$ such that $E_{i_0} \notin \mathcal{P}_0$. Thus, $E_{i_0} \cap V_i = \emptyset$ for every $i \leq r$. By Lemma 2.1, $\langle E_1, \dots, E_k \rangle_n \cap \langle V_1, \dots, V_r \rangle_n = \emptyset$. Hence, $\langle E_1, \dots, E_k \rangle_n \notin \{\mathcal{W} \in \mathfrak{P} \colon \mathcal{W} \cap \langle V_1, \dots, V_r \rangle_n \neq \emptyset\}$. This implies that

$$\{\mathcal{W} \in \mathfrak{P} \colon \mathcal{W} \cap \langle V_1, \dots, V_r \rangle_n \neq \emptyset\} \subset \{\langle P_1, \dots, P_s \rangle_n \colon P_1, \dots, P_s \in \mathcal{P}_0,\ s \leq n\}.$$

Furthermore, since $\mathcal{P}_0$ is finite, $\{\mathcal{W} \in \mathfrak{P} \colon \mathcal{W} \cap \langle V_1, \dots, V_r \rangle_n \neq \emptyset\}$ is finite. Hence, $\mathfrak{P}$ is locally finite in $\mathcal{F}_n(X)$.

Case 6. (P) is locally countable. Similar to the proof of Case 5. $\square$

Lemma 2.3. (1) *If $\mathcal{P}$ is a cn-network then so is $\mathfrak{P}$.*

(2) *If X is regular and $\mathcal{P}$ is a ck-network then so is $\mathfrak{P}$.*

Proof: Let $F = \{x_1, \dots, x_r\} \in \mathcal{F}_n(X)$ and $\mathcal{U}$ be an open neighborhood of F in $\mathcal{F}_n(X)$. Then, there exist open subsets $U_1, \dots, U_s$ of X such that

$$F \in \langle U_1, \dots, U_s \rangle_n \subset \mathcal{U}.$$

It follows from Notation 1.3 that there exist open subsets $U_{x_1}, \dots, U_{x_r}$ of X such that $x_j \in U_{x_j}$ for each $j \leq r$, and

$$F \in \langle U_{x_1}, \dots, U_{x_r} \rangle_n \subset \langle U_1, \dots, U_s \rangle_n \subset \mathcal{U}.$$

(1) For each $j \leq r$, we put

$$\mathcal{P}_j = \{P \in \mathcal{P} \colon x_j \in P \subset U_{x_j}\}.$$

Then, for each $j \leq r$, since $\mathcal{P}$ is a *cn*-network in X, $\bigcup \mathcal{P}_j$ is a neighborhood of x_j in X. This implies that for each $j \leq r$, there is V_j open in X such that

$$x_j \in V_j \subset \bigcup \mathcal{P}_j.$$

Moreover, if we put $\mathcal{R} = \bigcup_{j \leq r} \mathcal{P}_j$ then

$$\begin{aligned} F \in \langle V_1, \dots, V_r \rangle_n &\subset \Big\langle \bigcup \mathcal{P}_1, \dots, \bigcup \mathcal{P}_r \Big\rangle_n \\ &\subset \bigcup \{\langle P_1, \dots, P_s \rangle_n \colon F \in \langle P_1, \dots, P_s \rangle_n,\ P_1, \dots, P_s \in \mathcal{R},\ s \leq n\} \\ &\subset \bigcup \{\mathcal{W} \in \mathfrak{P} \colon F \in \mathcal{W} \subset \mathcal{U}\}. \end{aligned}$$

On the other hand, since $\langle V_1, \dots, V_r \rangle_n$ is open in $\mathcal{F}_n(X)$, we have $\bigcup \{\mathcal{W} \in \mathfrak{P} \colon F \in \mathcal{W} \subset \mathcal{U}\}$ is a neighborhood of F in $\mathcal{F}_n(X)$. Therefore, $\mathfrak{P}$ is a *cn*-network in $\mathcal{F}_n(X)$.

(2) For each $j \leq r$, since $\mathcal{P}$ is a *ck*-network in X, there exists a neighborhood $V_{x_j} \subset U_{x_j}$ such that for each compact subset $A_j \subset V_{x_j}$, there exists a finite

subfamily $\mathcal{A}_j$ of $\mathcal{P}$ satisfying

$$x_j \in \bigcap \mathcal{A}_j \qquad \text{and} \qquad A_j \subset \bigcup \mathcal{A}_j \subset U_{x_j}.$$

Next, for each $j \le r$, since X is regular, there exists W_{x_j} open in X such that

$$x_j \in W_{x_j} \subset \overline{W}_{x_j} \subset V_{x_j}.$$

Now, if we put $\mathcal{V}_F = \langle W_{x_1}, \dots, W_{x_r} \rangle_n$ then for each compact subset $\mathcal{K} \subset \mathcal{V}_F$, we have

$$\bigcup \mathcal{K} \subset \bigcup_{j \le r} \overline{W}_{x_j}.$$

Moreover, since $\bigcup \mathcal{K}$ is compact in X by Lemma 1.4, we have $K_j = (\bigcup \mathcal{K}) \cap \overline{W}_{x_j}$ is compact in X and $K_j \subset V_{x_j}$. Thus, there exists a finite subfamily $\mathcal{F}_j \subset \mathcal{P}$ such that

$$x_j \in \bigcap \mathcal{F}_j \qquad \text{and} \qquad K_j \subset \bigcup \mathcal{F}_j \subset U_{x_j}.$$

Lastly, if we put $\mathcal{R} = \bigcup_{j \le r} \mathcal{F}_j$ and

$$\mathcal{F} = \{\langle P_1, \dots, P_s \rangle_n \colon F \in \langle P_1, \dots, P_s \rangle_n,\ P_1, \dots, P_s \in \mathcal{R},\ s \le n\}$$

then $\mathcal{F}$ is finite, $F \in \bigcap \mathcal{F}$ and $\bigcup \mathcal{F} \subset \langle U_{x_1}, \dots, U_{x_r} \rangle_n$. Furthermore, $\mathcal{K} \subset \bigcup \mathcal{F}$. In fact, for any $\{y_1, \dots, y_p\} \in \mathcal{K}$, we have $\{y_1, \dots, y_p\} \subset \bigcup \mathcal{K}$. For each $k \le p$, since $\bigcup \mathcal{K} = \bigcup_{j \le r} K_j$, there exists $j_0 \le r$ such that $y_k \in K_{j_0} \subset \bigcup \mathcal{F}_{j_0}$. This implies that $\{y_1, \dots, y_p\} \in \bigcup \mathcal{F}$. Thus, $\mathcal{K} \subset \bigcup \mathcal{F} \subset \langle U_{x_1}, \dots, U_{x_r} \rangle_n$.

Therefore, $\mathfrak{P}$ is a *ck*-network in $\mathcal{F}_n(X)$. □

Theorem 2.4. *Let X be a space and let $n \in \mathbb{N}$. Then:*

1. *Space X has a σ-(P)-property cn-network if and only if so does $\mathcal{F}_n(X)$.*
2. *If X is regular, then X has a σ-(P)-property ck-network if and only if so does $\mathcal{F}_n(X)$.*

Proof: *Necessity.* Assume that $\mathcal{P} = \bigcup\{\mathcal{P}_k \colon k \in \mathbb{N}\}$ is a *cn*-network (*ck*-network) in X, where each $\mathcal{P}_k$ has (P)-property and $\mathcal{P}_k \subset \mathcal{P}_{k+1}$ for each $k \in \mathbb{N}$. By Lemma 2.2, we have

$$\mathfrak{P}_k = \{\langle P_1, \dots, P_s \rangle_n \colon P_1, \dots, P_s \in \mathcal{P}_k,\ s \le n\}$$

has the (P)-property, and $\mathfrak{P}_k \subset \mathfrak{P}_{k+1}$ for each $k \in \mathbb{N}$. Therefore, $\mathfrak{P} = \bigcup\{\mathfrak{B}_k \colon k \in \mathbb{N}\}$ is a cover for $\mathcal{F}_n(X)$ having σ-(P)-property. Moreover, observe that

$$\mathfrak{P} \subset \{\langle P_1, \dots, P_s \rangle_n \colon P_1, \dots, P_s \in \mathcal{P},\ s \le n\}.$$

Now, let $\mathcal{W} \in \{\langle P_1, \dots, P_s \rangle_n \colon P_1, \dots, P_s \in \mathcal{P},\ s \le n\}$. Then, there exist $P_1, \dots, P_s \in \mathcal{P}$ such that $\mathcal{W} = \langle P_1, \dots, P_s \rangle_n$. Since $\mathcal{P} = \bigcup\{\mathcal{P}_k \colon k \in \mathbb{N}\}$, there exist $k_i \in \mathbb{N}$ such that $P_i \in \mathcal{P}_{k_i}$ for each $i \leq s$. If we put $m = \max\{k_i : i \leq s\}$ then $P_1, \dots, P_s \in \mathcal{P}_m$ and $m \in \mathbb{N}$. This implies that $\mathcal{W} \in \mathfrak{P}_m \subset \mathfrak{P}$. Thus,

$$\mathfrak{P} = \{\langle P_1, \dots, P_s \rangle_n : P_1, \dots, P_s \in \mathcal{P},\ s \leq n\}.$$

It follows from Lemma 2.3 that $\mathfrak{P}$ is a cn-network (ck-network) in $\mathcal{F}_n(X)$.

Sufficiency. Let $\mathfrak{B} = \bigcup\{\mathfrak{B}_k : k \in \mathbb{N}\}$ be a cn-network (ck-network) in $\mathcal{F}_n(X)$ with σ-(P)-property. Then,

$$\mathcal{P} = \bigcup\{\mathfrak{B}_k|_{\mathcal{F}_1(X)} : k \in \mathbb{N}\}$$

is a cn-network (ck-network) in $\mathcal{F}_1(X)$ with σ-(P)-property, where $\mathfrak{B}_k|_{\mathcal{F}_1(X)} = \{P \cap \mathcal{F}_1(X) : P \in \mathfrak{B}_k\}$ for each $k \in \mathbb{N}$. On the other hand, it follows from Remark 1.1 that $\xi \colon \mathcal{F}_1(X) \twoheadrightarrow X$ given by $\xi(\{x\}) = x$ is a homeomorphism. Therefore, X has a σ-(P)-property cn-network (ck-network). $\square$

By Theorem 2.4, we obtain the following corollary.

Corollary 2.5. *Let X be a space and let $n \in \mathbb{N}$. Then:*

(1) *Space X is strict σ-space if and only if so is $\mathcal{F}_n(X)$.*
(2) *If X is regular, then X is strict $\aleph$-space if and only if so is $\mathcal{F}_n(X)$.*

Question 2.6. *Let X be a Hausdorff space and let $n \in \mathbb{N}$. If X has a σ-(P)-property ck-network, then does $\mathcal{F}_n(X)$ have a σ-(P)-property ck-network?*

References

[1] Borsuk K., Ulam S., *On symmetric products of topological spaces*, Bull. Amer. Math. Soc. **37** (1931), no. 12, 875–882.

[2] Gabriyelyan S. S., Kakol J., *On $\mathfrak{P}$-spaces and related concepts*, Topology Appl. **191** (2015), 178–198.

[3] Good C., Macías S., *Symmetric products of generalized metric spaces*, Topology Appl. **206** (2016), 93–114.

[4] Peng L.-X., Sun Y., *A study on symmetric products of generalized metric spaces*, Topology Appl. **231** (2017), 411–429.

[5] Tang Z., Lin S., Lin F., *Symmetric products and closed finite-to-one mappings*, Topology Appl. **234** (2018), 26–45.

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