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GAUSSIAN AND PRÜFER CONDITIONS IN BI-AMALGAMATED ALGEBRAS

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Abstract. Let $f\colon A\to B$ and $g\colon A\to C$ be two ring homomorphisms and let J and J' be ideals of B and C, respectively, such that $f^{-1}(J)=g^{-1}(J')$. In this paper, we investigate the transfer of the notions of Gaussian and Prüfer rings to the bi-amalgamation of A with (B,C) along (J,J') with respect to (f,g) (denoted by $A\bowtie^{f,g}(J,J')$), introduced and studied by S. Kabbaj, K. Louartiti and M. Tamekkante in 2013. Our results recover well known results on amalgamations in C. A. Finocchiaro (2014) and generate new original examples of rings possessing these properties.

Keywords: bi-amalgamation; amalgamated algebra; Gaussian ring; Prüfer ring

MSC 2010: 13F05, 13A15, 13E05, 13F20, 13C10, 13C11, 13F30, 13D05, 16D40, 16E10, 16E60

1. Introduction

All rings considered in this paper are assumed to be commutative and have identity element, and all modules are unitary.

In 1932, Prüfer introduced and studied in [24] integral domains in which every finitely generated ideal is invertible. In 1936, Krull [21] named these rings after H. Prüfer and stated equivalent conditions that make a domain Prüfer. Through the years, Prüfer domains acquired a great many equivalent characterizations, each of which was extended to rings with zero-divisors in different ways. In their paper devoted to Gaussian properties, Bazzoni and Glaz have proved that a Prüfer ring satisfies any of the other four Prüfer conditions if and only if its total ring of quotients satisfies that same condition, see [4], Theorems 3.3, 3.6, 3.7, 3.12. In 1970, Koehler studied associative rings for which every cyclic module is quasi-projective and she noticed that any commutative ring satisfies this property, see [20]. Recall that for a commutative ring R, an R-module V is *quasi-projective* if the map

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$\mathrm{Hom}_R(V,V) \to \mathrm{Hom}_R(V,V/N)$ is surjective for every submodule N of V (see [1]). In [2], the authors examined the transfer of the Prüfer conditions and obtained further evidence for the validity of Bazzoni-Glaz conjecture sustaining that "the weak global dimension of a Gaussian ring is 0, 1, or ∞", see [4]. Notice that both conjectures share the common context of rings. Abuihlail, Jarrar and Kabbaj studied in [1] the multiplicative ideal structure of commutative rings in which every finitely generated ideal is quasi-projective. Furthermore, they provided some preliminaries for quasi-projective modules over commutative rings and investigated the correlation with well-known Prüfer conditions; namely, they proved that this class of rings stands strictly between the two classes of arithmetical rings and Gaussian rings. Thereby, they generalized Osofsky's theorem on the weak global dimension of arithmetical rings and partially resolved Bazzoni-Glaz's related conjecture on Gaussian rings. They also established an analogue of Bazzoni-Glaz results on the transfer of Prüfer conditions between a ring and its total ring of quotients. In [8], the authors studied the transfer of the notions of local Prüfer ring and total ring of quotients. They examined the arithmetical, Gaussian, fqp conditions to amalgamated duplication along an ideal. At this point, we recall the following definitions:

Definition 1.1. Let R be a commutative ring.

(1) R is called an *arithmetical ring* if the lattice formed by its ideals is distributive, see [14].
(2) R is called a *Gaussian ring* if for every $f, g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$, see [25].
(3) R is called a *Prüfer ring* if every finitely generated regular ideal of R is invertible (equivalently, every two-generated regular ideal is invertible), see [6], [17].

In the domain context, all these forms coincide with the definition of a Prüfer domain. Glaz in [15] provides examples which show that all these notions are distinct in the context of arbitrary rings. The following diagram of implications summarizes the relations between them, see [3], [4], [16], [15], [22], [23], [25]:

$$\text{arithmetical} \Rightarrow \text{Gaussian} \Rightarrow \text{Prüfer}$$

and examples are given in [15] to show that, in general, the implications cannot be reversed.

In this paper, we investigate the transfer of Gaussian and Prüfer properties in bi-amalgamation of rings, introduced and studied by Kabbaj, Louartiti and Tamekkante in [18] and defined as follows: Let $f\colon A \to B$ and $g\colon A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that

$I_0 := f^{-1}(J) = g^{-1}(J')$. The *bi-amalgamation* (or *bi-amalgamated algebra*) *of* A *with* (B, C) *along* (J, J') *with respect to* (f, g) is the subring of $B \times C$ given by

$$A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j')\colon\ a \in A, (j, j') \in J \times J'\}.$$

This construction was introduced in [18] as a natural generalization of duplications (see [9], [12]) and amalgamations (see [10], [11]). Note that some of the results of the present paper (as pointed out later) overlap very recent results obtained independently by Campanini-Finocchiaro in [7].

In [18], the authors provide original examples of bi-amalgamations and, in particular, show that Boisen-Sheldon's CPI-extensions (see [5]) can be viewed as bi-amalgamations (notice that [10], Example 2.7 shows that CPI-extensions can be viewed as quotient rings of amalgamated algebras). They also show how every bi-amalgamation can arise as a natural pullback (or even as a conductor square) and then characterize pullbacks that can arise as bi-amalgamations. Then, the last two sections of [18] deal, respectively, with the transfer of some basic ring theoretic properties to bi-amalgamations and the study of their prime ideal structures. All their results recover known results on duplications and amalgamations. Recently in [19], the authors established necessary and sufficient conditions for a bi-amalgamation to inherit the arithmetical property, with applications to the weak global dimension and transfer of the semihereditary property. Throughout, we will adopt the following notations:

For a ring A, $\mathrm{Spec}(A)$ and $\mathrm{Max}(A)$ will denote, respectively, the sets of all prime and maximal ideals of A, and for any ideal I of A, $\mathrm{Spec}(A, I)$ and $\mathrm{Max}(A, I)$ will denote, respectively, the sets of all prime and maximal ideals of A containing I. For any $p \in \mathrm{Spec}(A, I_0)$ or $p \in \mathrm{Max}(A, I_0)$, consider the multiplicative subsets

$$S_p := f(A - p) + J \quad \text{and} \quad S'_p := g(A - p) + J'$$

of B and C, respectively, and let

$$f_p\colon\ A_p \to B_{S_p} \quad \text{and} \quad g_p\colon\ A_p \to C_{S'_p}$$

be the canonical ring homomorphisms induced by f and g. One can easily check that

$$f_p^{-1}(J_{S_p}) = g_p^{-1}(J'_{S'_p}) = (I_0)_p.$$

Moreover, by [18], Lemma 5.1, $P := p \bowtie^{f,g} (J, J')$ is a prime (or maximal) ideal of $A \bowtie^{f,g} (J, J')$ and, by [18], Proposition 5.7, we have

$$(A \bowtie^{f,g} (J, J'))_P \cong A_p \bowtie^{f_p, g_p} (J_{S_p}, J'_{S'_p}).$$

For a ring R, we denote by $\mathrm{Jac}(R)$, the Jacobson radical of R.

2. Results

Let $f\colon A \to B$ and $g\colon A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C, respectively, such that $I_0 := f^{-1}(J) = g^{-1}(J')$. All along this section, $A \bowtie^{f,g} (J, J')$ will denote the bi-amalgamation of A with (B, C) along (J, J') with respect to (f, g). Our first result investigates the transfer of Gaussian and Prüfer properties in bi-amalgamated algebras, in case $J \times J'$ contains a regular element. Observe that this happens only in the trivial case for which the bi-amalgamation degenerates in the direct product $B \times C$.

Theorem 2.1. *Assume $J \times J'$ is a regular ideal of $(f(A)+J) \times (g(A)+J')$. Then $A \bowtie^{f,g} (J, J')$ is Gaussian or Prüfer if and only if $J = B$, $J' = C$ and B and C are Gaussian or Prüfer, respectively.*

Proof. Assume that $A \bowtie^{f,g} (J, J')$ is Gaussian (Prüfer). We claim that $I_0 = f^{-1}(J) = g^{-1}(J') = A$. Deny, suppose that there exists a maximal ideal m of A such that $I_0 \subseteq m$. From [18], Lemma 5.1, $M := m \bowtie^{f,g} (J, J')$ is a maximal ideal of $A \bowtie^{f,g} (J, J')$ and we have

$$(A \bowtie^{f,g} (J, J'))_M \cong A_m \bowtie^{f_m, g_m} (J_{S_m}, J'_{S'_m}) =: D.$$

Let (j, j') be a regular element of $J \times J'$. It is easy to see that $j/1$ and $j'/1$ are also regular elements of B_{S_m} or C_{S_m}, respectively. Using the fact $A \bowtie^{f,g} (J, J')$ is Gaussian (Prüfer), then by [17], Theorem 13, the principal ideals $(j/1, 0)D$ and $(j/1, j'/1)D$ are comparable. Since $0 \neq j'/1$, then necessarily $(j/1, 0)D \subseteq (j/1, j'/1)D$. Thus, there exist $\alpha \in A_m$, $\beta \in J_{S_m}$ and $\gamma \in J'_{S_m}$ such that $(j/1, 0) = (j/1, j'/1)(f_m(\alpha)+\beta, g_m(\alpha) + \gamma)$. Hence, it follows that $f_m(\alpha) + \beta = 1$ and $g_m(\alpha) + \gamma = 0$. Thus, $\alpha \in (I_0)_m$ and so $f_m(\alpha) \in J_{S_m}$ and $1 = f_m(\alpha) + \beta \in J_{S_m}$. Therefore, $J_{S_m} = B_{S_m}$. Then $(I_0)_m = A_m$, which is a contradiction since $I_0 \subseteq m$. Hence, $I_0 = f^{-1}(J) = A$ and so $J = B$ and $J' = C$ and $A \bowtie^{f,g} (J, J') = B \times C$ which is Gaussian (Prüfer). It is known that Gaussian (Prüfer) notion is stable under finite products. It follows that B and C are Gaussian (Prüfer). The converse is straightforward. $\square$

Remark 2.2. It is worth mentioning that very recently, a result similar to Theorem 2.1 was independently obtained by Campanini and Finocchiaro in [7], Proposition 4.10, using different notation.

Recall that the amalgamation of A with B along an ideal J of B with respect to the ring homomorphism $f\colon A \to B$ is given by

$$A \bowtie^f J := \{(a, f(a) + j)\colon\ a \in A,\ j \in J\}.$$

Clearly, every amalgamation can be viewed as a special bi-amalgamation, since $A \bowtie^f J = A \bowtie^{\mathrm{id}_A, f} (f^{-1}(J), J)$.

The following result is an immediate consequence of Theorem 2.1 and recovers [13], Theorem 3.1.

Corollary 2.3. *Under the above notation, assume that $f^{-1}(J) \times J$ is a regular ideal of $A \times f(A) + J$. Then $A \bowtie^f J$ is Gaussian (Prüfer) if and only if $f^{-1}(J) = A$ and $J = B$ and both A and B are Gaussian (Prüfer).*

Let I be a *proper* ideal of A. The (amalgamated) duplication of A along I is a special amalgamation given by

$$A \bowtie I := A \bowtie^{\mathrm{id}_A} I = \{(a, a+i)\colon\ a \in A,\ i \in I\}.$$

The next corollary is an immediate consequence of Corollary 2.3 on the transfer of Gaussian and Prüfer properties to duplications and capitalizes, see [13], Corollary 3.3.

Corollary 2.4. *Let A be a ring and I a regular ideal of A. Then $A \bowtie I$ is Gaussian (Prüfer) if and only if A is Gaussian (Prüfer) and $I = A$.*

The next result investigates when the bi-amalgamation is local Gaussian in case $J \times J'$ is not a regular ideal. We recall an important characterization of a local Gaussian ring A. Namely, for any two elements a and b in the ring A, we have $(a,b)^2 = (a^2)$ or (b^2); moreover if $ab = 0$ and $(a,b)^2 = (a^2)$, then $b^2 = 0$ (see [4], Theorem 2.2).

Proposition 2.5. *Assume that (A, m) is a local ring and J and J' are nonzero proper ideal of B and C, respectively, such that $J \times J' \subseteq \mathrm{Jac}(B \times C)$. Then the following statements hold:*

1. *If $A \bowtie^{f,g} (J, J')$ is Gaussian, then so are $f(A) + J$ and $g(A) + J'$.*
2. *If A, $f(A) + J$ and $g(A) + J'$ are Gaussian, $J^2 = 0$, $J'^2 = 0$ for all $a \in m$, $f(a)J = f(a)^2 J$ and $g(a)J' = g(a)^2 J'$, then $A \bowtie^{f,g} (J, J')$ is Gaussian.*
3. *Assume that A is Gaussian, $J^2 = 0$, $J'^2 = 0$ and I_0 is a prime ideal of A. Then $A \bowtie^{f,g} (J, J')$ is Gaussian if and only if $f(A) + J$, $g(A) + J'$ are Gaussian for all $a \in m$, $f(a)J = f(a)^2 J$ and $g(a)J' = g(a)^2 J'$.*

Proof. Notice that from [18], Proposition 5.4 (2), $(A \bowtie^{f,g} (J, J'), m \bowtie^{f,g} (J, J'))$ is local since (A, m) is local and $J \times J' \subseteq \mathrm{Jac}(B \times C)$.

(1) Since the Gaussian property is stable under factor rings (here, $f(A) + J \simeq A \bowtie^{f,g} (J, J')/(0 \times J')$ and $g(A) + J' \simeq A \bowtie^{f,g} (J, J')/(J \times 0)$ by [18], Proposition 4.1 (2)), the result is straightforward.

(2) Assume that A, $f(A)+J$ and $g(A)+J'$ are Gaussian, $J^2=0$, $J'^2=0$ and for all $a \in m$, $f(a)J = f(a)^2 J$ and $g(a)J' = g(a)^2 J'$. Our aim is to show that $A \bowtie^{f,g} (J, J')$ is Gaussian. Let $(f(a)+i, g(a)+i')$ and $(f(b)+j, g(b)+j') \in A \bowtie^{f,g} (J, J')$. Two cases are possible:

Case 1: a or $b \notin m$. Assume without loss of generality that $a \notin m$. Then $(f(a)+i, g(a)+i') \notin m \bowtie^{f,g} (J, J')$. So $(f(a)+i, g(a)+i')$ is invertible in $A \bowtie^{f,g} (J, J')$. Therefore, $((f(a)+i, g(a)+i'), (f(b)+j, g(b)+j'))^2 = ((f(a)+i, g(a)+i')^2) = A \bowtie^{f,g} (J, J')$. Moreover, if $((f(a)+i, g(a)+i'), (f(b)+j, g(b)+j'))^2 = ((f(a)+i, g(a)+i')^2) = A \bowtie^{f,g} (J, J')$ and $(f(a)+i, g(a)+i')(f(b)+j, g(b)+j') = (0,0)$, then it follows that $(f(b)+j, g(b)+j') = (0,0)$, making $(f(b)+j, g(b)+j')^2 = (0,0)$, as desired.

Case 2: a and $b \in m$. Using the fact that A is local Gaussian, we have $(a,b)^2 = (a^2)$ or (b^2). We may assume that $(a,b)^2 = (a^2)$. So we have $b^2 = a^2 x$ and $ab = a^2 y$ for some $x, y \in A$. Moreover, $ab = 0$ implies that $b^2 = 0$. So $f(b)^2 = f(a)^2 f(x)$, $g(b)^2 = g(a)^2 g(x)$ and $f(a)f(b) = f(a)^2 f(y)$, $g(a)g(b) = g(a)^2 g(y)$. By assumption, $2f(b)j, f(b)i \in f(b)^2 J$ and $2f(a)if(x), f(a)j, 2f(a)if(y) \in f(a)^2 J$. Therefore, there exist $j_1, i_1, j_2, i_2, i_3 \in J$ such that $2f(b)j = f(a)^2 f(x) j_1$, $2f(a)if(x) = f(a)^2 i_1$, $f(a)j = f(a)^2 j_2$, $f(b)i = f(a)^2 f(x) i_2$, $2f(a)if(y) = f(a)^2 i_3$ and similarly, there exist $j'_1, i'_1, j'_2, i'_2, i'_3 \in J'$ such that $2g(b)j' = g(a)^2 g(x) j'_1$, $2g(a)i'g(x) = g(a)^2 i'_1$, $g(a)j' = g(a)^2 j'_2$, $g(b)i' = g(a)^2 g(x) i'_2$ and $2g(a)i'g(y) = g(a)^2 i'_3$. In view of the fact that $J^2 = 0$ and $J'^2 = 0$, one can easily check that $(f(b)+j, g(b)+j')^2 = (f(a)+i, g(a)+i')^2 (f(x)+f(x)j_1-i_1, g(x)+g(x)j'_1-i'_1)$ and $(f(b)+j, g(b)+j')(f(a)+i, g(a)+i') = (f(a)+i, g(a)+i')^2 (f(y)+f(x)i_2+j_2-i_3, g(y)-g(x)i'_2+j'_2-i'_3)$. Consequently, $((f(a)+i, g(a)+i'), (f(b)+j, g(b)+j'))^2 = ((f(a)+i, g(a)+i')^2)$. Moreover, assume that $(f(a)+i, g(a)+i')(f(b)+j, g(b)+j') = (0,0)$. Hence, $(f(a)+i)(f(b)+j) = 0$ and $(g(a)+i')(g(b)+j') = 0$. Since $((f(a)+i), (f(b)+j))^2 = ((f(a)+i)^2)$, $((g(a)+i'), (g(b)+j'))^2 = ((g(a)+i')^2)$, and $f(A)+J$ and $g(A)+J'$ are local Gaussian, we have $(f(b)+j)^2 = 0$ and $(g(b)+j')^2 = 0$. Thus, $(f(b)+j, g(b)+j')^2 = (0,0)$. Finally, $A \bowtie^{f,g} (J, J')$ is Gaussian, as desired.

(3) If A, $f(A)+J$, $g(A)+J'$ are Gaussian, $J^2 = 0$ for all $a \in m$, $f(a)J = f(a)^2 J$, $J'^2 = 0$ and $g(a)J' = g(a)^2 J'$, then by statement (2) above, $A \bowtie^{f,g} (J, J')$ is Gaussian. Conversely, assume that $A \bowtie^{f,g} (J, J')$ is Gaussian. Then by statement (1) above, $f(A)+J$ and $g(A)+J'$ are Gaussian. Next, we show that for all $a \in m$, $f(a)J = f(a)^2 J$. It is clear that $f(a)^2 J \subseteq f(a)J$. On the other hand, let $a \in m$ and $0 \neq x \in J$. If $f(a) = 0$, then $f(a)J = f(a)^2 J$, as desired. We may assume that $f(a) \neq 0$. Then obviously, $(0,0) \neq (f(a), g(a))$ and $(0,0) \neq (x, 0)$ are elements of $A \bowtie^{f,g} (J, J')$. Using the fact $A \bowtie^{f,g} (J, J')$ is (local) Gaussian yields that $((f(a), g(a)), (x,0))^2 = ((f(a), g(a))^2$ or $((x,0))^2$. Since $J^2 = 0$, say $((f(a), g(a)), (x,0))^2 = ((f(a), g(a))^2)$. If $(f(a), g(a))^2 = (0,0)$, it fol-

lows that $xf(a) = 0$ and so $f(a)J \subseteq f(a)^2J$, as desired. We may assume that $(f(a), g(a))^2 \neq (0, 0)$. And so there exists $(f(b) + j, g(b) + j') \in A \bowtie^{f,g} (J, J')$ such that $(xf(a), 0) = (f(a^2), g(a^2))(f(b) + j, g(b) + j')$. Therefore,

$$\begin{cases} xf(a) = (f(a^2)(f(b) + j), & \text{see (i)} \\ 0 = (g(a^2b) + g(a^2)j'), & \text{see (ii).} \end{cases}$$

From equation (ii) it follows that $a^2b \in I_0$ which is a prime ideal of A. So $a^2 \in I_0$ or $b \in I_0$. Two cases are possible:

Case 1: $a^2 \in I_0$. Then $a \in I_0$ and $f(a) \in J$. Therefore, $f(a)J = f(a)^2J = 0$ (as $J^2 = 0$).

Case 2: $b \in I_0$. Then $f(b) \in J$ and $f(b) + j \in J$. Consequently, $xf(a) = f(a^2)(f(b) + j) \in f(a)^2J$. Hence, $f(a)J \subseteq f(a)^2J$, as desired. Next, it remains to show that for all $a \in m$, $g(a)J' = g(a)^2J'$. Clearly, $g(a)^2J' \subseteq g(a)J'$. On the other hand, let $a \in m$ and $0 \neq x' \in J'$. By a similar argument as previously, it follows that $g(a)J' \subseteq g(a)^2J'$, as desired. □

Remark 2.6. It is worth mentioning that there is some overlapping between the assertion (2) of Proposition 2.5 and [7], Theorem 5.4, although the assumptions and proofs are different. Indeed, in the proof of [7], Theorem 5.4, the authors use the fact that if A is Gaussian, then B and C are Gaussian, which requires further assumptions on f and g (f and g are surjective). Our argument to prove the assertion (2) of Proposition 2.5 is different. Indeed, the use of $f(A)+J$ and $g(A)+J'$ in our statement seems more natural in our context (without any assumption on f and g), due to the assertion (1) of Proposition 2.5. Moreover, the assertion (2) of Proposition 2.5 allows us to construct new examples that are different from those issued from [7], Theorem 5.4, which require both f and g to be surjective. Furthermore, it is also interesting to notice that the assertion (3) of Proposition 2.5 which is not stated in [7], gives a characterization of Gaussian rings issued from the bi-amalgamation in case A is local, $J^2 = 0$, $J'^2 = 0$ and I_0 is a prime ideal of A.

Proposition 2.5 enriches the literature with new original examples of non-arithmetical Gaussian rings. Recall that for a ring A and an A-module E, the *trivial ring extension of A by E* (also called *idealization of E over A*) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + ea')$.

Example 2.7. Let $(A, m) := (A_1 \propto E_1, m_1 \propto E_1)$ be the trivial ring extension of A_1 by E_1, where A_1 is supposed to be a non-arithmetical Gaussian ring with $m_1^2 = 0$, (for instance $(A_1, m_1) := (\mathbb{Z}/4\mathbb{Z}, 2.\mathbb{Z}/4\mathbb{Z})$) and E_1 is a nonzero A_1/m_1-vector space (for instance $E_1 = (\mathbb{Z}/4\mathbb{Z})/(2.\mathbb{Z}/4\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$). By [19], Theorem 2.1 (2) and (3), A is a non-arithmetical Gaussian ring, as A_1 is not a field. Let $B := A \propto E$ be the trivial ring extension of A by a nonzero A/m-vector space E. Consider

$$f\colon\ A \hookrightarrow B,$$
$$(a_1, e_1) \hookrightarrow f((a_1, e_1)) = ((a_1, e_1), 0);$$

note that f is an injective ring homomorphism and $J := m \propto E = (m_1 \propto E_1) \propto E$ is the maximal ideal of B. Let $C := A_1$ and let

$$g\colon\ A \to C,$$
$$(a_1, e_1) \to g((a_1, e_1)) = a_1;$$

observe that g is a surjective ring homomorphism and $J' := m_1$ is the maximal ideal of C. Clearly, $f^{-1}(J) = g^{-1}(J') = m_1 \propto E_1$. Then:
(1) $A \bowtie^{f,g} (J, J')$ is Gaussian;
(2) $A \bowtie^{f,g} (J, J')$ is not arithmetical.

Proof. (1) One can verify that $J^2 = 0$, $J'^2 = 0$, $f(a)J = f(a)^2 J = 0$, $g(a)J' = g(a)^2 J' = 0$ for all $a \in m$. Hence by using statement (2) of Proposition 2.5, it follows that $A \bowtie^{f,g} (J, J')$ is Gaussian.

(2) By [19], Theorem 2.1 (2), $A \bowtie^{f,g} (J, J')$ is not arithmetical since $f(A) + J = A \propto 0 + m \propto E = A \propto E$ which is not arithmetical (by [2], Theorem 3.1 (3), as A is not a field). $\square$

Recall that a ring R is said to be *a total quotient ring* if every element of R is invertible or a zero-divisor. Total quotient rings are an important source of Prüfer rings. Now, we study the transfer of this notion to bi-amalgamated algebras, in case $J \times J'$ is not a regular ideal of $(f(A) + J) \times (g(A) + J')$. For any ring R and J an ideal of R, we denote by $Z(R)$ and $\mathrm{Ann}(J)$ the set of zero-divisor elements of R and the annihilator of J, respectively.

Proposition 2.8. *Let (A, m) be a local total ring of quotients, let $f\colon A \to B$, $g\colon A \to C$ be two ring homomorphisms, and let J and J' be nonzero proper ideals of B and C, respectively, such that $f^{-1}(J) = g^{-1}(J')$, $J \times J' \subseteq \mathrm{Jac}(B \times C)$. Assume that f is injective, $J^2 = 0$ and $J'^2 = 0$. Then $A \bowtie^{f,g} (J, J')$ is a local total ring of quotients.*

Proof. Assume that f is injective, $J^2 = 0$ and $J'^2 = 0$. By [18], Proposition 5.4 (2), $(A \bowtie^{f,g} (J, J'), m \bowtie^{f,g} (J, J'))$ is local since (A, m) is local and $J \times J' \subseteq \mathrm{Jac}(B \times C)$.

Our aim is to show that $A \bowtie^{f,g} (J, J')$ is a total ring of quotients; we have to prove that each element $(f(a)+i, g(a)+i')$ of $A \bowtie^{f,g} (J, J')$, is invertible or a zero-divisor element.

Let $(f(a)+i, g(a)+i')$ be an element of $A \bowtie^{f,g} (J, J')$. If $a \notin m$, then a is invertible. And so $(f(a)+i, g(a)+i') \notin m \bowtie^{f,g} (J, J')$. Consequently, $(f(a)+i, g(a)+i')$ is invertible in $A \bowtie^{f,g} (J, J')$, as desired.

Now, we may assume that $a \in m$. If $a = 0$, then $(f(a)+i, g(a)+i') = (i, i') \in Z(A \bowtie^{f,g} (J, J'))$, since $J^2 = J'^2 = 0$. We may assume $a \neq 0$. Since A is local total ring of quotients, there exists $0 \neq b \in A$ such that $ab = 0$. So $f(a)f(b) = 0$ and $g(a)g(b) = 0$. Two cases are then possible:

Case 1: $f(b) \in \mathrm{Ann}(J)$ and $g(b) \in \mathrm{Ann}(J')$. Using the fact that f is injective, there exists $(0,0) \neq (f(b), g(b)) \in A \bowtie^{f,g} (J, J')$ such that $(f(a)+i, g(a)+i')(f(b), g(b)) = (0,0)$. Consequently, $(f(a)+i, g(a)+i') \in Z(A \bowtie^{f,g} (J, J'))$.

Case 2: Assume that $f(b) \notin \mathrm{Ann}(J)$ or $g(b) \notin \mathrm{Ann}(J')$. Then there exists $0 \neq k \in J$ or $0 \neq k' \in J'$ such that $f(b)k \neq 0$ or $g(b)k' \neq 0$. So, $(f(a)+i, g(a)+i')(f(b)k, 0) = (0,0)$ or $(f(a)+i, g(a)+i')(0, g(b)k') = (0,0)$. Hence, $(f(a)+i, g(a)+i') \in Z(A \bowtie^{f,g} (J, J'))$. Thus, $A \bowtie^{f,g} (J, J')$ is a local total ring of quotients. □

Remark 2.9. It is worth noting that total rings of quotients of bi-amalgamation were also studied in [7], Proposition 4.12. However, the proofs in [7], Proposition 4.12 and Proposition 2.8 are different, because of different hypotheses. Observe that in Proposition 2.8, our approach concerns the local case of bi-amalgamation where f is injective, $J^2 = 0$ and $J'^2 = 0$ and our argument in the proof is mainly based on the use of $\mathrm{Ann}(J)$ and $\mathrm{Ann}(J')$, which diverges with the proof [7], Proposition 4.12, where the authors use the argument that the ideals J and J' are torsion $A/(\ker(f) \cap \ker(g))$-modules by their assumptions.

Proposition 2.8 enriches the current literature with new original examples of Prüfer rings which are not Gaussian rings.

Example 2.10. Let (A, m) be a non Gaussian local total ring of quotients (for instance take $(A, m) := (A_1 \propto A_1/m_1, m_1 \propto A_1/m_1)$ with (A_1, m_1) a local ring that is not Gaussian, by using [2], Theorem 3.1 (1) and (2)). Let $(B, N) := (A \propto E, m \propto E)$ be the trivial ring extension of A by a nonzero (A/m)-vector space E and let $C := B \propto E'$ be the trivial ring extension of B by a nonzero (B/N)-vector space E'. Consider

$$f\colon\ A \hookrightarrow B,$$
$$(a_1, e_1) \hookrightarrow f((a_1, e_1)) = ((a_1, e_1), 0);$$

note that f is an injective ring homomorphism and $J := 0 \propto E$ is a nonzero proper ideal of B and let

$$g\colon A \hookrightarrow C,$$
$$(a_1, e_1) \hookrightarrow g((a_1, e_1)) = (((a_1, e_1), 0), 0);$$

observe that g is an injective ring homomorphism and $J' := J \propto E'$ is a proper ideal of C. Obviously, $f^{-1}(J) = g^{-1}(J') = 0$. Then:
(1) $A \bowtie^{f,g} (J, J')$ is Prüfer.
(2) $A \bowtie^{f,g} (J, J')$ is not Gaussian.

Proof. (1) We claim that $A \bowtie^{f,g} (J, J')$ is a local total ring of quotients. Indeed, by [18], Proposition 5.3, $A \bowtie^{f,g} (J, J')$ is local since A is local and $J \times J' \subseteq \mathrm{Jac}(B \times C)$. One can easily check that $J^2 = 0$, $J'^2 = 0$. Hence, by using Proposition 2.8, it follows that $A \bowtie^{f,g} (J, J')$ is a total ring of quotients. Hence, $A \bowtie^{f,g} (J, J')$ is Prüfer.

(2) By (1) of Proposition 2.5, $A \bowtie^{f,g} (J, J')$ is not Gaussian since $f(A) + J = (A \propto 0) + (0 \propto E) = (A + 0) \propto (0 + E) = A \propto E$ is not Gaussian (by [2], Theorem 3.1 (2), since A is not Gaussian, as A_1 is not Gaussian). $\square$

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