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ONE-SIDED GORENSTEIN SUBCATEGORIES

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Abstract. We introduce the right (left) Gorenstein subcategory relative to an additive subcategory \mathcal{C} of an abelian category \mathcal{A} , and prove that the right Gorenstein subcategory $r\mathcal{G}(\mathcal{C})$ is closed under extensions, kernels of epimorphisms, direct summands and finite direct sums. When \mathcal{C} is self-orthogonal, we give a characterization for objects in $r\mathcal{G}(\mathcal{C})$, and prove that any object in \mathcal{A} with finite $r\mathcal{G}(\mathcal{C})$ -projective dimension is isomorphic to a kernel (or a cokernel) of a morphism from an object in \mathcal{A} with finite \mathcal{C} -projective dimension to an object in $r\mathcal{G}(\mathcal{C})$. As an application, we obtain a weak Auslander-Buchweitz context related to the kernel of a hereditary cotorsion pair in \mathcal{A} having enough injectives.

Keywords: right Gorenstein subcategory; self-orthogonal subcategory; relative projective dimension; cotorsion pair; kernel; (weak) Auslander-Buchweitz context

MSC 2010: 18G25, 16E10, 18G10

1. INTRODUCTION

As a nice generalization of finitely generated projective modules, Auslander and Bridger in [2] introduced finitely generated modules having Gorenstein dimension zero over commutative Noetherian rings. For arbitrary modules over general rings, Enochs and Jenda in [10] introduced Gorenstein projective dimension, which coincides with Gorenstein dimension for finitely generated modules over commutative Noetherian rings; meanwhile, they also introduced Gorenstein injective modules as the dual of Gorenstein projective modules. Since then, these modules have become crucial research objects in Gorenstein homological algebra, and have been studied extensively, see [2], [4], [6], [7], [8], [9], [10], [11], [16], and the references therein.

Let \mathcal{A} be an abelian category and \mathcal{C} an additive subcategory of \mathcal{A} . As a common generalization of Gorenstein projective and Gorenstein injective modules, Sather-

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Wagstaff et al. in [20] introduced the Gorenstein subcategory $\mathcal{G}(\mathcal{C})$ of \mathcal{A} relative to \mathcal{C} . It is shown that Gorenstein subcategories share many nice properties of the categories of Gorenstein projective modules and Gorenstein injective modules, see [14], [17] and [20]. In [20], Section 5 the authors asked: do some important properties of the category of Gorenstein projective (injective) modules such as exactness, closure of kernels of epimorphisms and cokernels of monomorphisms, hold true for Gorenstein subcategories? In fact, we will answer this question negatively. Why are the answers negative? From the definition of the Gorenstein subcategory $\mathcal{G}(\mathcal{C})$, we know that \mathcal{C} should be simultaneously a generator and a cogenerator for $\mathcal{G}(\mathcal{C})$. It leads to some limitations. The aim of this paper is to overcome such limitations by modifying the definition of Gorenstein subcategories. The paper is organized as follows.

In Section 2, we give some terminology and notation.

In Section 3, we introduce right (left) Gorenstein subcategories relative to a subcategory \mathcal{C} of an abelian category \mathcal{A} , such that when \mathcal{C} is self-orthogonal, the Gorenstein subcategory coincides with the intersection of the left and the right Gorenstein subcategories. We prove that the right Gorenstein subcategory $r\mathcal{G}(\mathcal{C})$ is exact and closed under kernels of epimorphisms (see Proposition 3.3), however, the Gorenstein subcategory does not possess such properties in general (see Example 3.4), which answers in [20], Question 5.8 negatively. Under the assumption that \mathcal{C} is self-orthogonal, we give some equivalent conditions for objects in $r\mathcal{G}(\mathcal{C})$ (see Theorem 3.7), which shows that the subcategory $r\mathcal{G}(\mathcal{C})$ has some kind of stability. Moreover, we prove that an object in \mathcal{A} with finite $r\mathcal{G}(\mathcal{C})$ -projective dimension is isomorphic to a kernel (or a cokernel) of a morphism from an object in \mathcal{A} with finite \mathcal{C} -projective dimension to an object in $r\mathcal{G}(\mathcal{C})$ (see Theorem 3.11).

In Section 4, as an application of the above results, we prove that if a hereditary cotorsion pair $(\mathcal{U}, \mathcal{V})$ with kernel \mathcal{C} has enough injectives then $(r\mathcal{G}(\mathcal{C}), \mathcal{C}\text{-pd}^{<\infty}, \mathcal{C})$ is a weak Auslander-Buchweitz context, where $\mathcal{C}\text{-pd}^{<\infty}$ is the subcategory of \mathcal{A} consisting of objects with finite \mathcal{C} -projective dimension (see Theorem 4.8).

2. PRELIMINARIES

In this paper, \mathcal{A} is an abelian category and all subcategories of \mathcal{A} are full, additive and closed under isomorphisms.

For a subcategory \mathcal{X} of \mathcal{A} , we write

$$\begin{aligned} {}^\perp\mathcal{X} &:= \{A \in \mathcal{A} : \text{Ext}_{\mathcal{A}}^{\geq 1}(A, X) = 0 \text{ for } X \in \mathcal{X}\}, \\ \mathcal{X}^\perp &:= \{A \in \mathcal{A} : \text{Ext}_{\mathcal{A}}^{\geq 1}(X, A) = 0 \text{ for } X \in \mathcal{X}\}, \\ {}^\perp_1\mathcal{X} &:= \{A \in \mathcal{A} : \text{Ext}_{\mathcal{A}}^1(A, X) = 0 \text{ for } X \in \mathcal{X}\}, \\ \mathcal{X}^{\perp_1} &:= \{A \in \mathcal{A} : \text{Ext}_{\mathcal{A}}^1(X, A) = 0 \text{ for } X \in \mathcal{X}\}. \end{aligned}$$

For subcategories \mathcal{X}, \mathcal{Y} of \mathcal{A} , we write $\mathcal{X} \perp \mathcal{Y}$ if $\text{Ext}_{\mathcal{A}}^{\geq 1}(X, Y) = 0$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$; and \mathcal{X} is called *self-orthogonal* if $\mathcal{X} \perp \mathcal{X}$.

Let $\mathcal{C} \subseteq \mathcal{X}$ be subcategories of \mathcal{A} . We say that \mathcal{C} is a *generator* for \mathcal{X} if for any $X \in \mathcal{X}$ there exists an exact sequence

$$0 \rightarrow X' \rightarrow C \rightarrow X \rightarrow 0$$

in \mathcal{A} with $X' \in \mathcal{X}$ and $C \in \mathcal{C}$. We say that \mathcal{C} is a *projective generator* for \mathcal{X} if \mathcal{C} is a generator for \mathcal{X} and $\mathcal{C} \perp \mathcal{X}$. Dually, we say that \mathcal{C} is a *cogenerator* for \mathcal{X} if for any $X \in \mathcal{X}$ there exists an exact sequence

$$0 \rightarrow X \rightarrow C \rightarrow X' \rightarrow 0$$

in \mathcal{A} with $C \in \mathcal{C}$ and $X' \in \mathcal{X}$. We say that \mathcal{C} is an *injective cogenerator* for \mathcal{X} if \mathcal{C} is a cogenerator for \mathcal{X} and $\mathcal{X} \perp \mathcal{C}$.

Let \mathcal{X} be a subcategory of \mathcal{A} and $A \in \mathcal{A}$. The \mathcal{X} -*projective dimension* $\mathcal{X}\text{-pd } A$ of A is defined to be the infimum integer n for which there exists an exact sequence

$$0 \rightarrow X_n \rightarrow \dots \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with all X_i in \mathcal{X} , and we set $\mathcal{X}\text{-pd } A = \infty$ if no such integer exists. Dually, the \mathcal{X} -*injective dimension* $\mathcal{X}\text{-id } A$ of A is defined to be the infimum integer n for which there exists an exact sequence

$$0 \rightarrow A \rightarrow X^0 \rightarrow X^1 \rightarrow \dots \rightarrow X^n \rightarrow 0$$

in \mathcal{A} with all X^i in \mathcal{X} , and we set $\mathcal{X}\text{-id } A = \infty$ if no such integer exists. We use $\mathcal{X}\text{-pd}^{<\infty}$ and $\mathcal{X}\text{-id}^{<\infty}$ to denote the subcategories of \mathcal{A} consisting of objects with finite \mathcal{X} -projective and \mathcal{X} -injective dimensions, respectively.

A sequence \mathbb{E} in \mathcal{A} is called $\text{Hom}_{\mathcal{A}}(\mathcal{X}, -)$ -*exact* or $\text{Hom}_{\mathcal{A}}(-, \mathcal{X})$ -*exact* if $\text{Hom}_{\mathcal{A}}(X, \mathbb{E})$ or $\text{Hom}_{\mathcal{A}}(\mathbb{E}, X)$, respectively, is exact for any $X \in \mathcal{X}$. Following [20], we write

$$\begin{aligned} \text{res } \widetilde{\mathcal{X}} := \{A \in \mathcal{A} : \text{there exists an exact } \text{Hom}_{\mathcal{A}}(\mathcal{X}, -)\text{-exact sequence} \\ \dots \rightarrow X_1 \rightarrow X_0 \rightarrow A \rightarrow 0 \text{ in } \mathcal{A} \text{ with all } X_i \text{ in } \mathcal{X}\}, \end{aligned}$$

and

$$\begin{aligned} \text{cores } \widetilde{\mathcal{X}} := \{A \in \mathcal{A} : \text{there exists an exact } \text{Hom}_{\mathcal{A}}(-, \mathcal{X})\text{-exact sequence} \\ 0 \rightarrow A \rightarrow X^0 \rightarrow X^1 \rightarrow \dots \text{ in } \mathcal{A} \text{ with all } X_i \text{ in } \mathcal{X}\}. \end{aligned}$$

Definition 2.1 ([20]). Let \mathcal{C} be a subcategory of \mathcal{A} . The *Gorenstein subcategory* $\mathcal{G}(\mathcal{C})$ of \mathcal{A} relative to \mathcal{C} is defined to be all $G \in \mathcal{A}$ such that there exists an exact $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact and $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$\dots \rightarrow C_1 \rightarrow C_0 \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$$

in \mathcal{A} with all C_i, C^i in \mathcal{C} , such that $G \cong \text{Im}(C_0 \rightarrow C^0)$.

The Gorenstein subcategory unifies the following notions: modules of Gorenstein dimension zero (see [2]), Gorenstein projective modules, Gorenstein injective modules ([10]), V -Gorenstein projective modules, V -Gorenstein injective modules (see [12]), \mathcal{W} -Gorenstein modules (see [14]), and so on; see [17] for the details.

Definition 2.2 ([19], Chapter 5.5). A category \mathcal{E} is called *exact* if \mathcal{E} is a full subcategory of some abelian category \mathcal{A} and it is closed under extensions.

3. ONE-SIDED GORENSTEIN SUBCATEGORIES

In this section, we fix a subcategory \mathcal{C} of an abelian category \mathcal{A} . Following [17], Lemma 5.7, if $\mathcal{C} \perp \mathcal{C}$ then

$$\mathcal{G}(\mathcal{C}) = ({}^\perp \mathcal{C} \cap \text{cores } \tilde{\mathcal{C}}) \cap (\mathcal{C}^\perp \cap \text{res } \tilde{\mathcal{C}}).$$

Motivated by this, we introduce right and left Gorenstein subcategories.

Definition 3.1. We call

$$r\mathcal{G}(\mathcal{C}) := {}^\perp \mathcal{C} \cap \text{cores } \tilde{\mathcal{C}} \quad \text{and} \quad l\mathcal{G}(\mathcal{C}) := \mathcal{C}^\perp \cap \text{res } \tilde{\mathcal{C}},$$

respectively, the *right* and *left Gorenstein subcategory* of \mathcal{A} relative to \mathcal{C} .

In the following example, all rings are associative with identities and all modules are unitary. For a ring R , $\text{Mod } R$ is the category of left R -modules.

Example 3.2.

- (1) In general, we have $r\mathcal{G}(\mathcal{C}) \neq l\mathcal{G}(\mathcal{C})$. Let R be a ring and let $\mathcal{A} = \text{Mod } R$. If \mathcal{C} is the subcategory of \mathcal{A} consisting of projective modules, then $r\mathcal{G}(\mathcal{C})$ is the subcategory of \mathcal{A} consisting of Gorenstein projective modules and $l\mathcal{G}(\mathcal{C}) = \mathcal{A}$. If \mathcal{C} is the subcategory of \mathcal{A} consisting of injective modules, then $r\mathcal{G}(\mathcal{C}) = \mathcal{A}$ and $l\mathcal{G}(\mathcal{C})$ is the subcategory of \mathcal{A} consisting of Gorenstein injective modules.
- (2) Let R, S be rings and ${}_R C_S$ a semidualizing (R, S) -bimodule.

- (2.1) If $\mathcal{A} = \text{Mod } R$ and \mathcal{C} is the subcategory of \mathcal{A} consisting of direct summands of direct sums of copies of C , then $r\mathcal{G}(\mathcal{C})$ is the subcategory of \mathcal{A} consisting of G_C -projective modules by [18], Proposition 2.4 (1), and $l\mathcal{G}(\mathcal{C})$ is the Bass class $\mathcal{B}_C(R)$ with respect to C by [21], Theorem 3.9.
- (2.2) If $\mathcal{A} = \text{Mod } S$ and \mathcal{C} is the subcategory of \mathcal{A} consisting of direct summands of direct products of copies of $\text{Hom}_{\mathbb{Z}}(C, \mathbb{Q}/\mathbb{Z})$, where \mathbb{Z} is the additive group of integers and \mathbb{Q} is the additive group of rational numbers, then $r\mathcal{G}(\mathcal{C})$ is the Auslander class $\mathcal{A}_C(S)$ with respect to C by [22], Theorem 3.11 (1) and [5], Proposition VI.5.1, and $l\mathcal{G}(\mathcal{C})$ is the subcategory of \mathcal{A} consisting of G_C -injective modules by [18], Proposition 2.4 (2).

In the following, we study the properties of $r\mathcal{G}(\mathcal{C})$. The dual versions of all the results on $r\mathcal{G}(\mathcal{C})$ also hold true on $l\mathcal{G}(\mathcal{C})$ by using completely dual arguments.

The following result shows that $r\mathcal{G}(\mathcal{C})$ is an exact category, and has a quasi-resolving structure that is closed under kernels of epimorphisms.

Proposition 3.3.

- (1) *The subcategory $r\mathcal{G}(\mathcal{C})$ is closed under extensions, direct summands and finite direct sums.*
- (2) *The subcategory $r\mathcal{G}(\mathcal{C})$ is closed under kernels of epimorphisms.*

Proof. (1) Let

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

be an exact sequence in \mathcal{A} with $L, N \in r\mathcal{G}(\mathcal{C})$. Then it is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact. Since $L, N \in r\mathcal{G}(\mathcal{C})$, we have $M \in {}^{\perp}\mathcal{C}$ and there exist exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequences

$$0 \rightarrow L \xrightarrow{d^0} C'^0 \xrightarrow{d^1} C'^1 \rightarrow \dots \quad \text{and} \quad 0 \rightarrow N \xrightarrow{d''^0} C''^0 \xrightarrow{d''^1} C''^1 \rightarrow \dots$$

in \mathcal{A} with all C'^i and C''^i in \mathcal{C} . It is trivial that all cokernels of d'^i and d''^i are in $\text{cores } \tilde{\mathcal{C}}$. Put $C^0 := C'^0 \oplus C''^0$ and consider the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N \longrightarrow 0 \\ & & \downarrow d'^0 & & \downarrow d''^0 & & \\ 0 & \longrightarrow & C'^0 & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & C^0 & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & C''^0 \longrightarrow 0. \end{array}$$

Since the upper row in this diagram is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact, we get an epimorphism $\text{Hom}_{\mathcal{A}}(M, C'^0) \rightarrow \text{Hom}_{\mathcal{A}}(L, C'^0)$ and there exists $\alpha : M \rightarrow C'^0$ such that $d'^0 = \alpha f$.

Putting $d^0 := \begin{pmatrix} \alpha \\ d''^0 g \end{pmatrix}$, we obtain the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N \longrightarrow 0 \\
 & & \downarrow d'^0 & \nearrow \alpha & \downarrow d^0 & & \downarrow d''^0 \\
 0 & \longrightarrow & C'^0 & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & C^0 & \xrightarrow{(0 \ 1)} & C''^0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \dashrightarrow & \text{Coker } d'^0 & \dashrightarrow & \text{Coker } d^0 & \dashrightarrow & \text{Coker } d''^0 \dashrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
 \end{array}$$

It is easy to check that the bottom row and the middle column in this diagram are $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact. Repeating this process, we get an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$$

in \mathcal{A} with all C^i in \mathcal{C} . Thus $M \in \text{cores } \tilde{\mathcal{C}}$ and $M \in r\mathcal{G}(\mathcal{C})$.

It is trivial that ${}^\perp \mathcal{C}$ is closed under direct summands. By [17], Theorem 4.6 (1), $\text{cores } \tilde{\mathcal{C}}$ is closed under direct summands. Thus $r\mathcal{G}(\mathcal{C})$ is also closed under direct summands.

It is easy to see that $r\mathcal{G}(\mathcal{C})$ is closed under finite direct sums.

(2) Consider an exact sequence

$$0 \rightarrow A \rightarrow A_1 \rightarrow A_2 \rightarrow 0$$

in \mathcal{A} with $A_1, A_2 \in r\mathcal{G}(\mathcal{C})$. First, it is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact and $A \in {}^\perp \mathcal{C}$. Moreover, there exist exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequences

$$\begin{aligned}
 0 &\rightarrow A_1 \rightarrow C_1^0 \rightarrow C_1^1 \rightarrow \dots \rightarrow C_1^n \rightarrow \dots, \\
 0 &\rightarrow A_2 \rightarrow C_2^0 \rightarrow C_2^1 \rightarrow \dots \rightarrow C_2^m \rightarrow \dots
 \end{aligned}$$

in \mathcal{A} with all C_m^i in \mathcal{C} . By [17], Theorem 3.8 (1), (5), we get an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$0 \rightarrow A \rightarrow C_1^0 \rightarrow C_1^1 \oplus C_2^0 \rightarrow \dots \rightarrow C_1^n \oplus C_1^{n-1} \rightarrow \dots$$

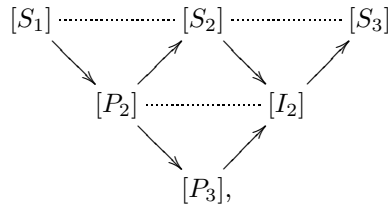
that is, $A \in \text{cores } \tilde{\mathcal{C}}$. Thus $A \in r\mathcal{G}(\mathcal{C})$, as desired. \square

Sather-Wagstaff, Sharif and White in [20], Question 5.8 posed the following questions: Is $\mathcal{G}(\mathcal{C})$ always exact? Is $\mathcal{G}(\mathcal{C})$ always closed under kernels of epimorphisms or cokernels of monomorphisms? The following example answers these two questions negatively.

Example 3.4. Let $\Lambda = kQ$ be a finite-dimensional hereditary path algebra over an algebraically closed field k , where Q is the quiver

$$1 \leftarrow 2 \leftarrow 3.$$

Then the Auslander-Reiten quiver of Λ is (see [1], Chapter IV, Example 4.10)



where the symbol $[M]$ denotes the isomorphism class of a module M , and

- (1) $S_1 = (k \leftarrow 0 \leftarrow 0)$, $S_2 = (0 \leftarrow k \leftarrow 0)$ and $S_3 = (0 \leftarrow 0 \leftarrow k)$ are all simple modules;
- (2) $P_1 = S_1$, $P_2 = (k \leftarrow k \leftarrow 0)$ and $P_3 = (k \leftarrow k \leftarrow k)$ are all indecomposable projective modules;
- (3) $I_1 = P_3$, $I_2 = (0 \leftarrow k \leftarrow k)$ and $I_3 = S_3$ are all indecomposable injective modules.

Let $\mathcal{A} = \Lambda\text{-mod}$ be the category of finitely generated left Λ -modules.

- (i) Let $\mathcal{C} = \text{add}(S_1 \oplus S_2 \oplus P_3 \oplus S_3)$ be the full subcategory of \mathcal{A} consisting of all direct summands of finite direct sum of $S_1 \oplus S_2 \oplus P_3 \oplus S_3$. By computation, $\mathcal{G}(\mathcal{C}) = \text{add}(S_1 \oplus S_2 \oplus P_3 \oplus S_3)$. From the exact sequence

$$0 \rightarrow S_1 \rightarrow P_2 \rightarrow S_2 \rightarrow 0$$

we see that $\mathcal{G}(\mathcal{C})$ is not exact; and from the exact sequences

$$0 \rightarrow P_2 \rightarrow P_3 \rightarrow S_3 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow S_1 \rightarrow P_3 \rightarrow I_2 \rightarrow 0$$

we see that $\mathcal{G}(\mathcal{C})$ is neither closed under kernels of epimorphisms nor closed under cokernels of monomorphisms.

(ii) Let $\mathcal{C} = \text{add}(S_1 \oplus P_3 \oplus S_3)$. In this case, $\mathcal{C} \perp \mathcal{C}$. To illustrate it, we only need to check $\text{Ext}_\Lambda^1(S_3, S_1) = 0$. In fact, by [1], Chapter IV, Corollary 2.14 we have

$$\text{Ext}_\Lambda^1(S_3, S_1) \cong D \text{Hom}_\Lambda(S_1, \tau S_3) \cong D \text{Hom}_\Lambda(S_1, S_2) = 0,$$

where $D := \text{Hom}_k(-, k)$. Moreover, by computation, $\mathcal{G}(\mathcal{C}) = \text{add}(S_1 \oplus P_3 \oplus S_3)$. Thus the second and third short exact sequences in (i) show that $\mathcal{G}(\mathcal{C})$ is still neither closed under kernels of epimorphisms nor closed under cokernels of monomorphisms.

In the rest of this section, we always assume $\mathcal{C} \perp \mathcal{C}$. In this setting, $\mathcal{C} \subseteq r\mathcal{G}(\mathcal{C})$. We have the following lemma.

Lemma 3.5. *For an object $L \in \mathcal{A}$, $L \in r\mathcal{G}(\mathcal{C})$ holds if and only if there exists an exact $(\text{Hom}_{\mathcal{A}}(-, \mathcal{C})\text{-exact})$ sequence*

$$0 \rightarrow L \rightarrow C \rightarrow N \rightarrow 0$$

in \mathcal{A} with $C \in \mathcal{C}$ and $N \in r\mathcal{G}(\mathcal{C})$.

Proof. The necessity follows from Proposition 3.3 (2).

For the sufficiency, since $L \in r\mathcal{G}(\mathcal{C})$, there exists an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})\text{-exact}$ sequence

$$0 \rightarrow L \rightarrow C \rightarrow N \rightarrow 0$$

in \mathcal{A} with $C \in \mathcal{C}$ and $N \in \text{cores } \tilde{\mathcal{C}}$. Since $\mathcal{C} \perp \mathcal{C}$, we have $N \in {}^\perp \mathcal{C}$ and thus $N \in r\mathcal{G}(\mathcal{C})$. \square

Proposition 3.6. *If*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

is an exact sequence in \mathcal{A} with $L, M \in r\mathcal{G}(\mathcal{C})$, then $N \in r\mathcal{G}(\mathcal{C})$ if and only if $N \in {}^\perp \mathcal{C}$.

Proof. The necessity is trivial. In the following we prove the sufficiency. Let

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

be an exact sequence in \mathcal{A} with $L, M \in r\mathcal{G}(\mathcal{C})$ and $\text{Ext}_{\mathcal{A}}^1(N, C) = 0$ for any $C \in \mathcal{C}$. Since $L \in r\mathcal{G}(\mathcal{C})$, by Lemma 3.5 there exists an exact sequence

$$0 \rightarrow L \rightarrow C \rightarrow L' \rightarrow 0$$

in \mathcal{A} with $C \in \mathcal{C}$ and $L' \in r\mathcal{G}(\mathcal{C})$. Consider the push-out diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \dashrightarrow & C & \dashrightarrow & T & \dashrightarrow & N \dashrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & L' & \equiv & L' & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

By Proposition 3.3(1), the middle column in this diagram shows that $T \in r\mathcal{G}(\mathcal{C})$. Since $\text{Ext}_{\mathcal{A}}^1(N, C) = 0$ by assumption, the middle row in the above diagram splits. Thus N is isomorphic to a direct summand of T , and hence it is in $r\mathcal{G}(\mathcal{C})$ by Proposition 3.3(1). \square

In the following, we give some equivalent conditions for objects in $r\mathcal{G}(\mathcal{C})$.

Theorem 3.7. *For any $M \in \mathcal{A}$, the following statements are equivalent.*

- (1) $M \in r\mathcal{G}(\mathcal{C})$.
- (2) $M \in {}^\perp\mathcal{C}$ and for any subcategory \mathcal{D} of \mathcal{A} such that \mathcal{C} is an injective cogenerator for \mathcal{D} there exists an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$0 \rightarrow M \rightarrow D^0 \rightarrow D^1 \rightarrow \dots$$

in \mathcal{A} with all D^i in \mathcal{D} .

- (3) There exists an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$\mathbb{G} := \dots \rightarrow G_1 \rightarrow G_0 \rightarrow G_{-1} \rightarrow G_{-2} \rightarrow \dots$$

in \mathcal{A} with all G_i in $r\mathcal{G}(\mathcal{C})$ such that $M \cong \text{Ker}(G_{-1} \rightarrow G_{-2})$.

- (4) For any subcategory \mathcal{D} of \mathcal{A} with $\mathcal{C} \subseteq \mathcal{D} \subseteq r\mathcal{G}(\mathcal{C})$ there exists an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{D})$ -exact sequence

$$\mathbb{G} := \dots \rightarrow G_1 \rightarrow G_0 \rightarrow G_{-1} \rightarrow G_{-2} \rightarrow \dots$$

in \mathcal{A} with all G_i in $r\mathcal{G}(\mathcal{C})$ such that $M \cong \text{Ker}(G_{-1} \rightarrow G_{-2})$.

- (5) There exists an exact $\text{Hom}_{\mathcal{A}}(-, r\mathcal{G}(\mathcal{C}))$ -exact sequence

$$\mathbb{G} := \dots \rightarrow G_1 \rightarrow G_0 \rightarrow G_{-1} \rightarrow G_{-2} \rightarrow \dots$$

in \mathcal{A} with all G_i in $r\mathcal{G}(\mathcal{C})$ such that $M \cong \text{Ker}(G_{-1} \rightarrow G_{-2})$.

Proof. The implications (1) \Rightarrow (2) and (5) \Rightarrow (4) \Rightarrow (3) are trivial.
(2) \Rightarrow (1) Let $M \in {}^\perp \mathcal{C}$ and

$$0 \rightarrow M \rightarrow D^0 \rightarrow D^1 \rightarrow \dots$$

be an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence in \mathcal{A} with all D^i in \mathcal{D} . Put $M^1 := \text{Im}(D^0 \rightarrow D^1)$. Since \mathcal{C} is an injective cogenerator for \mathcal{D} by assumption, we have $M^1 \in {}^\perp \mathcal{C}$ and there exists an exact sequence

$$0 \rightarrow D^0 \rightarrow C^0 \rightarrow D' \rightarrow 0$$

in \mathcal{A} with $C^0 \in \mathcal{C}$ and $D' \in \mathcal{D}$. Consider the push-out diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & M & \longrightarrow & D^0 & \longrightarrow & M^1 \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \dashrightarrow & M & \dashrightarrow & C^0 & \dashrightarrow & M' \dashrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & D' & = & D' \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

Since $D', M^1 \in {}^\perp \mathcal{C}$, we have that $M' \in {}^\perp \mathcal{C}$ and therefore the middle row is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact. Similarly, we get an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$0 \rightarrow M' \rightarrow C^1 \rightarrow M'' \rightarrow 0$$

in \mathcal{A} with $M'' \in {}^\perp \mathcal{C}$. Continuing this process, we obtain an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$$

in \mathcal{A} with all C^i in \mathcal{C} , as desired.

(1) \Rightarrow (5) It holds by setting

$$\mathbb{G} := \dots \rightarrow 0 \rightarrow M \xrightarrow{\text{id}} M \rightarrow 0 \rightarrow \dots$$

(3) \Rightarrow (1) Let

$$\mathbb{G} := \dots \rightarrow G_1 \rightarrow G_0 \rightarrow G_{-1} \rightarrow G_{-2} \rightarrow \dots$$

be an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence in \mathcal{A} with all G_i in $r\mathcal{G}(\mathcal{C})$ and $M \cong \text{Ker}(G_{-1} \rightarrow G_{-2})$. It is easy to see that $M_{-i} \in {}^\perp\mathcal{C}$, where $M_{-i} := \text{Ker}(G_{-(i+1)} \rightarrow G_{-(i+2)})$ for any $i \geq 0$ (note: $M_0 = M$).

Since $G_{-1} \in r\mathcal{G}(\mathcal{C})$, by Lemma 3.5 there exists an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$0 \rightarrow G_{-1} \rightarrow C_{-1} \rightarrow G'_{-1} \rightarrow 0$$

in \mathcal{A} with $C_{-1} \in \mathcal{C}$ and $G'_{-1} \in r\mathcal{G}(\mathcal{C})$. Consider the push-out diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & M & \longrightarrow & G_{-1} & \longrightarrow & M_{-1} \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \dashrightarrow & M & \dashrightarrow & C_{-1} & \dashrightarrow & Q_{-1} \dashrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & G'_{-1} & = & G'_{-1} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

Then $Q_{-1} \in {}^\perp\mathcal{C}$ and the middle row in this diagram is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact. Now consider the push-out diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & M_{-1} & \longrightarrow & Q_{-1} & \longrightarrow & G'_{-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \dashrightarrow & G_{-2} & \dashrightarrow & T_{-2} & \dashrightarrow & G'_{-1} \dashrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & M_{-2} & = & M_{-2} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Since $G'_{-1}, G_{-2} \in r\mathcal{G}(\mathcal{C})$, we have $T_{-2} \in r\mathcal{G}(\mathcal{C})$ by Proposition 3.3(1). Since $M_{-2} \in {}^\perp\mathcal{C}$, the middle column in the above diagram is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact and we get an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$0 \rightarrow Q_{-1} \rightarrow T_{-2} \rightarrow G_{-3} \rightarrow G_{-4} \rightarrow \dots$$

in \mathcal{A} . We repeat the argument by replacing M with Q_{-1} to get an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$0 \rightarrow Q_{-1} \rightarrow C_{-2} \rightarrow Q_{-2} \rightarrow 0$$

in \mathcal{A} with $C_{-2} \in \mathcal{C}$. Continuing this process, we obtain an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$0 \rightarrow M \rightarrow C_{-1} \rightarrow C_{-2} \rightarrow \dots$$

in \mathcal{A} with all C_i in \mathcal{C} . It implies $M \in r\mathcal{G}(\mathcal{C})$. □

We write $r\mathcal{G}^2(\mathcal{C})$ to be a subcategory of \mathcal{A} consists of all $A \in \mathcal{A}$ such that there exists an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$\dots \rightarrow G_1 \rightarrow G_0 \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$$

in \mathcal{A} with all G_i, G^i in $r\mathcal{G}(\mathcal{C})$ and $A \cong \text{Im}(G_0 \rightarrow G^0)$. The next result is an immediate consequence of Theorem 3.7, which generalizes [18], Theorem 2.9.

Corollary 3.8. $r\mathcal{G}^2(\mathcal{C}) = r\mathcal{G}(\mathcal{C})$.

The next result shows that under some conditions, objects in $r\mathcal{G}(\mathcal{C})$ possess the symmetry just as objects in $\mathcal{G}(\mathcal{C})$ do.

Corollary 3.9. *If $r\mathcal{G}(\mathcal{C})$ has a projective generator \mathcal{P} , then the following statements are equivalent for any $G \in \mathcal{A}$.*

- (1) $G \in r\mathcal{G}(\mathcal{C})$.
- (2) *There exists an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence*

$$\dots \rightarrow P_1 \rightarrow P_0 \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$$

in \mathcal{A} with all P_i in \mathcal{P} and C^i in \mathcal{C} such that $G \cong \text{Im}(P_0 \rightarrow C^0)$.

- (3) *There exists an exact $\text{Hom}_R(-, \mathcal{C})$ -exact sequence*

$$\dots \rightarrow W_1 \rightarrow W_0 \rightarrow W^0 \rightarrow W^1 \rightarrow \dots$$

in \mathcal{A} with all $W_i, W^i \in \mathcal{C} \cup \mathcal{P}$ and $G \cong \text{Im}(W_0 \rightarrow W^0)$.

Proof. The implications (1) \Leftrightarrow (2) \Rightarrow (3) are clear. Since $\mathcal{C} \cup \mathcal{P} \subseteq r\mathcal{G}(\mathcal{C})$ by assumption, the implication (3) \Rightarrow (1) follows from Corollary 3.8. □

In the following, we will give some criteria for computing the $\mathcal{G}(\mathcal{C})$ -projective dimension of an object in \mathcal{A} . We need the following lemma.

Lemma 3.10. *If*

$$0 \rightarrow K \rightarrow G_1 \xrightarrow{f} G_0 \rightarrow A \rightarrow 0$$

is an exact sequence in \mathcal{A} with $G_0, G_1 \in r\mathcal{G}(\mathcal{C})$, then there exists an exact sequence

$$0 \rightarrow K \rightarrow C \rightarrow G \rightarrow A \rightarrow 0$$

in \mathcal{A} with $C \in \mathcal{C}$ and $G \in r\mathcal{G}(\mathcal{C})$.

Proof. Since $G_1 \in r\mathcal{G}(\mathcal{C})$, by Lemma 3.5 there exists an exact sequence

$$0 \rightarrow G_1 \rightarrow C \rightarrow G' \rightarrow 0$$

in \mathcal{A} with $C \in \mathcal{C}$ and $G' \in r\mathcal{G}(\mathcal{C})$. The push-out diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & K & \longrightarrow & G_1 & \longrightarrow & \text{Im } f \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \dashrightarrow & K & \dashrightarrow & C & \dashrightarrow & L \dashrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & G' & \equiv \equiv \equiv & G' \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

yields a push-out diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \text{Im } f & \longrightarrow & G_0 & \longrightarrow & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \dashrightarrow & L & \dashrightarrow & G & \dashrightarrow & A \dashrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & G' & \equiv \equiv \equiv & G' & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

By Proposition 3.3(1), the middle column in the last diagram implies that $G \in r\mathcal{G}(\mathcal{C})$. Splicing the middle rows of the two diagrams above, we get the desired exact sequence. \square

The following result provides some criteria for computing the $\mathcal{G}(\mathcal{C})$ -projective dimension of an object in \mathcal{A} , which shows that any object in \mathcal{A} with finite $r\mathcal{G}(\mathcal{C})$ -projective dimension is isomorphic to a kernel (a cokernel) of a morphism from an object in \mathcal{A} with finite \mathcal{C} -projective dimension to an object in $r\mathcal{G}(\mathcal{C})$.

Theorem 3.11. *For any $A \in \mathcal{A}$ and $n \geq 0$, the following statements are equivalent.*

- (1) $r\mathcal{G}(\mathcal{C})\text{-pd } A \leq n$.
- (2) *There exists an exact sequence*

$$0 \rightarrow H \rightarrow G \rightarrow A \rightarrow 0$$

in \mathcal{A} with $G \in r\mathcal{G}(\mathcal{C})$ and $\mathcal{C}\text{-pd } H \leq n - 1$.

- (3) *There exists an exact ($\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact) sequence*

$$0 \rightarrow A \rightarrow H' \rightarrow G' \rightarrow 0$$

in \mathcal{A} with $G' \in r\mathcal{G}(\mathcal{C})$ and $\mathcal{C}\text{-pd } H' \leq n$.

Proof. (1) \Rightarrow (2) We proceed by induction on n . If $n = 0$ then $H = 0$ and $G = A$ give the desired exact sequence. If $n = 1$ then there exists an exact sequence

$$0 \rightarrow G_1 \rightarrow G_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with $G_0, G_1 \in r\mathcal{G}(\mathcal{C})$. Applying Lemma 3.10 with $K = 0$, we get an exact sequence

$$0 \rightarrow C \rightarrow G'_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with $C \in \mathcal{C}$ and $G'_0 \in r\mathcal{G}(\mathcal{C})$. Now suppose $n \geq 2$. Then there exists an exact sequence

$$0 \rightarrow G_n \rightarrow G_{n-1} \rightarrow \dots \rightarrow G_0 \rightarrow A \rightarrow 0$$

in \mathcal{A} with all G_i in $r\mathcal{G}(\mathcal{C})$. Set $T := \text{Im}(G_1 \rightarrow G_0)$. By the induction hypothesis, we get the exact sequence

$$0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \dots \rightarrow C_2 \rightarrow G'_1 \rightarrow T \rightarrow 0$$

in \mathcal{A} with all C_i in \mathcal{C} and $G'_1 \in r\mathcal{G}(\mathcal{C})$. Set $B := \text{Im}(C_2 \rightarrow G'_1)$. By Lemma 3.10, we get an exact sequence

$$0 \rightarrow B \rightarrow C_1 \rightarrow G \rightarrow A \rightarrow 0$$

in \mathcal{A} with $C_1 \in \mathcal{C}$ and $G \in r\mathcal{G}(\mathcal{C})$. Thus we get the desired exact sequence

$$0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow C_{n-2} \rightarrow \dots \rightarrow C_1 \rightarrow G \rightarrow A \rightarrow 0.$$

Then we get the desired exact sequence by putting $H := \text{Coker}(C_2 \rightarrow C_1)$.

(2) \Rightarrow (3) Let

$$0 \rightarrow H \rightarrow G \rightarrow A \rightarrow 0$$

be an exact sequence in \mathcal{A} with $G \in r\mathcal{G}(\mathcal{C})$ and $\mathcal{C}\text{-pd } H \leq n - 1$. Since $G \in r\mathcal{G}(\mathcal{C})$, by Lemma 3.5 there exists an exact $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact sequence

$$0 \rightarrow G \rightarrow C \rightarrow G' \rightarrow 0$$

in \mathcal{A} with $C \in \mathcal{C}$ and $G' \in r\mathcal{G}(\mathcal{C})$. Consider the following push-out diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \vdots & \\
 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & A \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \vdots \\
 0 & \dashrightarrow & H & \dashrightarrow & C & \dashrightarrow & H' \dashrightarrow 0 \\
 & & \parallel & & \downarrow & & \vdots \\
 & & & & G' & = & G' \\
 & & & & \downarrow & & \vdots \\
 & & & & 0 & & 0.
 \end{array}$$

By the middle row in this diagram, we have $\mathcal{C}\text{-pd } H' \leq n$. Moreover, since $G' \in r\mathcal{G}(\mathcal{C})$, the rightmost column is $\text{Hom}_{\mathcal{A}}(-, \mathcal{C})$ -exact. Thus we get the desired exact sequence.

(3) \Rightarrow (1) Let

$$0 \rightarrow A \rightarrow H' \rightarrow G' \rightarrow 0$$

be an exact sequence in \mathcal{A} with $G' \in r\mathcal{G}(\mathcal{C})$ and $\mathcal{C}\text{-pd } H' \leq n$. Then there exists an exact sequence

$$0 \rightarrow C_n \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow H' \rightarrow 0$$

in \mathcal{A} with all C_i in \mathcal{C} . Set $K := \text{Ker}(C_0 \rightarrow H')$. Then $\mathcal{C}\text{-pd } K \leq n - 1$. Consider the pull-back diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & K & = & = & = & K \\
 & & \downarrow & & \downarrow & & \\
 0 & \dashrightarrow & G & \dashrightarrow & C_0 & \dashrightarrow & G' \dashrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & A & \longrightarrow & H' & \longrightarrow & G' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Applying Proposition 3.3(2) to the middle row in this diagram yields $G \in r\mathcal{G}(\mathcal{C})$. Thus $r\mathcal{G}(\mathcal{C})\text{-pd } A \leq n$ by the leftmost column in the above diagram. \square

4. WEAK AUSLANDER-BUCHWEITZ CONTEXTS

Following the approximation theory of Auslander-Buchweitz (see [3]), Hashimoto (see [15]) introduced the following definition.

Definition 4.1 ([15]). A triple $(\mathcal{X}, \mathcal{Y}, \omega)$ of subcategories of \mathcal{A} is called a *weak Auslander-Buchweitz context* in \mathcal{A} if the following conditions are satisfied.

- (1) \mathcal{X} is closed under extensions, kernels of epimorphisms and direct summands.
- (2) $\mathcal{Y} \subseteq \mathcal{X}\text{-pd}^{<\infty}$ and \mathcal{Y} is closed under extensions, cokernels of monomorphisms and direct summands.
- (3) $\omega = \mathcal{X} \cap \mathcal{Y}$ and ω is an injective cogenerator for \mathcal{X} .

A weak Auslander-Buchweitz context $(\mathcal{X}, \mathcal{Y}, \omega)$ is called an *Auslander-Buchweitz context* if $\mathcal{A} = \mathcal{X}\text{-pd}^{<\infty}$.

Definition 4.2 ([13]). Let \mathcal{U}, \mathcal{V} be subcategories of \mathcal{A} .

- (1) The pair $(\mathcal{U}, \mathcal{V})$ is called a *cotorsion pair* in \mathcal{A} if $\mathcal{U} = {}^{\perp_1}\mathcal{V}$ and $\mathcal{V} = \mathcal{U}^{\perp_1}$; in this case, $\mathcal{C} := \mathcal{U} \cap \mathcal{V}$ is called the *kernel* of $(\mathcal{U}, \mathcal{V})$.
- (2) A cotorsion pair $(\mathcal{U}, \mathcal{V})$ is said to *have enough injectives* if for any $A \in \mathcal{A}$ there exists an exact sequence

$$0 \rightarrow A \rightarrow V \rightarrow U \rightarrow 0$$

in \mathcal{A} with $V \in \mathcal{V}$ and $U \in \mathcal{U}$. Dually, a cotorsion pair $(\mathcal{U}, \mathcal{V})$ is said to *have enough projectives* if for any $A \in \mathcal{A}$ there exists an exact sequence

$$0 \rightarrow V' \rightarrow U' \rightarrow A \rightarrow 0$$

in \mathcal{A} with $V' \in \mathcal{V}$ and $U' \in \mathcal{U}$.

(3) A cotorsion pair $(\mathcal{U}, \mathcal{V})$ is called *hereditary* if one of the following equivalent conditions is satisfied.

(3.1) $\mathcal{U} \perp \mathcal{V}$.

(3.2) \mathcal{U} is resolving in the sense that \mathcal{U} contains all projectives in \mathcal{A} , \mathcal{U} is closed under extensions and kernels of epimorphisms.

(3.3) \mathcal{V} is coresolving in the sense that \mathcal{V} contains all injectives in \mathcal{A} , \mathcal{V} is closed under extensions and cokernels of monomorphisms.

In what follows, $(\mathcal{U}, \mathcal{V})$ is a hereditary cotorsion pair and $\mathcal{C} := \mathcal{U} \cap \mathcal{V}$ is its kernel. The following is a standard observation.

Lemma 4.3. *For any $A \in \mathcal{A}$ and $n \geq 0$, the following statements are equivalent.*

- (1) $\mathcal{U}\text{-pd } A \leq n$.
- (2) $\text{Ext}_{\mathcal{A}}^{n+1}(A, V) = 0$ for any $V \in \mathcal{V}$.
- (3) $\text{Ext}_{\mathcal{A}}^{\geq n+1}(A, V) = 0$ for any $V \in \mathcal{V}$.

Lemma 4.4. *If $(\mathcal{U}, \mathcal{V})$ has enough injectives, then \mathcal{C} is an injective cogenerator for \mathcal{U} .*

Proof. If $(\mathcal{U}, \mathcal{V})$ has enough injectives, then for any $U \in \mathcal{U}$ there exists an exact sequence

$$0 \rightarrow U \rightarrow V \rightarrow U^1 \rightarrow 0$$

in \mathcal{A} with $V \in \mathcal{V}$ and $U^1 \in \mathcal{U}$. Since \mathcal{U} is closed under extensions, we have $V \in \mathcal{U}$ and so $V \in \mathcal{C}$. Since $\mathcal{U} \perp \mathcal{C}$ clearly, \mathcal{C} is an injective cogenerator for \mathcal{U} . \square

The following result is used for obtaining our main result (Theorem 4.8).

Proposition 4.5. *If $(\mathcal{U}, \mathcal{V})$ has enough injectives, then we have*

- (1) $r\mathcal{G}(\mathcal{C}) \cap \mathcal{U}\text{-pd}^{<\infty} = \mathcal{U}$.
- (2) $\mathcal{C}\text{-pd}^{<\infty} = \mathcal{U}\text{-pd}^{<\infty} \cap \mathcal{V}$.
- (3) $\mathcal{C}\text{-pd}^{<\infty}$ is closed under extensions, cokernels of monomorphisms and direct summands.

Proof. (1) It is easy to see that $\mathcal{U} \subseteq r\mathcal{G}(\mathcal{C}) \cap \mathcal{U}\text{-pd}^{<\infty}$ by Lemma 4.4.

Conversely, let $A \in r\mathcal{G}(\mathcal{C}) \cap \mathcal{U}\text{-pd}^{<\infty}$. We will prove $A \in \mathcal{U}$ by induction on $n := \mathcal{U}\text{-pd} A$. The case for $n = 0$ is trivial. Now let $n \geq 1$ and let

$$0 \rightarrow L \rightarrow U_0 \rightarrow A \rightarrow 0$$

be an exact sequence in \mathcal{A} with $U_0 \in \mathcal{U} (\subseteq r\mathcal{G}(\mathcal{C}))$ and $\mathcal{U}\text{-pd} L \leq n - 1$. By Proposition 3.3 (2), we have $L \in r\mathcal{G}(\mathcal{C})$. Thus $L \in \mathcal{U}$ by the induction hypothesis. By Lemma 4.4, we have an exact sequence

$$0 \rightarrow L \rightarrow C \rightarrow U \rightarrow 0$$

in \mathcal{A} with $C \in \mathcal{C}$ and $U \in \mathcal{U}$. Consider the push-out diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & L & \longrightarrow & U_0 & \longrightarrow & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \dashrightarrow & C & \dashrightarrow & Q & \dashrightarrow & A \dashrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & U & = & U & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

By the middle column in this diagram, we have $Q \in \mathcal{U}$. Since $A \in r\mathcal{G}(\mathcal{C})$ and $C \in \mathcal{C}$, it follows that $\text{Ext}_{\mathcal{A}}^1(A, C) = 0$ and the middle row in the above diagram splits. Thus A is isomorphic to a direct summand of Q and $A \in \mathcal{U}$.

(2) Clearly, $\mathcal{C}\text{-pd}^{<\infty} \subseteq \mathcal{U}\text{-pd}^{<\infty}$. Since \mathcal{V} is closed under cokernels of monomorphisms, we have $\mathcal{C}\text{-pd}^{<\infty} \subseteq \mathcal{V}$. Thus $\mathcal{C}\text{-pd}^{<\infty} \subseteq \mathcal{U}\text{-pd}^{<\infty} \cap \mathcal{V}$.

Conversely, let $A \in \mathcal{U}\text{-pd}^{<\infty} \cap \mathcal{V}$ and $\mathcal{U}\text{-pd} A = n (< \infty)$. Since $\mathcal{U} \subseteq r\mathcal{G}(\mathcal{C})$, by Theorem 3.11 there exists an exact sequence

$$0 \rightarrow H \rightarrow G \rightarrow A \rightarrow 0$$

in \mathcal{A} with $G \in r\mathcal{G}(\mathcal{C})$ and $\mathcal{C}\text{-pd} H \leq n - 1$. By Lemma 4.3, we have $\mathcal{U}\text{-pd} G \leq n$. Then $G \in \mathcal{U}$ by (1). Since \mathcal{V} is closed under cokernels of monomorphisms and extensions, we have $H \in \mathcal{V}$, and hence $G \in \mathcal{V}$. Thus $G \in \mathcal{C} (= \mathcal{U} \cap \mathcal{V})$ and $\mathcal{C}\text{-pd} A \leq n$.

(3) By Lemma 4.3, $\mathcal{U}\text{-pd}^{<\infty}$ is closed under direct summands. Since \mathcal{V} is closed under direct summands, so is $\mathcal{C}\text{-pd}^{<\infty}$ by (2).

Since $\mathcal{C} \perp \mathcal{C}$, we have $\mathcal{C}\text{-pd}^{<\infty} \subseteq \mathcal{C}^\perp$ by the dimension shifting. Then any short exact sequence in \mathcal{A} with the first term in $\mathcal{C}\text{-pd}^{<\infty}$ is $\text{Hom}_{\mathcal{A}}(\mathcal{C}, -)$ -exact. Therefore by [17], Lemma 3.1(1) we get $\mathcal{C}\text{-pd}^{<\infty}$ is closed under extensions, and by [17], Theorem 3.6(1), we get $\mathcal{C}\text{-pd}^{<\infty}$ is closed under cokernels of monomorphisms. \square

As a consequence, we have the following proposition.

Proposition 4.6. *Assume that $(\mathcal{U}, \mathcal{V})$ has enough injectives. If $(r\mathcal{G}(\mathcal{C}), \mathcal{C}\text{-pd}^{<\infty}, \mathcal{C})$ is an Auslander-Buchweitz context, then*

$$(r\mathcal{G}(\mathcal{C}), \mathcal{U}\text{-pd}^{<\infty} \cap \mathcal{V}) = (r\mathcal{G}(\mathcal{C}), \mathcal{C}\text{-pd}^{<\infty})$$

is a cotorsion pair.

Proof. By Proposition 4.5(2), $\mathcal{U}\text{-pd}^{<\infty} \cap \mathcal{V} = \mathcal{C}\text{-pd}^{<\infty}$. It is easy to see that $r\mathcal{G}(\mathcal{C}) \perp \mathcal{C}\text{-pd}^{<\infty}$ by the dimension shifting. Thus $r\mathcal{G}(\mathcal{C}) \subseteq {}^{\perp_1}(\mathcal{C}\text{-pd}^{<\infty})$ and $\mathcal{C}\text{-pd}^{<\infty} \subseteq r\mathcal{G}(\mathcal{C})^{\perp_1}$. Since $(r\mathcal{G}(\mathcal{C}), \mathcal{C}\text{-pd}^{<\infty}, \mathcal{C})$ is an Auslander-Buchweitz context by assumption, we have $\mathcal{A} = r\mathcal{G}(\mathcal{C})\text{-pd}^{<\infty}$.

Let $A \in {}^{\perp_1}(\mathcal{C}\text{-pd}^{<\infty})$. Then $r\mathcal{G}(\mathcal{C})\text{-pd} A < \infty$. By Theorem 3.11, there exists an exact sequence

$$0 \rightarrow H \rightarrow G \rightarrow A \rightarrow 0$$

in \mathcal{A} with $G \in r\mathcal{G}(\mathcal{C})$ and $\mathcal{C}\text{-pd} H < \infty$. Thus this exact sequence splits, and hence A is isomorphic to a direct summand of G . Then by Proposition 3.3(1) we have $A \in r\mathcal{G}(\mathcal{C})$ and ${}^{\perp_1}(\mathcal{C}\text{-pd}^{<\infty}) \subseteq r\mathcal{G}(\mathcal{C})$.

Now let $A \in r\mathcal{G}(\mathcal{C})^{\perp_1}$. Note that $r\mathcal{G}(\mathcal{C})\text{-pd} A < \infty$. By Theorem 3.11, there exists an exact sequence

$$0 \rightarrow A \rightarrow H' \rightarrow G' \rightarrow 0$$

in \mathcal{A} with $G' \in r\mathcal{G}(\mathcal{C})$ and $\mathcal{C}\text{-pd} H' < \infty$. Thus this exact sequence splits, and hence A is isomorphic to a direct summand of H' . Then by Proposition 4.5(3) we have $\mathcal{C}\text{-pd} A < \infty$ and $r\mathcal{G}(\mathcal{C})^{\perp_1} \subseteq \mathcal{C}\text{-pd}^{<\infty}$. \square

We also need the following easy observation.

Lemma 4.7.

- (1) \mathcal{C} is an injective cogenerator for $r\mathcal{G}(\mathcal{C})$.
- (2) $r\mathcal{G}(\mathcal{C}) \cap \mathcal{C}\text{-pd}^{<\infty} = \mathcal{C}$.

Proof. The assertion (1) is trivial. Since $\mathcal{C} \perp \mathcal{C}$, we have ${}^{\perp}\mathcal{C} \cap \mathcal{C}\text{-pd}^{<\infty} = \mathcal{C}$ by the dimension shifting, so the assertion (2) follows. \square

Now we are in a position to give the following theorem.

Theorem 4.8. *If $(\mathcal{U}, \mathcal{V})$ has enough injectives, then $(r\mathcal{G}(\mathcal{C}), \mathcal{C}\text{-pd}^{<\infty}, \mathcal{C})$ is a weak Auslander-Buchweitz context.*

Proof. It follows from Propositions 3.3 and 4.5 (3), and Lemma 4.7. □

By Theorem 4.8 and [15], Theorem 1.12.10, we immediately get the following corollary.

Corollary 4.9. *If $(\mathcal{U}, \mathcal{V})$ has enough injectives, then:*

- (1) \mathcal{C} is a unique additive injective cogenerator for $r\mathcal{G}(\mathcal{C})$ in the sense that if \mathcal{E} is an injective cogenerator for $r\mathcal{G}(\mathcal{C})$, then $\text{add } \mathcal{E} = \mathcal{C}$, where $\text{add } \mathcal{E}$ is the subcategory of \mathcal{A} consisting of direct summands of finite direct sums of objects in \mathcal{E} .
- (2) For any $A \in r\mathcal{G}(\mathcal{C})\text{-pd}^{<\infty}$, the following statements are equivalent.
 - (i) $A \in r\mathcal{G}(\mathcal{C})$;
 - (ii) $A \in {}^\perp(\mathcal{C}\text{-pd}^{<\infty})$;
 - (iii) $A \in {}^{\perp_1}(\mathcal{C}\text{-pd}^{<\infty})$;
 - (iv) $A \in {}^\perp\mathcal{C}$.
- (3) For any $A \in r\mathcal{G}(\mathcal{C})\text{-pd}^{<\infty}$, the following statements are equivalent.
 - (i) $A \in \mathcal{C}\text{-pd}^{<\infty}$;
 - (ii) $A \in r\mathcal{G}(\mathcal{C})^\perp$;
 - (iii) $A \in r\mathcal{G}(\mathcal{C})^{\perp_1}$;
 - (iv) $r\mathcal{G}(\mathcal{C})\text{-id } A < \infty$ and $A \in \mathcal{C}^\perp$.
- (4) For any $A \in r\mathcal{G}(\mathcal{C})\text{-pd}^{<\infty}$, we have

$$\begin{aligned} r\mathcal{G}(\mathcal{C})\text{-pd } A &= \inf\{n: \text{Ext}_{\mathcal{A}}^{n+1}(A, C) = 0 \text{ for any } C \in \mathcal{C}\} \\ &= \inf\{n: \text{Ext}_{\mathcal{A}}^{n+1}(A, W) = 0 \text{ for any } W \in \mathcal{C}\text{-pd}^{<\infty}\}. \end{aligned}$$

- (5) For any $A \in \mathcal{C}\text{-pd}^{<\infty}$, we have $r\mathcal{G}(\mathcal{C})\text{-pd } A = \mathcal{C}\text{-pd } A$.
- (6) For an exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

in \mathcal{A} , if any two of L , M and N are in $r\mathcal{G}(\mathcal{C})\text{-pd}^{<\infty}$, then so is the third.

As a dual of Definition 4.1, we give the following definition.

Definition 4.10. A triple $(\mathcal{X}, \mathcal{Y}, \omega)$ of subcategories of \mathcal{A} is called a *weak co-Auslander-Buchweitz context* if the following conditions are satisfied.

- (1) \mathcal{X} is closed under extensions, cokernels of monomorphisms and direct summands.
- (2) $\mathcal{Y} \subseteq \mathcal{X}\text{-id}^{<\infty}$ and \mathcal{Y} is closed under kernels of epimorphisms, extensions and direct summands.
- (3) $\omega = \mathcal{X} \cap \mathcal{Y}$ and ω is a projective generator for \mathcal{X} .

A weak co-Auslander-Buchweitz context $(\mathcal{X}, \mathcal{Y}, \omega)$ is called a *co-Auslander-Buchweitz context* if $\mathcal{X}\text{-id}^{<\infty} = \mathcal{A}$.

By using arguments completely dual to those in the proofs of Proposition 4.6 and Theorem 4.8, we have the following proposition.

Proposition 4.11. *Assume that $(\mathcal{U}, \mathcal{V})$ has enough projectives. If $(l\mathcal{G}(\mathcal{C}), \mathcal{C}\text{-id}^{<\infty}, \mathcal{C})$ is a co-Auslander-Buchweitz context, then*

$$(l\mathcal{G}(\mathcal{C}), \mathcal{U} \cap \mathcal{V}\text{-id}^{<\infty}) = (r\mathcal{G}(\mathcal{C}), \mathcal{C}\text{-id}^{<\infty})$$

is a cotorsion pair.

Theorem 4.12. *If $(\mathcal{U}, \mathcal{V})$ has enough projectives, then $(l\mathcal{G}(\mathcal{C}), \mathcal{C}\text{-id}^{<\infty}, \mathcal{C})$ is a weak co-Auslander-Buchweitz context.*

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