Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 70 (2020), No. 2, 339–368

Persistent URL: `http://dml.cz/dmlcz/148233`

Czechoslovak Mathematical Journal, 70 (145) (2020), 339–368

A UNIFIED ANALYSIS OF ELLIPTIC PROBLEMS WITH VARIOUS BOUNDARY CONDITIONS AND THEIR APPROXIMATION

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Received June 26, 2018. Published online November 4, 2019.

Abstract. We design an abstract setting for the approximation in Banach spaces of operators acting in duality. A typical example are the gradient and divergence operators in Lebesgue-Sobolev spaces on a bounded domain. We apply this abstract setting to the numerical approximation of Leray-Lions type problems, which include in particular linear diffusion. The main interest of the abstract setting is to provide a unified convergence analysis that simultaneously covers (i) all usual boundary conditions, (ii) several approximation methods. The considered approximations can be conforming (that is, the approximation functions can belong to the energy space relative to the problem) or not, and include classical as well as recent numerical schemes. Convergence results and error estimates are given. We finally briefly show how the abstract setting can also be applied to some models such as flows in fractured medium, elasticity equations and diffusion equations on manifolds.

Keywords: elliptic problem; various boundary conditions; gradient discretisation method; Leray-Lions problem

MSC 2010: 65J05, 65N99, 47A58

1. Introduction

We are interested in the approximation of linear and nonlinear elliptic problems with various boundary conditions.

Numerical schemes for the approximation of nonlinear diffusion problems of Leray-Lions type on standard meshes have already been proposed and studied in the past. Finite elements were proposed for the particular case of the p-Laplace problem (see [6], [7], [27], [28]) as well as for quasi-linear problems and non-Newtonian models in glaciology (see [8], [23]). More recently, non conforming numerical schemes defined on polytopal meshes were introduced; discrete duality finite volume schemes were studied in [1], [2], [3], [4]. Other schemes which have been shown to be part of

DOI: 10.21136/CMJ.2019.0312-18

the gradient discretisation method reviewed in the recent book (see [19]), were also studied for the Leray-Lions type problems, namely the SUSHI scheme in [20], the mixed finite volume scheme in [18], the mimetic finite difference method in [5]; the discontinuous Galerkin approximation was considered in [14], [21] and the hybrid high order scheme in [17]. In all these works, usually only one type of boundary conditions is considered (most often homogeneous Dirichlet boundary conditions). These schemes have been shown to be part of the GDM framework in [19], Part III; the convergence analysis of [19], Parts I and II holds for each of them. However, the analysis performed therein is done for each type of boundary conditions (Dirichlet, Neumann, Fourier). Our aim here is to provide a unified formulation of the continuous and discrete problems that covers all boundary conditions; this formulation is based on some abstract Banach spaces in which both the continuous and approximate problems are posed.

The present paper is organised as follows. The next section is devoted to an illustrative example, which shows how to build the abstract spaces and operators in order to express a variety of problems with a variety of boundary conditions. In Section 3, we provide the detailed framework concerning the function spaces, and the core properties of the Gradient Discretisation Method. In Section 4, we apply this framework to the approximation of an abstract Leray-Lions problem, and we prove the convergence of the approximation methods. Then we turn in Section 5 to the approximation of a linear elliptic problem, deduced from the abstract Leray-Lions problem, with $p = 2$. Note that in this case the framework becomes Hilbertian. Finally, in Section 6, we briefly review a series of applications of the unified discretisation setting.

2. An illustrative example

In this section, we take $p \in (1, \infty)$ and define $p' \in (1, \infty)$ by $1/p + 1/p' = 1$, and consider an archetypal example of elliptic problems, that is the anisotropic p-Laplace problem, which reads:

$$-\operatorname{div}(\Lambda|\nabla \bar{u}|_{\Lambda}^{p-2}\nabla \bar{u}) = r + \operatorname{div} \boldsymbol{F} \quad \text{in } \Omega, \tag{2.1}$$

where

(2.2a) Ω is an open bounded connected subset of $\mathbb{R}^d$ $(d \in \mathbb{N}^*)$ with boundary $\partial\Omega$,

(2.2b) Λ is a measurable function from Ω to the set of $d \times d$ symmetric matrices, and there exists $\underline{\lambda}$, $\overline{\lambda} > 0$ such that, for a.e. $\boldsymbol{x} \in \Omega$, $\Lambda(\boldsymbol{x})$ has eigenvalues in $[\underline{\lambda}, \overline{\lambda}]$,

(2.2c) for a.e. $\boldsymbol{x} \in \Omega$ $\forall \xi \in \mathbb{R}^d$, $|\xi|_{\Lambda(\boldsymbol{x})} = \sqrt{\Lambda(\boldsymbol{x})\xi \cdot \xi}$, $r \in L^{p'}(\Omega)$ and $\boldsymbol{F} \in L^{p'}(\Omega)^d$.

This problem can be considered with a variety of boundary conditions (BCs), with an additional condition on $\bar{u}$ in the case of Neumann boundary conditions. These conditions are summarised in Table 1, in which $\boldsymbol{n}$ denotes the outer normal to $\partial\Omega$.

	homogeneous Dirichlet	homogeneous Neumann
on $\partial\Omega$	$\bar{u}=0$	$(\Lambda\vert\nabla\bar{u}\vert_\Lambda^{p-2}\nabla\bar{u}+\boldsymbol{F})\cdot\boldsymbol{n}=0$
additional conditions		$\int_\Omega r(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0$ $\int_\Omega \overline{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=0$
	nonhomogeneous Neumann	**Fourier**
on $\partial\Omega$	$(\Lambda\vert\nabla\bar{u}\vert_\Lambda^{p-2}\nabla\bar{u}+\boldsymbol{F})\cdot\boldsymbol{n}=g$	$(\Lambda\vert\nabla\bar{u}\vert_\Lambda^{p-2}\nabla\bar{u}+\boldsymbol{F})\cdot\boldsymbol{n}$ $+b\vert\overline{u}\vert^{p-2}\overline{u}=g$
additional conditions	$g\in L^{p'}(\partial\Omega)$ $\int_\Omega r(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}+\int_{\partial\Omega} g(\boldsymbol{x})\,\mathrm{d}s(\boldsymbol{x})=0$ $\int_\Omega \overline{u}(\boldsymbol{x})\,\mathrm{d}x=0$	$g\in L^{p'}(\partial\Omega)$ $b\in L^\infty(\partial\Omega)$ $0<\underline{b}\leqslant b(\boldsymbol{x})$

Table 1. Various boundary conditions for (2.1).

The analysis of approximations of (2.1) can then be carried out, for each of these boundary conditions; a usual way is to first write a weak formulation of the problem and then design tools to approximate this formulation. For nonhomogeneous Neumann BCs and Fourier BCs, these tools must include the approximation of the trace on the boundary. Let us now describe a unified formulation of (2.1) that includes all considered boundary conditions, together with a generic approximation scheme based on this unified formulation.

Introduce two Banach spaces $\boldsymbol{L}=L^p(\Omega)^d$ and L, a space $W_{\mathrm{G}}\subset L$ (which is dense in L), an operator $\mathrm{G}\colon W_{\mathrm{G}}\to\boldsymbol{L}$, two mappings $\boldsymbol{a}\colon L\times\boldsymbol{L}\to\boldsymbol{L}'$ and $a\colon L\to L'$ and a right-hand-side $f\in L'$ as in Table 2. Here and in the rest of the paper, γu is the trace on $\partial\Omega$ of any function $u\in W^{1,p}(\Omega)$.

The weak formulation of Problem (2.1) with all considered BCs is then:

(2.3) Find $\bar{u}\in W_{\mathrm{G}}$ such that,

$$\forall\, v\in W_{\mathrm{G}},\quad \langle\boldsymbol{a}(\bar{u},\mathrm{G}\bar{u}),\mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}+\langle a(\bar{u}),v\rangle_{L',L}=\langle f,v\rangle_{L',L}-\langle\boldsymbol{F},\mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}.$$

Indeed:

▷ In the case of homogeneous Dirichlet boundary conditions, $\|\nabla\cdot\|_{L^p(\Omega)^d}$ is a norm on the space $W_{\mathrm{G}}=W_0^{1,p}(\Omega)$ (owing to Poincaré's inequality) and there is no need for an additional condition: we can then let $a=0$.

	homogeneous Dirichlet	homogeneous Neumann
$L =$	$L^p(\Omega)$	$L^p(\Omega)$
$W_{\mathrm{G}} =$	$W_0^{1,p}(\Omega)$	$W^{1,p}(\Omega)$
$\mathrm{G}\colon$	$u \mapsto \nabla u$	$u \mapsto \nabla u$
$\boldsymbol{a}\colon$	$(u,\boldsymbol{v}) \mapsto \Lambda\|\boldsymbol{v}\|_\Lambda^{p-2}\boldsymbol{v}$	$(u,\boldsymbol{v}) \mapsto \Lambda\|\boldsymbol{v}\|_\Lambda^{p-2}\boldsymbol{v}$
$a\colon$	$u \mapsto 0$	$u \mapsto \|\int_\Omega u\|^{p-2}(\int_\Omega u)1_\Omega$
$f =$	r	r
	nonhomogeneous Neumann	**Fourier**
$L =$	$L^p(\Omega) \times L^p(\partial\Omega)$	$L^p(\Omega) \times L^p(\partial\Omega)$
$W_{\mathrm{G}} =$	$\{(u,\gamma u)\colon\ u \in W^{1,p}(\Omega)\}$	$\{(u,\gamma u)\colon\ u \in W^{1,p}(\Omega)\}$
$\mathrm{G}\colon$	$(u,w) \mapsto \nabla u$	$(u,w) \mapsto \nabla u$
$\boldsymbol{a}\colon$	$((u,w),\boldsymbol{v}) \mapsto \Lambda\|\boldsymbol{v}\|_\Lambda^{p-2}\boldsymbol{v}$	$((u,w),\boldsymbol{v}) \mapsto \Lambda\|\boldsymbol{v}\|_\Lambda^{p-2}\boldsymbol{v}$
$a\colon$	$(u,w) \mapsto \|\int_\Omega u\|^{p-2}(\int_\Omega u)(1_\Omega, 0)$	$(u,w) \mapsto (0, b\|w\|^{p-2}w)$
$f =$	(r,g)	(r,g)

Table 2. Abstract operators for various boundary conditions.

- ▷ In the case of homogeneous Neumann conditions, multiplying (2.1) by $v = 1_\Omega$ and integrating over Ω shows that the condition $\int_\Omega r(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$ is necessary for the existence of at least one solution; this solution is defined up to an additive constant which is fixed by imposing, for example, $\int_\Omega \bar{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$. A classical technique to write a weak formulation that embeds this condition, and has the required coercivity property, is to introduce an additional term $\langle a(\bar{u}), v\rangle_{L',L}$ on the left-hand side of this formulation, where $a(\bar{u}) = |\int_\Omega \bar{u}|^{p-2}(\int_\Omega \bar{u})1_\Omega$. Nonhomogeneous Neumann BCs are handled in a similar way.
- ▷ In the case of Fourier boundary conditions, the term $\langle a(\bar{u},\gamma\bar{u}), (v,\gamma v)\rangle_{L',L} = \int_{\partial\Omega} b|\gamma\bar{u}|^{p-2}\gamma\bar{u}\gamma v\,\mathrm{ds}$ naturally appears when multiplying (2.1) by a test function v and formally integrating by parts.

Problem (2.3) can be re-formulated by introducing a space $\boldsymbol{W}_{\mathrm{D}} \subset \boldsymbol{L}'$ and the dual operator $\mathrm{D}\colon\ \boldsymbol{W}_{\mathrm{D}} \to L'$ to G as per Table 3. In this table, we set

$$\begin{aligned} W^{p'}_{\mathrm{div}}(\Omega) &= \{\boldsymbol{\varphi} \in L^{p'}(\Omega)^d\colon\ \operatorname{div}\boldsymbol{\varphi} \in L^{p'}(\Omega)\},\\ W^{p'}_{\mathrm{div},0}(\Omega) &= \{\boldsymbol{\varphi} \in W^{p'}_{\mathrm{div}}(\Omega)\colon\ \gamma_{\mathbf{n}}\boldsymbol{\varphi} = 0\},\\ W^{p'}_{\mathrm{div},\partial}(\Omega) &= \{\boldsymbol{\varphi} \in W^{p'}_{\mathrm{div}}(\Omega)\colon\ \gamma_{\mathbf{n}}\boldsymbol{\varphi} \in L^{p'}(\partial\Omega)\}, \end{aligned}$$

where $\gamma_{\mathbf{n}}\boldsymbol{\varphi}$ is the normal trace of $\boldsymbol{v}$ on $\partial\Omega$. The space $\boldsymbol{W}_{\mathrm{D}}$ and the operator D are defined such that the following formula, which generalises the Stokes formula to all types of boundary conditions, holds:

$$\forall u \in W_{\mathrm{G}},\ \forall \boldsymbol{v} \in \boldsymbol{W}_{\mathrm{D}}, \quad \langle \boldsymbol{v}, \mathrm{G}u\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \mathrm{D}\boldsymbol{v}, u\rangle_{L',L} = 0. \tag{2.4}$$

	homogeneous Dirichlet	homogeneous Neumann	nonhomogeneous Neumann	Fourier
$\boldsymbol{W}_{\mathrm{D}} =$	$W^{p'}_{\mathrm{div}}(\Omega)$	$W^{p'}_{\mathrm{div},0}(\Omega)$	$W^{p'}_{\mathrm{div},\partial}(\Omega)$	$W^{p'}_{\mathrm{div},\partial}(\Omega)$
D:	$\boldsymbol{v} \mapsto \mathrm{div}\,\boldsymbol{v}$	$\boldsymbol{v} \mapsto \mathrm{div}\,\boldsymbol{v}$	$\boldsymbol{v} \mapsto (\mathrm{div}\,\boldsymbol{v}, -\gamma_{\mathbf{n}}\boldsymbol{v})$	$\boldsymbol{v} \mapsto (\mathrm{div}\,\boldsymbol{v}, -\gamma_{\mathbf{n}}\boldsymbol{v})$

Table 3. Dual space and operators for various boundary conditions.

Problem (2.3) is then equivalent to

$$\begin{aligned}&\text{Find } \bar{u} \in W_{\mathrm{G}} \text{ such that}\\ &\boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u}) + \boldsymbol{F} \in \boldsymbol{W}_{\mathrm{D}} \quad \text{and} \quad -\mathrm{D}(\boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u}) + \boldsymbol{F}) + a(\bar{u}) = f \quad \text{in } L'.\end{aligned} \tag{2.5}$$

This equivalence is proved in Section 4 in the general abstract setting. Thanks to the above introduced framework, approximations of Problem (2.3) can be designed by drawing inspiration from the Gradient Discretisation Method (GDM), see [19]. Three discrete objects $\mathcal{D} = (X_{\mathcal{D}}, \mathrm{P}_{\mathcal{D}}, \mathrm{G}_{\mathcal{D}})$, forming altogether a *gradient discretisation*, are introduced: a finite dimensional vector space $X_{\mathcal{D}}$ meant to contain the families of discrete unknowns, a linear mapping $\mathrm{P}_{\mathcal{D}}\colon X_{\mathcal{D}} \to L$ that reconstructs an element in L from an element of $X_{\mathcal{D}}$, and a "gradient" reconstruction $\mathrm{G}_{\mathcal{D}}\colon X_{\mathcal{D}} \to \boldsymbol{L}$, which is a linear mapping that reconstructs an element in $\boldsymbol{L}$ from an element of $X_{\mathcal{D}}$. The *gradient scheme* for the approximation of Problem (2.3) is then obtained by replacing the continuous space and operators by the discrete ones:

$$\begin{aligned}&\text{Find } u \in X_{\mathcal{D}} \text{ such that,}\\ &\forall v \in X_{\mathcal{D}}, \quad \langle \boldsymbol{a}(\mathrm{P}_{\mathcal{D}}u, \mathrm{G}_{\mathcal{D}}u), \mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\mathrm{P}_{\mathcal{D}}u), \mathrm{P}_{\mathcal{D}}v\rangle_{L',L}\\ &\qquad\qquad = \langle f, \mathrm{P}_{\mathcal{D}}v\rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}.\end{aligned} \tag{2.6}$$

Note that $P_{\mathcal{D}}$ denotes either a reconstructed function over Ω (Dirichlet or homogeneous Neumann conditions), or a pair of the reconstructed function on Ω and the reconstructed trace on $\partial\Omega$ (nonhomogeneous Neumann and Fourier conditions, see Table 4).

	homogeneous Dirichlet	homogeneous Neumann	nonhomogeneous Neumann	Fourier
$\mathrm{P}_{\mathcal{D}}$:	$u \mapsto \Pi_{\mathcal{D}}u$	$u \mapsto \Pi_{\mathcal{D}}u$	$u \mapsto (\Pi_{\mathcal{D}}u, \mathbb{T}_{\mathcal{D}}u)$	$u \mapsto (\Pi_{\mathcal{D}}u, \mathbb{T}_{\mathcal{D}}u)$

Table 4. Function ($\Pi_{\mathcal{D}}$) and trace ($\mathbb{T}_{\mathcal{D}}$) reconstructions for various boundary conditions.

3. Continuous and discrete settings

The examples in Section 2 gave a flavour of a general setting we now describe.

3.1. Continuous spaces and operators. Let L and $\boldsymbol{L}$ be separable reflexive Banach spaces, with the respective topological dual spaces L' and $\boldsymbol{L}'$. Let $W_{\mathrm{G}} \subset L$ be a dense subspace of L and let $\mathrm{G}\colon W_{\mathrm{G}} \to \boldsymbol{L}$ be a linear operator whose graph $\mathcal{G} = \{(u, \mathrm{G}u), u \in W_{\mathrm{G}}\}$ is closed in $L \times \boldsymbol{L}$. As a consequence, W_{G} endowed with the graph norm $\|u\|_{W_{\mathrm{G}},\mathcal{G}} = \|u\|_L + \|\mathrm{G}u\|_{\boldsymbol{L}}$ is a Banach space continuously embedded in L. Since $L \times \boldsymbol{L}$ is separable, W_{G} is also separable for the norm $\|\cdot\|_{W_{\mathrm{G}},\mathcal{G}}$ (see [11], Chapter III).

Define $\boldsymbol{W}_{\mathrm{D}}$ by

$$\boldsymbol{W}_{\mathrm{D}} = \{\boldsymbol{v} \in \boldsymbol{L}'\colon \exists w \in L' \quad \forall\, u \in W_{\mathrm{G}}, \langle \boldsymbol{v}, \mathrm{G}u\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle w, u\rangle_{L',L} = 0\}. \tag{3.1}$$

The density of W_{G} in L implies (and is actually equivalent to) the following property.

$$\forall\, w \in L', \quad (\forall\, u \in W_{\mathrm{G}}, \langle w, u\rangle_{L',L} = 0) \Rightarrow w = 0. \tag{3.2}$$

Therefore, for any $\boldsymbol{v} \in \boldsymbol{W}_{\mathrm{D}}$, the element $w \in L'$ whose existence is assumed in (3.1) is unique; this defines a linear operator $\mathrm{D}\colon \boldsymbol{W}_{\mathrm{D}} \to L'$, the adjoint operator of $-\mathrm{G}$ in the sense of [24], page 167 or [11], page 43, such that $w = \mathrm{D}\boldsymbol{v}$, that is,

$$\forall\, u \in W_{\mathrm{G}},\ \forall\, \boldsymbol{v} \in \boldsymbol{W}_{\mathrm{D}}, \quad \langle \boldsymbol{v}, \mathrm{G}u\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \mathrm{D}\boldsymbol{v}, u\rangle_{L',L} = 0. \tag{3.3}$$

It easily follows from this that the graph of D is closed in $\boldsymbol{L}' \times L'$, and therefore that, endowed with the graph norm $\|\boldsymbol{v}\|_{\boldsymbol{W}_{\mathrm{D}}} = \|\boldsymbol{v}\|_{\boldsymbol{L}'} + \|\mathrm{D}\boldsymbol{v}\|_{L'}$, $\boldsymbol{W}_{\mathrm{D}}$ is a Banach space continuously embedded and dense in $\boldsymbol{L}'$ (see [24], Theorem 5.29, page 168).

Remark 3.1 (Reverse construction of the dual operators)**.** Since the spaces L and $\boldsymbol{L}$ are reflexive, see [24], Theorem 5.29, page 168 also states that

$$W_{\mathrm{G}} = \{u \in L\colon \exists \boldsymbol{u} \in \boldsymbol{L} \quad \forall\, \boldsymbol{v} \in \boldsymbol{W}_{\mathrm{D}}, \langle \boldsymbol{v}, \boldsymbol{u}\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \mathrm{D}\boldsymbol{v}, u\rangle_{L',L} = 0\},$$

for any $u \in W_{\mathrm{G}}$, $\mathrm{G}u$ is the element $\boldsymbol{u} \in \boldsymbol{L}$ in the definition of W_{G}. It is therefore equivalent to begin with the construction of $(W_{\mathrm{G}}, \mathrm{G})$ or that of $(\boldsymbol{W}_{\mathrm{D}}, \mathrm{D})$.

Let V be a closed subspace of L' and denote by $|\cdot|_{L,V}$ the semi-norm on L defined by

$$\forall\, u \in L, \quad |u|_{L,V} = \begin{cases} \displaystyle\sup_{\mu \in V\setminus\{0\}} \frac{|\langle \mu, u\rangle_{L',L}|}{\|\mu\|_{L'}} & \text{if } V \neq \{0\},\\ 0 & \text{if } V = \{0\}. \end{cases} \tag{3.4}$$

By construction, for all $u \in L$, $|u|_{L,V} \leqslant \sup_{\mu \in L' \setminus \{0\}} |\langle \mu, u\rangle_{L',L}|/\|\mu\|_{L'} = \|u\|_L$. Defining, for $u \in W_{\mathrm{G}}$,

$$\|u\|_{W_{\mathrm{G}}} = |u|_{L,V} + \|\mathrm{G}u\|_{\boldsymbol{L}}, \tag{3.5}$$

we therefore have

$$\forall\, u \in W_{\mathrm{G}}, \quad \|u\|_{W_{\mathrm{G}}} \leqslant \|u\|_{W_{\mathrm{G}},\mathcal{G}}. \tag{3.6}$$

A necessary and sufficient condition on V for the norm $\|\cdot\|_{W_{\mathrm{G}},\mathcal{G}}$ and semi-norm $\|\cdot\|_{W_{\mathrm{G}}}$ to be equivalent is that $L' = \mathrm{Im}(\mathrm{D}) + V$ as stated in the next theorem, which is an extension of [11], Theorem 2.20 to the case $V \neq \{0\}$.

Theorem 3.2 (Equivalence of the norms)**.** *Under the above assumptions of the present section, the norms $\|\cdot\|_{W_{\mathrm{G}},\mathcal{G}}$ and $\|\cdot\|_{W_{\mathrm{G}}}$ are equivalent, that is*

$$\forall\, u \in W_{\mathrm{G}}, \quad \|u\|_{W_{\mathrm{G}},\mathcal{G}} \leqslant C_{W_{\mathrm{G}},V} \|u\|_{W_{\mathrm{G}}} \tag{3.7}$$

if and only if

$$L' = \mathrm{Im}(\mathrm{D}) + V. \tag{3.8}$$

Proof. Let us assume that (3.7) holds. Owing to [26] (see Remark 4.4), we can assume that $(L, \|\cdot\|_L)$ and $(\boldsymbol{L}, \|\cdot\|_{\boldsymbol{L}})$ are smooth. Lemma 4.5 can then be applied to define $\boldsymbol{a}$ by (4.3). Let $a : L \to V \subset L'$ be defined as in Lemma 4.7. Thanks to Lemma 4.11, for any $f \in L'$, there exists a solution $\bar{u}$ to (4.6) with $\boldsymbol{F} = 0$. Setting $\boldsymbol{v} = -\boldsymbol{a}(\mathrm{G}\bar{u})$, Lemma 4.10 shows that $f = \mathrm{D}\boldsymbol{v} + a(\bar{u}) \in \mathrm{Im}(\mathrm{D}) + V$.

Reciprocally, let us assume that (3.8) holds.

Since $\|\cdot\|_{W_{\mathrm{G}},\mathcal{G}}$ is a norm, proving its equivalence with $\|\cdot\|_{W_{\mathrm{G}}}$ establishes that this latter semi-norm is also a norm. Half of the equivalence has already been established in (3.6); to prove the other half, we just need to show that

$$E = \{u \in W_{\mathrm{G}} \colon \; \|u\|_{W_{\mathrm{G}}} = 1\}$$

is bounded in L. Indeed, this establishes the existence of $M \geqslant 0$ such that, for all $u \in E$, $\|u\|_L \leqslant M$ and thus, since $\|\mathrm{G}u\|_{\boldsymbol{L}} \leqslant \|u\|_{W_{\mathrm{G}}} = 1$,

$$\|u\|_{W_{\mathrm{G}},\mathcal{G}} \leqslant 1 + M = (1 + M)\|u\|_{W_{\mathrm{G}}}.$$

By homogeneity of the semi-norms, this concludes the proof that $\|\cdot\|_{W_{\mathrm{G}},\mathcal{G}}$ and $\|\cdot\|_{W_{\mathrm{G}}}$ are equivalent on W_{G}.

To prove that E is bounded, take $f \in L'$ and apply (3.8) to get $\boldsymbol{v}_f \in \boldsymbol{W}_{\mathrm{D}}$ and $\mu_f \in V$ such that $f = \mathrm{D}\boldsymbol{v}_f + \mu_f$. Then, for any $u \in E$, by definition of the semi-norm $|\cdot|_{L,V}$ and since $\|\mathrm{G}u\|_{\boldsymbol{L}} \leqslant 1$ and $|u|_{L,V} \leqslant 1$,

$$
\begin{aligned}
|\langle f, u\rangle_{L',L}| &= |\langle \mathrm{D}\boldsymbol{v}_f, u\rangle_{L',L} + \langle \mu_f, u\rangle_{L',L}| = |-\langle \boldsymbol{v}_f, \mathrm{G}u\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \mu_f, u\rangle_{L',L}| \\
&\leqslant \|\boldsymbol{v}_f\|_{\boldsymbol{L}'}\|\mathrm{G}u\|_{\boldsymbol{L}} + \|\mu_f\|_{L'}|u|_{L,V} \leqslant \|\boldsymbol{v}_f\|_{\boldsymbol{L}'} + \|\mu_f\|_{L'}.
\end{aligned}
$$

This shows that $\{\langle f, u\rangle_{L',L}\colon\ u \in E\}$ is bounded by some constant depending on f. Since this is valid for any $f \in L'$, the Banach-Steinhaus theorem, see [11], Theorem 2.2 shows that E is bounded in L.

Let $\pi\colon\ L' \to L'/V$ be the canonical projector that associates to every $v \in L'$ its equivalence class. We denote by $\widetilde{\mathrm{D}}\colon\ \boldsymbol{W}_{\mathrm{D}} \to L'/V$ the operator $\pi \circ \mathrm{D}$. Its graph is closed and $\widetilde{\mathrm{D}}$ is densely defined in $\boldsymbol{L}'$. It is then possible to define its adjoint $-\widetilde{\mathrm{G}}$ in the sense of [11], page 44, which is a mapping from $(L'/V)'$ to $\boldsymbol{L}$, uniquely defined thanks to the density of $\boldsymbol{W}_{\mathrm{D}}$ in $\boldsymbol{L}'$. Identifying $(L'/V)'$ with $V^{\perp} \subset L$ by [11], Proposition 11.9, we find that the domain of $\widetilde{\mathrm{G}}$, defined as $\{u \in V^{\perp}\colon\ \exists C \geqslant 0$ for all $\boldsymbol{v} \in \boldsymbol{W}_{\mathrm{D}}$, $|\langle \mathrm{D}\boldsymbol{v}, u\rangle_{L',L}| \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{L}'}\}$, is in fact equal to $W_{\mathrm{G}} \cap V^{\perp}$. Moreover, for all $u \in W_{\mathrm{G}} \cap V^{\perp}$, the unique element $\widetilde{\mathrm{G}}u \in \boldsymbol{L}$ such that $\langle \boldsymbol{v}, -\widetilde{\mathrm{G}}u\rangle_{\boldsymbol{L}',\boldsymbol{L}} = \langle \mathrm{D}\boldsymbol{v}, u\rangle_{L',L}$ (for all $\boldsymbol{v} \in \boldsymbol{W}_{\mathrm{D}}$) is therefore equal to $\mathrm{G}u$; note that $W_{\mathrm{G}} \cap V^{\perp}$ may be not dense in $V^{\perp}$, see [11], Remark 15, page 44.

Now, (3.7) shows that for all $u \in W_{\mathrm{G}} \cap V^{\perp}$,

$$
C_{W_{\mathrm{G}},V}\|\mathrm{G}u\|_{\boldsymbol{L}} = C_{W_{\mathrm{G}},V}\|u\|_{W_{\mathrm{G}}} \geqslant \|u\|_{L}.
$$

Using [11], Theorem 2.20 we infer that $\widetilde{\mathrm{D}}$ is surjective. This means that, for any $f \in L'$, there exists $\boldsymbol{v} \in \boldsymbol{L}'$ such that the equivalence classes of f and $\mathrm{D}\boldsymbol{v}$ in L'/V are identical; in other words, $f - \mathrm{D}\boldsymbol{v} \in V$. This proves (3.8). □

Remark 3.3 (Poincaré inequalities)**.** In the particular context of Sobolev spaces, Theorem 3.2 proves that there is equivalence between the so-called "mean" Poincaré-Wirtinger inequality and the surjectivity of the divergence operator.

Remark 3.4 (Examples of spaces V)**.** In the examples of Section 2, as well as in the example of continuum mechanics (see Section 6.2), in the case of Neumann boundary conditions the dimension of the kernel of G is finite (in the latter case, it is equal to 6, see [15]). These examples are such that Im(G) is closed (or equivalently Im(D) is closed, as proved in [11], Theorem 2.19). Then $\mathrm{Ker(G)}^{\perp} = \mathrm{Im(D)}$, and one can construct V as a finite dimensional space complementary to Im(D) in L', with the same dimension as Ker(G), following the method given in [11], page 39 and in [15].

In the case of Fourier boundary conditions, $V = \{0\} \times L^{p'}(\partial\Omega)$, and $\|u\|_{W_{\mathrm{G}}} = \|\nabla u\|_{L^p(\Omega)^d} + \|\gamma u\|_{L^p(\partial\Omega)}$.

In the remainder of the paper, we will assume that the norm $\|\cdot\|_{W_{\mathrm{G}},\mathcal{G}}$ and seminorm $\|\cdot\|_{W_{\mathrm{G}}}$ are actually equivalent, i.e. that (3.7) holds.

3.2. Gradient discretisations. Based on the previous definitions, we generalise the concept of gradient discretisation of [19] and the key notions of coercivity, limit-conformity, consistency and compactness to the present abstract setting. These properties enable us, in Section 4, to design converging approximation schemes for an abstract monotonous problem.

3.2.1. Key definitions.

Definition 3.5 (Gradient Discretisation). In the setting described in Section 3.1, a gradient discretisation is defined by $\mathcal{D} = (X_{\mathcal{D}}, \mathrm{P}_{\mathcal{D}}, \mathrm{G}_{\mathcal{D}})$, where:

(1) The set of discrete unknowns $X_{\mathcal{D}}$ is a finite dimensional vector space on $\mathbb{R}$.
(2) The "function" reconstruction $\mathrm{P}_{\mathcal{D}}\colon X_{\mathcal{D}} \to L$ is a linear mapping that reconstructs, from an element of $X_{\mathcal{D}}$, an element in L.
(3) The "gradient" reconstruction $\mathrm{G}_{\mathcal{D}}\colon X_{\mathcal{D}} \to \boldsymbol{L}$ is a linear mapping that reconstructs, from an element of $X_{\mathcal{D}}$, an element of $\boldsymbol{L}$.
(4) The mappings $\mathrm{P}_{\mathcal{D}}$ and $\mathrm{G}_{\mathcal{D}}$ are such that the following quantity is a norm on $X_{\mathcal{D}}$:

$$\|v\|_{\mathcal{D}} := |\mathrm{P}_{\mathcal{D}} v|_{L,V} + \|\mathrm{G}_{\mathcal{D}} v\|_{\boldsymbol{L}}.$$

Definition 3.6 (Coercivity). If $\mathcal{D}$ is a gradient discretisation in the sense of Definition 3.6, let $C_{\mathcal{D}}$ be the norm of $\mathrm{P}_{\mathcal{D}}$:

$$C_{\mathcal{D}} = \max_{v \in X_{\mathcal{D}} \setminus \{0\}} \frac{\|\mathrm{P}_{\mathcal{D}} v\|_L}{\|v\|_{\mathcal{D}}}. \tag{3.9}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretisations is *coercive* if there exists $C_P \in \mathbb{R}_+$ such that $C_{\mathcal{D}_m} \leqslant C_P$ for all $m \in \mathbb{N}$.

Definition 3.7 (Limit-conformity). If $\mathcal{D}$ is a gradient discretisation in the sense of Definition 3.6, let $W_{\mathcal{D}}\colon \boldsymbol{W}_{\mathrm{D}} \to [0,\infty)$ be given by

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{W}_{\mathrm{D}}, \quad W_{\mathcal{D}}(\boldsymbol{\varphi}) = \sup_{u \in X_{\mathcal{D}} \setminus \{0\}} \frac{|\langle \boldsymbol{\varphi}, \mathrm{G}_{\mathcal{D}} u\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \mathrm{D}\boldsymbol{\varphi}, \mathrm{P}_{\mathcal{D}} u\rangle_{L',L}|}{\|u\|_{\mathcal{D}}}. \tag{3.10}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretisations is *limit-conforming* if

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{W}_{\mathrm{D}}, \quad \lim_{m\to\infty} W_{\mathcal{D}_m}(\boldsymbol{\varphi}) = 0. \tag{3.11}$$

Once L, $\boldsymbol{L}$, $\boldsymbol{W}_{\mathrm{D}}$ and D are chosen, the Definition 3.8 of limit-conformity is constrained by the continuous duality formula (3.3); as a consequence of Lemma 3.11

below, the definition of coercivity is also constrained by this formula. These two notions therefore naturally follow from the continuous setting. On the contrary, the following two definitions of consistency and compactness are disconnected from the duality formula. Various choices for these notions are possible, we describe here one that is in particular adapted to the monotonous problem in Section 4.

Definition 3.8 (Consistency)**.** If $\mathcal{D}$ is a gradient discretisation in the sense of Definition 3.6, let $S_{\mathcal{D}}\colon W_{\mathrm{G}} \to [0,\infty)$ be given by

$$\forall\,\varphi \in W_{\mathrm{G}},\quad S_{\mathcal{D}}(\varphi) = \min_{v\in X_{\mathcal{D}}} (\|\mathrm{P}_{\mathcal{D}}v - \varphi\|_L + \|\mathrm{G}_{\mathcal{D}}v - \mathrm{G}\varphi\|_{\boldsymbol{L}}). \tag{3.12}$$

A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretisations is *consistent* if

$$\forall\,\varphi \in W_{\mathrm{G}},\quad \lim_{m\to\infty} S_{\mathcal{D}_m}(\varphi) = 0. \tag{3.13}$$

Definition 3.9 (Compactness)**.** A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretisations in the sense of Definition 3.6 is *compact* if, for any sequence $u_m \in X_{\mathcal{D}_m}$ such that $(\|u_m\|_{\mathcal{D}_m})_{m\in\mathbb{N}}$ is bounded, the sequence $(\mathrm{P}_{\mathcal{D}_m}u_m)_{m\in\mathbb{N}}$ is relatively compact in L.

3.2.2. Main properties. The following result uses the surjectivity of the divergence operator proved in Theorem 3.2.

Lemma 3.10 (Limit-conformity implies coercivity)**.** *If a sequence of gradient discretisations is limit-conforming in the sense of Definition* 3.8*, then it is also coercive in the sense of Definition* 3.7*.*

Proof. Consider a limit-conforming sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ and set

$$E = \left\{\frac{\mathrm{P}_{\mathcal{D}_m}v}{\|v\|_{\mathcal{D}_m}} \in L\colon\ m\in\mathbb{N},\ v\in X_{\mathcal{D}_m}\setminus\{0\}\right\}.$$

Proving the coercivity of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ consists in proving that E is bounded in L. Let $f\in L'$. By Theorem 3.2, there exist $\boldsymbol{v}_f \in \boldsymbol{W}_{\mathrm{D}}$ and $\mu_f\in V$ such that $f = \mathrm{D}\boldsymbol{v}_f + \mu_f$. The definition of $|\cdot|_{L,V}$ shows that $|\langle\mu_f,\cdot\rangle_{L',L}| \leqslant \|\mu_f\|_{L'}|\cdot|_{L,V}$. For $z\in E$, take $m\in\mathbb{N}$ and $v\in X_{\mathcal{D}_m}\setminus\{0\}$ such that $z = \mathrm{P}_{\mathcal{D}_m}v/\|v\|_{\mathcal{D}_m}$ and write

$$\begin{aligned}
|\langle f,z\rangle_{L',L}| &\leqslant \frac{1}{\|v\|_{\mathcal{D}_m}}|\langle \mathrm{D}\boldsymbol{v}_f,\mathrm{P}_{\mathcal{D}_m}v\rangle_{L',L}| + \frac{1}{\|v\|_{\mathcal{D}_m}}|\langle \mu_f,\mathrm{P}_{\mathcal{D}_m}v\rangle_{L',L}|\\
&\leqslant \frac{1}{\|v\|_{\mathcal{D}_m}}|\langle \mathrm{D}\boldsymbol{v}_f,\mathrm{P}_{\mathcal{D}_m}v\rangle_{L',L} + \langle \boldsymbol{v}_f,\mathrm{G}_{\mathcal{D}_m}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}|\\
&\quad + \frac{1}{\|v\|_{\mathcal{D}_m}}|\langle \boldsymbol{v}_f,\mathrm{G}_{\mathcal{D}_m}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}| + \frac{1}{\|v\|_{\mathcal{D}_m}}\|\mu_f\|_{L'}|\mathrm{P}_{\mathcal{D}_m}v|_{L,V}\\
&\leqslant W_{\mathcal{D}_m}(\boldsymbol{v}_f) + \|\boldsymbol{v}_f\|_{\boldsymbol{L}'} + \|\mu_f\|_{L'}.
\end{aligned} \tag{3.14}$$

In the last inequality we used $|\mathrm{P}_{\mathcal{D}_m} v|_{L,V} \leqslant \|v\|_{\mathcal{D}_m}$ and $\|\mathrm{G}_{\mathcal{D}_m} v\|_{\boldsymbol{L}} \leqslant \|v\|_{\mathcal{D}_m}$. Since $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is limit-conforming, $(W_{\mathcal{D}_m}(\boldsymbol{v}_f))_{m\in\mathbb{N}}$ converges to 0 and is therefore bounded. Estimate (3.14) thus shows that $\{\langle f, z\rangle_{L',L} \colon z \in E\}$ is bounded by some constant depending on f. Since this is valid for any $f \in L'$, we infer from the Banach-Steinhaus theorem (see [11], Theorem 2.2) that E is bounded in L. $\square$

Checking limit-conformity is made easier by the following result, which reduces the set of elements $\boldsymbol{\varphi}$ on which the convergence in (3.11) has to be asserted.

Lemma 3.11 (Equivalent condition for limit-conformity). *A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretisations is limit-conforming in the sense of Definition* 3.8 *if and only if it is coercive in the sense of Definition* 3.7 *and there exists a dense subset $\widetilde{\boldsymbol{W}}_{\mathrm{D}}$ of $\boldsymbol{W}_{\mathrm{D}}$ such that*

$$\forall\, \boldsymbol{\psi} \in \widetilde{\boldsymbol{W}}_{\mathrm{D}}, \quad \lim_{m\to\infty} W_{\mathcal{D}_m}(\boldsymbol{\psi}) = 0. \tag{3.15}$$

P r o o f. If $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is limit-conforming, then it is coercive by Lemma 3.11, and (3.15) is satisfied with $\widetilde{\boldsymbol{W}}_{\mathrm{D}} = \boldsymbol{W}_{\mathrm{D}}$, so that (3.11) is also satisfied.

Conversely, assume that $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is coercive and that (3.15) holds. Let $C_P \in \mathbb{R}_+$ be an upper bound of $(C_{\mathcal{D}_m})_{m\in\mathbb{N}}$. To prove (3.11), let $\boldsymbol{\varphi} \in \boldsymbol{W}_{\mathrm{D}}$ and $\varepsilon > 0$, and take $\boldsymbol{\psi} \in \widetilde{\boldsymbol{W}}_{\mathrm{D}}$ such that $\|\boldsymbol{\varphi} - \boldsymbol{\psi}\|_{\boldsymbol{W}_{\mathrm{D}}} \leqslant \varepsilon$. By definition of the norm in $\boldsymbol{W}_{\mathrm{D}}$, this means that $\|\boldsymbol{\varphi} - \boldsymbol{\psi}\|_{\boldsymbol{L}'} + \|\mathrm{D}\boldsymbol{\varphi} - \mathrm{D}\boldsymbol{\psi}\|_{L'} \leqslant \varepsilon$. Hence, for any $u \in X_{\mathcal{D}_m} \setminus \{0\}$,

$$\begin{aligned}
&\frac{|\langle \boldsymbol{\varphi} - \boldsymbol{\psi}, \mathrm{G}_{\mathcal{D}_m} u\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \mathrm{D}\boldsymbol{\varphi} - \mathrm{D}\boldsymbol{\psi}, \mathrm{P}_{\mathcal{D}_m} u\rangle_{L',L}|}{\|u\|_{\mathcal{D}_m}} \\
&\qquad \leqslant \|\boldsymbol{\varphi} - \boldsymbol{\psi}\|_{\boldsymbol{L}'} \frac{\|\mathrm{G}_{\mathcal{D}_m} u\|_{\boldsymbol{L}}}{\|u\|_{\mathcal{D}_m}} + \|\mathrm{D}\boldsymbol{\varphi} - \mathrm{D}\boldsymbol{\psi}\|_{L'} \frac{\|\mathrm{P}_{\mathcal{D}_m} u\|_{L}}{\|u\|_{\mathcal{D}_m}} \leqslant \max(1, C_P)\varepsilon.
\end{aligned}$$

Introducing $\boldsymbol{\psi}$ and $\mathrm{D}\boldsymbol{\psi}$ in the definition (3.10) of $W_{\mathcal{D}_m}(\boldsymbol{\varphi})$, we infer

$$\begin{aligned}
W_{\mathcal{D}_m}(\boldsymbol{\varphi}) &\leqslant \sup_{u\in X_{\mathcal{D}_m}\setminus\{0\}} \frac{|\langle \boldsymbol{\psi}, \mathrm{G}_{\mathcal{D}_m} u\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \mathrm{D}\boldsymbol{\psi}, \mathrm{P}_{\mathcal{D}_m} u\rangle_{L',L}|}{\|u\|_{\mathcal{D}_m}} + \max(1, C_P)\varepsilon \\
&= W_{\mathcal{D}_m}(\boldsymbol{\psi}) + \max(1, C_P)\varepsilon.
\end{aligned}$$

Invoking (3.15) we deduce that $\limsup_{m\to\infty} W_{\mathcal{D}_m}(\boldsymbol{\varphi}) \leqslant \max(1, C_P)\varepsilon$, and the proof is concluded by letting $\varepsilon \to 0$. $\square$

The next lemma is an essential tool to use compactness techniques in the convergence analysis of approximation methods for nonlinear problems.

Lemma 3.12 (Regularity of the limit). *Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a limit-conforming sequence of gradient discretisations, in the sense of Definition* 3.8. *For any $m \in \mathbb{N}$, take*

$u_m \in X_{\mathcal{D}_m}$ and assume that $(\|u_m\|_{\mathcal{D}_m})_{m\in\mathbb{N}}$ is bounded. Then there exists $u \in W_{\mathrm{G}}$ such that, along a subsequence as $m \to \infty$, $(\mathrm{P}_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ converges weakly in L to u, and $(\mathrm{G}_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ converges weakly in $\boldsymbol{L}$ to $\mathrm{G}u$.

P r o o f. By definition of $\|\cdot\|_{\mathcal{D}_m}$, $(\mathrm{G}_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ is bounded in $\boldsymbol{L}$. By Lemma 3.11, $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is coercive and therefore $(\mathrm{P}_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ is bounded in L. The reflexivity of L and $\boldsymbol{L}$ thus gives a subsequence of $(\mathcal{D}_m, u_m)_{m\in\mathbb{N}}$, denoted in the same way, and elements $u \in L$ and $\boldsymbol{u} \in \boldsymbol{L}$ such that $(\mathrm{P}_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ converges weakly in L to u and $(\mathrm{G}_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ converges weakly in $\boldsymbol{L}$ to $\boldsymbol{u}$. Hence, the limit-conformity of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ and the boundedness of $(\|u_m\|_{\mathcal{D}_m})_{m\in\mathbb{N}}$ give

$$\forall\, \boldsymbol{\varphi} \in \boldsymbol{W}_{\mathrm{D}}, \quad \langle \boldsymbol{\varphi}, \boldsymbol{u}\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \mathrm{D}\boldsymbol{\varphi}, u\rangle_{L',L} = 0.$$

Following Remark 3.1, this relation simultaneously proves that $u \in W_{\mathrm{G}}$ and that $\boldsymbol{u} = \mathrm{G}u$. □

Lemma 3.13 (Equivalent condition for the consistency)**.** *A sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of gradient discretisations is consistent in the sense of Definition* 3.9 *if and only if there exists a dense subset $\widetilde{W}_{\mathrm{G}}$ of W_{G} such that*

$$\forall\, \psi \in \widetilde{W}_{\mathrm{G}}, \quad \lim_{m\to\infty} S_{\mathcal{D}_m}(\psi) = 0. \tag{3.16}$$

P r o o f. Let us assume that (3.16) holds and let us prove (3.13) (the converse is straightforward, take $\widetilde{W}_{\mathrm{G}} = W_{\mathrm{G}}$). Observe first that, since W_{G} is continuously embedded in L, there exists $C_{W_{\mathrm{G}}} > 0$ such that

$$\forall\, \varphi \in W_{\mathrm{G}}, \quad \|\varphi\|_L \leqslant C_{W_{\mathrm{G}}} \|\varphi\|_{W_{\mathrm{G}}}.$$

Let $\varphi \in W_{\mathrm{G}}$. Take $\varepsilon > 0$ and $\psi \in \widetilde{W}_{\mathrm{G}}$ such that $\|\varphi - \psi\|_{W_{\mathrm{G}}} \leqslant \varepsilon$. For $v \in X_{\mathcal{D}_m}$, the triangle inequality and the definition (3.5) of the norm in W_{G} yield

$$\begin{aligned}
&\|\mathrm{P}_{\mathcal{D}_m} v - \varphi\|_L + \|\mathrm{G}_{\mathcal{D}_m} v - \mathrm{G}\varphi\|_{\boldsymbol{L}} \\
&\qquad \leqslant \|\mathrm{P}_{\mathcal{D}_m} v - \psi\|_L + \|\psi - \varphi\|_L + \|\mathrm{G}_{\mathcal{D}_m} v - \mathrm{G}\psi\|_{\boldsymbol{L}} + \|\mathrm{G}\psi - \mathrm{G}\varphi\|_{\boldsymbol{L}} \\
&\qquad \leqslant \|\mathrm{P}_{\mathcal{D}_m} v - \psi\|_L + \|\mathrm{G}_{\mathcal{D}_m} v - \mathrm{G}\psi\|_{\boldsymbol{L}} + (C_{W_{\mathrm{G}}} + 1)\|\psi - \varphi\|_{W_{\mathrm{G}}}.
\end{aligned}$$

Taking the infimum over $v \in X_{\mathcal{D}_m}$ leads to $S_{\mathcal{D}_m}(\varphi) \leqslant S_{\mathcal{D}_m}(\psi) + (C_{W_{\mathrm{G}}} + 1)\varepsilon$. Assumption (3.16) then shows that $\limsup\limits_{m\to\infty} S_{\mathcal{D}_m}(\varphi) \leqslant (C_{W_{\mathrm{G}}} + 1)\varepsilon$, and letting $\varepsilon \to 0$ concludes the proof that $S_{\mathcal{D}_m}(\varphi) \to 0$ as $m \to \infty$. □

Lemma 3.14 (Compactness implies coercivity)**.** *If a sequence of gradient discretisations is compact in the sense of Definition* 3.10, *then it is coercive in the sense of Definition* 3.7.

P r o o f. Assume that $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is compact but not coercive. Then there exists a subsequence of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ (denoted in the same way) such that, for all $m \in \mathbb{N}$, we can find $v_m \in X_{\mathcal{D}_m} \setminus \{0\}$ satisfying

$$\lim_{m\to\infty} \frac{\|\mathrm{P}_{\mathcal{D}_m} v_m\|_L}{\|v_m\|_{\mathcal{D}_m}} = \infty.$$

Setting $u_m = v_m/\|v_m\|_{\mathcal{D}_m}$, this gives $\lim\limits_{m\to\infty} \|\mathrm{P}_{\mathcal{D}_m} u_m\|_L = \infty$. But $\|u_m\|_{\mathcal{D}_m} = 1$ for all $m \in \mathbb{N}$ and the compactness of the sequence of gradient discretisations therefore implies that $(\mathrm{P}_{\mathcal{D}_m} u_m)_{m\in\mathbb{N}}$ is relatively compact in L, which is a contradiction. $\square$

The next two lemmas show that the compactness of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is strongly related to some compactness property of W_{G}.

Lemma 3.15 (Existence of a compact sequence of GDs implies compact embedding of W_{G})**.** *Let us assume the existence of a sequence of gradient discretisations which is consistent in the sense of Definition* 3.9 *and compact in the sense of Definition* 3.10. *Then the embedding of W_{G} in L is compact.*

P r o o f. Let $(\mathcal{D}_m)_{m\in\mathbb{N}}$ be a consistent and compact sequence of gradient discretisations, and let $(\overline{u}_m)_{m\in\mathbb{N}}$ be a bounded sequence in W_{G}. For $m = 0$, let $N_0 \in \mathbb{N}$ be such that there exists $u_{N_0} \in X_{\mathcal{D}_{N_0}}$ satisfying

$$\|\mathrm{P}_{\mathcal{D}_{N_0}} u_{N_0} - \overline{u}_0\|_L + \|\mathrm{G}_{\mathcal{D}_{N_0}} u_{N_0} - \mathrm{G}\overline{u}_0\|_{\boldsymbol{L}} \leqslant 1.$$

We then build a bounded sequence $(u_{N_m})_{m\in\mathbb{N}}$ by induction. For any $m \geqslant 1$, let $N_m > N_{m-1}$ such that there exists $u_{N_m} \in X_{D_{N_m}}$ satisfying

$$\|\mathrm{P}_{\mathcal{D}_{N_m}} u_{N_m} - \overline{u}_m\|_L + \|\mathrm{G}_{\mathcal{D}_{N_m}} u_{N_m} - \mathrm{G}\overline{u}_m\|_{\boldsymbol{L}} \leqslant \frac{1}{m+1}.$$

Then the sequence $(\|u_{N_m}\|_{\mathcal{D}_{N_m}})_{m\in\mathbb{N}}$ is bounded. Using the compactness hypothesis of $(\mathcal{D}_m)_{m\in\mathbb{N}}$, there exists a subsequence, denoted $(\mathcal{D}_{N_{\varphi(m)}}, u_{N_{\varphi(m)}})_{m\in\mathbb{N}}$ and $\overline{u} \in L$ such that $\mathrm{P}_{\mathcal{D}_{N_{\varphi(m)}}} u_{N_{\varphi(m)}}$ converges to $\overline{u}$ in L. We then have

$$\begin{aligned}\|\overline{u} - \overline{u}_{\varphi(m)}\|_L &= \|\overline{u} - \mathrm{P}_{\mathcal{D}_{N_{\varphi(m)}}} u_{N_{\varphi(m)}} + \mathrm{P}_{\mathcal{D}_{N_{\varphi(m)}}} u_{N_{\varphi(m)}} - \overline{u}_{\varphi(m)}\|_L \\ &\leqslant \frac{1}{\varphi(m)+1} + \|\mathrm{P}_{\mathcal{D}_{N_{\varphi(m)}}} u_{N_{\varphi(m)}} - \overline{u}\|_L,\end{aligned}$$

which shows that the subsequence $(\overline{u}_{\varphi(m)})_{m\in\mathbb{N}}$ converges to $\overline{u}$ in L. $\square$

3.2.3. A generic example of gradient discretisation. A series of examples of nonconforming GDs for usual second order elliptic problems (which enter the setting of this paper) may be found in [19]:

(1) Nonconforming finite elements,
(2) Discontinuous Galerkin methods,
(3) Hybrid Mimetic and Mixed methods.

Definition 3.17 below gives a very simple example (the classical Galerkin approximation) of a conforming GD which satisfies all the required properties.

Definition 3.16 (Galerkin gradient discretisation). Let $(u_i)_{i\in\mathbb{N}}$ be a dense sequence in W_{G} (whose existence is ensured by the separability of W_{G}). For all $m \in \mathbb{N}$, define a conforming Galerkin gradient discretisation $\mathcal{D}_m = (X_{\mathcal{D}_m}, \mathrm{P}_{\mathcal{D}_m}, \mathrm{G}_{\mathcal{D}_m})$, in the sense of Definition 3.6, in the following way:

(1) $X_{\mathcal{D}_m}$ is the vector space spanned by $(u_i)_{i=0,\dots,m}$,
(2) for all $u \in X_{\mathcal{D}_m}$, $\mathrm{P}_{\mathcal{D}_m} u = u$,
(3) for all $u \in X_{\mathcal{D}_m}$, $\mathrm{G}_{\mathcal{D}_m} u = \mathrm{G}u$.

Lemma 3.17 (Existence of a coercive, consistent and limit-conforming (and compact) sequence of GDs). *The sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ defined by Definition* 3.17 *is coercive, limit-conforming and consistent in the sense of the Definitions* 3.7, 3.8 *and* 3.9. *If, moreover, the embedding of W_{G} in L is compact, then $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is also compact in the sense of Definition* 3.10.

Proof. By definition, for all $v \in X_{\mathcal{D}_m}$ we have $\|v\|_{\mathcal{D}_m} = \|v\|_{W_{\mathrm{G}}}$, which proves that $\|\cdot\|_{\mathcal{D}_m}$ is a norm on $X_{\mathcal{D}_m}$. The coercivity is then a consequence of Assumption (3.7). Relation (3.3) implies that $W_{\mathcal{D}}$ defined by (3.10) is identically null, which implies the limit-conformity property. The consistency is a consequence of the assumption that $(u_i)_{i\in\mathbb{N}}$ is a dense sequence in W_{G}. The compactness of the sequence is a straightforward consequence of the compact embedding of W_{G} in L. □

4. Approximation of an abstract Leray-Lions problem

In this section, we generalise the problem presented in the introduction of this paper and provide a convergence analysis based on the GDM. In the whole section, $p \in (1,\infty)$ is given. Our general assumptions are similar to the assumptions considered in [25]:

(4.1a) $\boldsymbol{a}\colon L \times \boldsymbol{L} \to \boldsymbol{L}'$ is such that $\boldsymbol{a}(\cdot,\boldsymbol{v})$ is continuous for the strong topology of $\boldsymbol{L}'$, and $\boldsymbol{a}(v,\cdot)$ is continuous for the weak-$*$ topology of $\boldsymbol{L}'$, (4.1b) $\quad \boldsymbol{a}$ is monotonous in the sense:

$\forall v \in L, \quad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{L}, \langle \boldsymbol{a}(v, \boldsymbol{v}) - \boldsymbol{a}(v, \boldsymbol{w}), \boldsymbol{v} - \boldsymbol{w} \rangle_{\boldsymbol{L}', \boldsymbol{L}} \geqslant 0,$

(4.1c) $\quad \boldsymbol{a}$ is coercive in the sense that there exists $\underline{\boldsymbol{\alpha}} > 0$ such that:

$\forall v \in L, \quad \forall \boldsymbol{v} \in \boldsymbol{L},\ \underline{\boldsymbol{\alpha}} \|\boldsymbol{v}\|_{\boldsymbol{L}}^p \leqslant \langle \boldsymbol{a}(v, \boldsymbol{v}), \boldsymbol{v} \rangle_{\boldsymbol{L}', \boldsymbol{L}},$

(4.1d) $\quad$ there exists a function $\overline{\boldsymbol{\alpha}}\colon\ \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, nondecreasing with respect to its arguments, such that:

$\forall v \in L, \quad \forall \boldsymbol{v} \in \boldsymbol{L}, \|\boldsymbol{a}(v, \boldsymbol{v})\|_{\boldsymbol{L}'} \leqslant \overline{\boldsymbol{\alpha}}(\|v\|_L, \|\boldsymbol{v}\|_{\boldsymbol{L}}),$

(4.2a) $\quad a\colon\ L \to V$ is continuous for the weak-$*$ topology of L',

(4.2b) $\quad a$ is monotonous in the sense: $\forall v, w \in L, \langle a(v) - a(w), v - w \rangle_{L', L} \geqslant 0,$

(4.2c) $\quad a$ is "V-coercive" in the sense that there exists $\underline{\alpha} > 0$ such that:

$\forall v \in L,\ \underline{\alpha} |v|_{L,V}^p \leqslant \langle a(v), v \rangle_{L', L},$

(4.2d) $\quad$ there exists a nondecreasing function $\overline{\alpha}\colon\ \mathbb{R}^+ \to \mathbb{R}^+$such that:

$\forall v \in L,\ \|a(v)\|_{L'} \leqslant \overline{\alpha}(\|v\|_L).$

Remark 4.1. The existence of the nondecreasing functions $\overline{\boldsymbol{\alpha}}$, $\overline{\alpha}$ is equivalent to the boundedness of the mappings $\boldsymbol{a}$ and a, in the sense of [25] (a bounded mapping transforms any bounded set into a bounded set); this equivalence can be seen by setting, for instance, $\overline{\boldsymbol{\alpha}}(s,t) = \sup\{\|\boldsymbol{a}(v, \boldsymbol{v})\|_{\boldsymbol{L}'}\colon\ (v, \boldsymbol{v}) \in L \times \boldsymbol{L}$ with $\|v\|_L \leqslant s$ and $\|\boldsymbol{v}\|_{\boldsymbol{L}} \leqslant t\}$ and $\overline{\alpha}(s) = \sup\{a(v)\colon\ v \in L$ with $\|v\|_L \leqslant s\}$.

Remark 4.2. The framework of Section 3.2 can be extended, assuming that there exists a Banach space $\widehat{L}$ which is continuously embedded in L, such that the reconstruction operator $\mathrm{P}_{\mathcal{D}}$ has co-domain $\widehat{L}$. The coercivity Definition 3.7 is then modified, setting $C_{\mathcal{D}} = \max\limits_{v \in X_{\mathcal{D}} \setminus \{0\}} \|\mathrm{P}_{\mathcal{D}} v\|_{\widehat{L}} / \|v\|_{\mathcal{D}}$; the compactness Definition 3.10 is modified by requesting that the relative compactness of $(\mathrm{P}_{\mathcal{D}_m} u_m)_{m \in \mathbb{N}}$ holds in $\widehat{L}$. In this extended framework, replacing the space L by $\widehat{L}$ in Hypotheses (4.1), the convergence Theorem 4.12 still holds. An interesting application of this modified framework is the case, where $W_{\mathrm{G}} = W^{1,p}(\Omega)$, $L = L^p(\Omega)$ and $\widehat{L} = L^q(\Omega)$ with $q \in [p, pd/(d-p))$ for $p < d$ and $\widehat{L} = L^q(\Omega)$ with $q \in [p, \infty)$ for $p \geqslant d$; most of the numerical methods included in the GDM framework satisfy these extended coercivity and compactness definitions, see [19], Part III and Appendix B.

The next two results ensure that for any separable reflexive smooth Banach spaces, there exist operators $\boldsymbol{a}$ and a with the required properties. Let us recall the definition of a smooth Banach space.

Definition 4.3 (Strictly convex and smooth Banach spaces).

(1) A Banach space $(B, \|\cdot\|_B)$ is said to be strictly convex if $\|\cdot\|_B$ is a strictly convex mapping from B to $\mathbb{R}$.

(2) A Banach space $(B, \|\cdot\|_B)$ is said to be smooth if, for any $x \in B$ with $\|x\|_B = 1$, there exists one and only one $f \in B'$ such that $f(x) = \|f\|_{B'} = 1$.

(3) If $(B, \|\cdot\|_B)$ is smooth or strictly convex, then $(B', \|\cdot\|_{B'})$ is strictly convex or smooth, respectively.

Remark 4.4 (Equivalent strictly convex and smooth norms). Lindenstrauss proved in [26] that any reflexive Banach space has an equivalent strictly convex norm and an equivalent smooth norm.

Lemma 4.5 (Existence of $\boldsymbol{a}$). *Assume that $\boldsymbol{L}$ is smooth and define the duality mapping $T\colon \boldsymbol{L} \to \boldsymbol{L}'$ associated with the gauge $\mu(s) = s^{p-1}$. We recall that this mapping is defined by: for any $\boldsymbol{v} \in \boldsymbol{L}$, $T(\boldsymbol{v})$ is the unique element in $\boldsymbol{L}'$ such that*

$$\forall\, \boldsymbol{v} \in \boldsymbol{L}, \quad \langle T(\boldsymbol{v}), \boldsymbol{v}\rangle_{\boldsymbol{L}',\boldsymbol{L}} = \|\boldsymbol{v}\|_{\boldsymbol{L}}\mu(\|\boldsymbol{v}\|_{\boldsymbol{L}}) \quad \text{and} \quad \|T(\boldsymbol{v})\|_{\boldsymbol{L}'} = \mu(\|\boldsymbol{v}\|_{\boldsymbol{L}}). \tag{4.3}$$

Then $\boldsymbol{a}$, defined for all $(v, \boldsymbol{v}) \in L \times \boldsymbol{L}$ by $\boldsymbol{a}(v, \boldsymbol{v}) = T(\boldsymbol{v})$, satisfies Assumptions (4.1a)–(4.1d).

Proof. From [9], [12], [13], the mapping T exists and is continuous for the weak-$*$ topology of $\boldsymbol{L}'$ (its uniqueness is a consequence of the fact that the norm of $\boldsymbol{L}'$ is strictly convex).

The boundedness mentioned in (4.1d) is obvious (with $\overline{\boldsymbol{\alpha}}(s,t) = t^{p-1}$), as well as the coercivity (4.1c) (with $\underline{\boldsymbol{\alpha}} = 1$). It remains to check the monotonicity of T, which in turn implies (4.1b). By developing the duality product and using the definition of T, we have

$$\langle T(\boldsymbol{v}) - T(\boldsymbol{w}), \boldsymbol{v} - \boldsymbol{w}\rangle_{\boldsymbol{L}',\boldsymbol{L}} = \|\boldsymbol{v}\|_{\boldsymbol{L}}^p + \|\boldsymbol{w}\|_{\boldsymbol{L}}^p - \langle T(\boldsymbol{v}), \boldsymbol{w}\rangle_{\boldsymbol{L}',\boldsymbol{L}} - \langle T(\boldsymbol{w}), \boldsymbol{v}\rangle_{\boldsymbol{L}',\boldsymbol{L}}.$$

Therefore

$$\begin{aligned}\langle T(\boldsymbol{v}) - T(\boldsymbol{w}), \boldsymbol{v} - \boldsymbol{w}\rangle_{\boldsymbol{L}',\boldsymbol{L}} &\geqslant \|\boldsymbol{v}\|_{\boldsymbol{L}}^p + \|\boldsymbol{w}\|_{\boldsymbol{L}}^p - \|\boldsymbol{v}\|_{\boldsymbol{L}}^{p-1}\|\boldsymbol{w}\|_{\boldsymbol{L}} - \|\boldsymbol{w}\|_{\boldsymbol{L}}^{p-1}\|\boldsymbol{v}\|_{\boldsymbol{L}} \\ &= (\|\boldsymbol{v}\|_{\boldsymbol{L}}^{p-1} - \|\boldsymbol{w}\|_{\boldsymbol{L}}^{p-1})(\|\boldsymbol{v}\|_{\boldsymbol{L}} - \|\boldsymbol{w}\|_{\boldsymbol{L}}) \geqslant 0,\end{aligned}$$

since the function $s \mapsto s^{p-1}$ is increasing on $\mathbb{R}^+$. □

Remark 4.6. In the case $\boldsymbol{L} = L^p(\Omega)^d$, the operator T defined by (4.3) is $\boldsymbol{v} \mapsto T(\boldsymbol{v}) = |\boldsymbol{v}|^{p-2}\boldsymbol{v}$.

Lemma 4.7 (Existence of a). *Assume that L is smooth. Define $\widetilde{a}\colon L \to L'$ by*

$$u \mapsto \widetilde{a}(u) := \arg\max\{\langle \mu, u\rangle_{L',L}; \mu \in V \text{ such that } \|\mu\|_{L'} = 1\}.$$

Then for any $u \in L$ one has $\langle \widetilde{a}(u), u\rangle_{L',L} = |u|_{L,V}$ and the mapping $a\colon L \to L'$ defined by $a(u) := |\langle \widetilde{a}(u), u\rangle_{L',L}|^{p-1}\widetilde{a}(u)$ satisfies Hypotheses (4.2a)–(4.2d).

Proof. The relation $\langle \widetilde{a}(u), u\rangle_{L',L} = |u|_{L,V}$ is an immediate consequence of the definitions of $\widetilde{a}$ and $|\cdot|_{L,V}$. The proof that a satisfies the required properties is similar to that of Lemma 4.5. $\square$

Remark 4.8. If $V = \operatorname{span}(\mu_1, \ldots, \mu_r)$, a possible choice of a that satisfies (4.2a)–(4.2d) is given by

$$a(u) = \sum_{i=1}^{r} |\langle \mu_i, u\rangle_{L',L}|^{p-2}\langle \mu_i, u\rangle_{L',L}\mu_i.$$

In the case $r = 1$, this operator a is the one defined in Lemma 4.7.

For any $b \in (W_{\mathrm{G}})'$ (the space of linear continuous forms for the norm $\|\cdot\|_{W_{\mathrm{G}},\mathcal{G}}$), the abstract Leray-Lions problem reads in its weak form

(4.4) Find $\bar{u} \in W_{\mathrm{G}}$ such that

$$\forall\, v \in W_{\mathrm{G}}, \quad \langle \boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u}), \mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\bar{u}), v\rangle_{L',L} = \langle b, v\rangle_{(W_{\mathrm{G}})',W_{\mathrm{G}}}.$$

The following lemma will enable us to write an equivalent form for this problem.

Lemma 4.9. *If $b \in (W_{\mathrm{G}})'$ then there exists $(f, \boldsymbol{F}) \in L' \times \boldsymbol{L}'$ such that*

$$\forall\, v \in W_{\mathrm{G}}, \quad \langle b, v\rangle_{(W_{\mathrm{G}})',W_{\mathrm{G}}} = \langle f, v\rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}.$$

Proof. Let $I\colon W_{\mathrm{G}} \to L \times \boldsymbol{L}$ be the embedding $I(v) = (v, \mathrm{G}v)$. Define $\widetilde{b}\colon \operatorname{Im}(I) \to \mathbb{R}$ by $\widetilde{b}(I(v)) = \langle b, v\rangle_{(W_{\mathrm{G}})',W_{\mathrm{G}}}$. Then $\widetilde{b}$ is linear and $|\widetilde{b}(I(v))| \leqslant \|b\|_{(W_{\mathrm{G}})'}\|v\|_{W_{\mathrm{G}},\mathcal{G}} = \|b\|_{(W_{\mathrm{G}})'}(\|v\|_L + \|\mathrm{G}v\|_{\boldsymbol{L}})$. The Hahn-Banach extension theorem then enables us to extend $\widetilde{b}$ as a continuous linear form on $L \times \boldsymbol{L}$. Any such form can be represented as $\widetilde{b}(v, \boldsymbol{v}) = \langle f, v\rangle_{L',L} - \langle \boldsymbol{F}, \boldsymbol{v}\rangle_{\boldsymbol{L}',\boldsymbol{L}}$ for some $(f, \boldsymbol{F}) \in L' \times \boldsymbol{L}'$, and the proof is completed by the choice of $\widetilde{b}$ on $\operatorname{Im}(I)$. $\square$

Using $(f, \boldsymbol{F})$ provided by the preceding lemma, without loss of generality the problem (4.4) can be re-written as

(4.5) Find $\bar{u} \in W_{\mathrm{G}}$ such that

$$\forall\, v \in W_{\mathrm{G}}, \quad \langle \boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u}), \mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\bar{u}), v\rangle_{L',L} = \langle f, v\rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}.$$

As proved in Lemma 4.10 below, an equivalent form of Problem (4.5) reads:

(4.6) Find $\bar{u} \in W_{\mathrm{G}}$ such that

$$\boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u}) + \boldsymbol{F} \in \boldsymbol{W}_{\mathrm{D}} \quad \text{and} \quad -\mathrm{D}(\boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u}) + \boldsymbol{F}) + a(\bar{u}) = f.$$

Lemma 4.10. *Problems* (4.6) *and* (4.5) *are equivalent.*

Proof. Let $\bar{u} \in W_{\mathrm{G}}$ be a solution to Problem (4.6). The equation in this formulation is a relation between elements of L'. Applying this equation to a generic $v \in W_{\mathrm{G}}$ and using (3.3) shows that $\bar{u}$ is a solution to Problem (4.5).

Reciprocally, take a solution $\bar{u} \in W_{\mathrm{G}}$ to Problem (4.5). Then the equation in (4.5) shows that, for all $v \in W_{\mathrm{G}}$,

$$\langle \boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u}) + \boldsymbol{F}, \mathrm{G}v \rangle_{\boldsymbol{L}', \boldsymbol{L}} + \langle a(\bar{u}) - f, v \rangle_{L', L} = 0.$$

By definition (3.1) of $\boldsymbol{W}_{\mathrm{D}}$, this shows that $\boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u}) + \boldsymbol{F} \in \boldsymbol{W}_{\mathrm{D}}$ and that $\mathrm{D}(\boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u}) + \boldsymbol{F}) = a(\bar{u}) - f$, which is exactly (4.6). □

Lemma 4.11 (Existence of a solution to (4.5))**.** *Under Assumptions* (4.1a)–(4.2d), *there exists at least one solution to Problem* (4.5).

Proof. The fact that in the framework of this section, there exists at least one solution to Problem (4.5), is a by-product of the convergence Theorem 4.12 below and of the existence result given in Lemma 3.18. But it is also a consequence of [25], Théorème 1, in which the Banach space denoted by V corresponds to W_{G} here, and in which the operators denoted by $\boldsymbol{A}(u)$ and $A(u, v)$ are defined as follows.

▷ If we assume that $\boldsymbol{a}$ only depends on its second argument, we define $\boldsymbol{A}\colon W_{\mathrm{G}} \to W'_{\mathrm{G}}$ by:

$$\forall\, u, w \in W_{\mathrm{G}}, \quad \langle \boldsymbol{A}(u), w \rangle_{W_{\mathrm{G}}, W'_{\mathrm{G}}} = \langle \boldsymbol{a}(\mathrm{G}u), \mathrm{G}w \rangle_{\boldsymbol{L}', \boldsymbol{L}} + \langle a(u), w \rangle_{L', L}.$$

Then, owing to the monotony and boundedness hypotheses on $\boldsymbol{a}$ and a, Hypothèse I in [25], is satisfied.

▷ In the case, where $\boldsymbol{a}$ may also depend on its first argument, if we moreover assume that the embedding of W_{G} in L is compact, we define $A\colon W_{\mathrm{G}} \times W_{\mathrm{G}} \to W'_{\mathrm{G}}$, by:

$$\forall\, u, v, w \in W_{\mathrm{G}}, \quad \langle A(u, v), w \rangle_{W_{\mathrm{G}}, W'_{\mathrm{G}}} = \langle \boldsymbol{a}(u, \mathrm{G}v), \mathrm{G}w \rangle_{\boldsymbol{L}', \boldsymbol{L}} + \langle a(v), w \rangle_{L', L}.$$

Then, owing to Assumptions (4.1a)–(4.2d), Hypothèse II in [25], is satisfied.

This justifies the fact that we call Problem (4.5) an abstract Leray-Lions problem. □

Given a gradient discretisation $\mathcal{D}$, the gradient scheme (GS) for Problem (4.5) is: find $u \in X_{\mathcal{D}}$ such that

$$\forall v \in X_{\mathcal{D}}, \quad \langle \boldsymbol{a}(\mathrm{P}_{\mathcal{D}}u, \mathrm{G}_{\mathcal{D}}u), \mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\mathrm{P}_{\mathcal{D}}u), \mathrm{P}_{\mathcal{D}}v\rangle_{L',L} = \langle f, \mathrm{P}_{\mathcal{D}}v\rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}. \tag{4.7}$$

Theorem 4.12 (Convergence of the GS, abstract Leray-Lions problems)**.** *Under Assumptions* (4.1a)–(4.2d)*, take a sequence* $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *of GDs in the sense of Definition* 3.6*, which is consistent and limit-conforming in the sense of Definitions* 3.8 *and* 3.9*.*

Then for any $m \in \mathbb{N}$ *there exists at least one* $u_m \in X_{\mathcal{D}_m}$ *solution to the gradient scheme* (4.7)*. Moreover:*

- ▷ *If we assume that* $\boldsymbol{a}$ *only depends on its second argument, then there exists a solution* $\bar{u}$ *of* (4.5) *such that, up to a subsequence,* $\mathrm{P}_{\mathcal{D}_m}u_m$ *converges weakly in* L *to* $\bar{u}$ *and* $\mathrm{G}_{\mathcal{D}_m}u_m$ *converges weakly in* $\boldsymbol{L}$ *to* $\mathrm{G}\bar{u}$ *as* $m \to \infty$*.*
- ▷ *In the case where* $\boldsymbol{a}$ *may also depend on its first argument, if we moreover assume that the sequence* $(\mathcal{D}_m)_{m\in\mathbb{N}}$ *of GDs is compact in the sense of Definition* 3.10 *(this assumption implies that the embedding of* W_{G} *in* L *is compact, see Lemma* 3.16*), then there exists a solution* $\bar{u}$ *of* (4.5) *such that, up to a subsequence,* $\mathrm{P}_{\mathcal{D}_m}u_m$ *converges strongly in* L *to* $\bar{u}$ *and* $\mathrm{G}_{\mathcal{D}_m}u_m$ *converges weakly in* $\boldsymbol{L}$ *to* $\mathrm{G}\bar{u}$ *as* $m \to \infty$*.*

In the case when the solution $\bar{u}$ *of* (4.5) *is unique, the above convergence results hold for the whole sequence.*

Proof. *Step 1*: Existence of a solution to the scheme.

Let $\mathcal{D}$ be a GD in the sense of Definition 3.6. We endow the finite dimensional space $X_{\mathcal{D}}$ with an inner product $\langle,\rangle$ and denote by $|\cdot|$ its related norm. Define $F\colon X_{\mathcal{D}} \to X_{\mathcal{D}}$ as the function such that, if $u \in X_{\mathcal{D}}$, then $F(u)$ is the unique element in $X_{\mathcal{D}}$ which satisfies

$$\forall v \in X_{\mathcal{D}}, \quad \langle F(u), v\rangle = \langle \boldsymbol{a}(\mathrm{P}_{\mathcal{D}}u, \mathrm{G}_{\mathcal{D}}u), \mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\mathrm{P}_{\mathcal{D}}u), \mathrm{P}_{\mathcal{D}}v\rangle_{L',L}.$$

Likewise, we denote by $w \in X_{\mathcal{D}}$ the unique element such that

$$\forall v \in X_{\mathcal{D}}, \quad \langle w, v\rangle = \langle f, \mathrm{P}_{\mathcal{D}}v\rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}.$$

The assumptions on $\boldsymbol{a}$ and a show that F is continuous and that for all $u \in X_{\mathcal{D}}$ $\langle F(u), u\rangle \geqslant \underline{\boldsymbol{\alpha}}\|\mathrm{G}_{\mathcal{D}}u\|^p_{\boldsymbol{L}} + \underline{\alpha}|\mathrm{P}_{\mathcal{D}}u|^p_{L,V} \geqslant 2^{1-p}\min(\underline{\boldsymbol{\alpha}}, \underline{\alpha})\|u\|^p_{\mathcal{D}}$. By equivalence of the norms on the finite dimensional space $X_{\mathcal{D}}$, this shows that $\langle F(u), u\rangle \geqslant C_1|u|^p$ with C_1 not depending on u. Hence $\lim\limits_{|u|\to\infty} \langle F(u), u\rangle/|u| = \infty$ and F is surjective (see [25] or [16], Theorem 3.3, page 19). Therefore there exists $u \in X_{\mathcal{D}}$ such that $F(u) = w$, which means that u is a solution to (4.7).

Step 2: Convergence to a solution of the continuous problem.

As in the statement of the theorem, assume that u_m is a solution to (4.7) with $\mathcal{D} = \mathcal{D}_m$. Letting $v = u_m$ in (4.7) with $\mathcal{D} = \mathcal{D}_m$ and using (3.9), (4.1c) and (4.2c), we get

$$\begin{aligned} 2^{1-p}\min(\underline{\boldsymbol{\alpha}}, \underline{\alpha})\|u_m\|_{\mathcal{D}_m}^p &\leqslant \underline{\boldsymbol{\alpha}}\|\mathrm{G}_{\mathcal{D}_m}u_m\|_{\boldsymbol{L}}^p + \underline{\alpha}|\mathrm{P}_{\mathcal{D}_m}u_m|_{L,V}^p \\ &\leqslant (C_{\mathcal{D}_m}\|f\|_{L'} + \|\boldsymbol{F}\|_{\boldsymbol{L}'})\|u_m\|_{\mathcal{D}_m}. \end{aligned}$$

Thanks to the coercivity of the sequence of GDs, this provides an estimate on $\mathrm{G}_{\mathcal{D}_m}u_m$ in $\boldsymbol{L}$ and on $\mathrm{P}_{\mathcal{D}_m}u_m$ in L. Lemma 3.13 then gives $\bar{u} \in W_{\mathrm{G}}$ such that, up to a subsequence, $\mathrm{P}_{\mathcal{D}_m}u_m \to \bar{u}$ weakly in L and $\mathrm{G}_{\mathcal{D}_m}u_m \to \mathrm{G}\bar{u}$ weakly in $\boldsymbol{L}$. In the case when $\boldsymbol{a}$ may depend on its first argument, by compactness of the sequence of GDs, we can also assume that the convergence of $\mathrm{P}_{\mathcal{D}_m}u_m$ to $\bar{u}$ is strong in L.

By Hypothesis (4.1d), the sequence $(\boldsymbol{a}(\mathrm{P}_{\mathcal{D}_m}u_m, \mathrm{G}_{\mathcal{D}_m}u_m))_{m\in\mathbb{N}}$ of elements of $\boldsymbol{L}'$ remains bounded in $\boldsymbol{L}'$ and converges therefore, up to a subsequence, to some $\boldsymbol{A}$ weakly in $\boldsymbol{L}'$, as $m \to \infty$. Similarly, by Hypothesis (4.2d), the sequence $a(\mathrm{P}_{\mathcal{D}_m}u_m)$ of elements of L' remains bounded in L' and converges therefore, up to a subsequence, to some A weakly in L', as $m \to \infty$.

Let us now show that $\bar{u}$ is a solution to (4.5), using the well-known Minty trick, see [29]. For a given $\varphi \in W_{\mathrm{G}}$ and for any gradient discretisation $\mathcal{D}$ in the sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$, we introduce

$$I_{\mathcal{D}}\varphi \in \operatorname*{arg\,min}_{v\in X_{\mathcal{D}}}(\|\mathrm{P}_{\mathcal{D}}v - \varphi\|_L + \|\mathrm{G}_{\mathcal{D}}v - \mathrm{G}\varphi\|_{\boldsymbol{L}})$$

as a test function in (4.7). By the consistency of $(\mathcal{D}_m)_{m\in\mathbb{N}}$, $\mathrm{P}_{\mathcal{D}_m}I_{\mathcal{D}_m}\varphi \to \varphi$ in L and $\mathrm{G}_{\mathcal{D}_m}I_{\mathcal{D}_m}\varphi \to \mathrm{G}\varphi$ in $\boldsymbol{L}$, as $m \to \infty$. Hence, letting $m \to \infty$ in the gradient scheme, we obtain

$$\forall \varphi \in W_{\mathrm{G}}, \quad \langle \boldsymbol{A}, \mathrm{G}\varphi\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle A, \varphi\rangle_{L',L} = \langle f, \varphi\rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}\varphi\rangle_{\boldsymbol{L}',\boldsymbol{L}}. \tag{4.8}$$

On the other hand, we may take u_m as a test function in (4.7) and let $m \to \infty$. Using (4.8) with $\varphi = \bar{u}$, this leads to

$$\begin{aligned} &\lim_{m\to\infty}(\langle \boldsymbol{a}(\mathrm{P}_{\mathcal{D}_m}u_m, \mathrm{G}_{\mathcal{D}_m}u_m), \mathrm{G}_{\mathcal{D}_m}u_m\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\mathrm{P}_{\mathcal{D}_m}u_m), \mathrm{P}_{\mathcal{D}_m}u_m\rangle_{L',L}) \\ &\qquad = \langle f, \bar{u}\rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}\bar{u}\rangle_{\boldsymbol{L}',\boldsymbol{L}} = \langle \boldsymbol{A}, \mathrm{G}\bar{u}\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\bar{u}), \bar{u}\rangle_{L',L}. \end{aligned} \tag{4.9}$$

Hypotheses (4.1b) and (4.2b) give, for any $\overline{v} \in W_{\mathrm{G}}$,

$$\begin{aligned} &\langle \boldsymbol{a}(\mathrm{P}_{\mathcal{D}_m}u_m, \mathrm{G}_{\mathcal{D}_m}u_m) - \boldsymbol{a}(\mathrm{P}_{\mathcal{D}_m}u_m, \mathrm{G}\overline{v}), \mathrm{G}_{\mathcal{D}_m}u_m - \mathrm{G}\overline{v}\rangle_{\boldsymbol{L}',\boldsymbol{L}} \\ &\qquad + \langle a(\mathrm{P}_{\mathcal{D}_m}u_m) - a(\overline{v}), \mathrm{P}_{\mathcal{D}_m}u_m - \overline{v}\rangle_{L',L} \geqslant 0. \end{aligned} \tag{4.10}$$

Developing this, using (4.9) to identify the limit of the sole term

$$\langle \boldsymbol{a}(\mathrm{P}_{\mathcal{D}_m}u_m, \mathrm{G}_{\mathcal{D}_m}u_m), \mathrm{G}_{\mathcal{D}_m}u_m\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\mathrm{P}_{\mathcal{D}_m}u_m), \mathrm{P}_{\mathcal{D}_m}u_m\rangle_{L',L}$$

involving a product of two weak convergences and using the (strong) continuity of $\boldsymbol{a}$ with respect to its first argument (the second argument is $\mathrm{G}\overline{v}$), we may let $m\to\infty$ to get

$$\langle \boldsymbol{A} - \boldsymbol{a}(\bar{u}, \mathrm{G}\overline{v}), \mathrm{G}\bar{u} - \mathrm{G}\overline{v}\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle A - a(\overline{v}), \bar{u} - \overline{v}\rangle_{L',L} \geqslant 0.$$

Set $\overline{v} = \bar{u} + sv$ in the preceding inequality, where $v\in W_{\mathrm{G}}$ and $s>0$. Dividing by s, we get

$$\langle \boldsymbol{A} - \boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u} + s\mathrm{G}v), \mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle A - a(\bar{u}+sv), v\rangle_{L',L} \geqslant 0.$$

Letting $s\to 0$ and using the continuity of $\boldsymbol{a}(\bar{u},\cdot)$ for the weak topology of $\boldsymbol{L}'$ and the continuity of a for the weak topology of L' leads to

$$\forall\, v\in W_{\mathrm{G}},\quad \langle \boldsymbol{A} - \boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u}), \mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle A - a(\bar{u}), v\rangle_{L',L} \geqslant 0.$$

Changing v into $-v$ shows that $\langle \boldsymbol{A}, \mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle A, v\rangle_{L',L} = \langle \boldsymbol{a}(\bar{u}, \mathrm{G}\bar{u}), \mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\bar{u}), v\rangle_{L',L}$. Using this relation in (4.8) with $\varphi = v$, this concludes the proof that $\bar{u}$ is a solution of (4.5). □

5. Approximation of a linear elliptic problem

We consider here a particular case of Problem (4.5) or(4.6). We take $p=2$ and assume that there exist $\overline{\alpha}>0$ and $\underline{\alpha}>0$ such that

(5.1a) $\boldsymbol{a}\colon \boldsymbol{L}\to\boldsymbol{L}'$ is linear continuous with norm bounded by $\overline{\alpha}$,

(5.1b) $\boldsymbol{a}$ is $\underline{\alpha}$-coercive: $\forall\, v\in L,\quad \forall\, \boldsymbol{v}\in\boldsymbol{L},\ \underline{\alpha}\|\boldsymbol{v}\|_{\boldsymbol{L}}^2 \leqslant \langle \boldsymbol{a}(\boldsymbol{v}), \boldsymbol{v}\rangle_{\boldsymbol{L}',\boldsymbol{L}}$,

(5.1c) $a\colon L\to L'$ is linear and continuous with norm bounded by $\overline{\alpha}$,

(5.1d) a is $\underline{\alpha}$-coercive: $\forall\, v\in L,\ \underline{\alpha}|v|_{L,V}^2 \leqslant \langle a(v), v\rangle_{L',L}$.

Then $\boldsymbol{L}$ is a Hilbert space when endowed with the scalar product

$$(\boldsymbol{v},\boldsymbol{w})\mapsto \tfrac{1}{2}(\langle \boldsymbol{a}(\boldsymbol{v}), \boldsymbol{w}\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \boldsymbol{a}(\boldsymbol{w}), \boldsymbol{v}\rangle_{\boldsymbol{L}',\boldsymbol{L}}).$$

Hypotheses (5.1a)–(5.1d) imply

(5.2) $$\forall\, u\in W_{\mathrm{G}},\quad \underline{\alpha}\|u\|_{W_{\mathrm{G}}}^2 \leqslant \langle \boldsymbol{a}(\mathrm{G}u), \mathrm{G}u\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(u), u\rangle_{L',L} \leqslant \overline{\alpha}\|u\|_{W_{\mathrm{G}}}^2,$$

which shows that W_{G} is a Hilbert space when endowed with the scalar product

$$(5.3)\qquad (u,v)\mapsto \langle u,v\rangle_{W_{\mathrm{G}}} := \tfrac{1}{2}(\langle \boldsymbol{a}(\mathrm{G}u),\mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \boldsymbol{a}(\mathrm{G}v),\mathrm{G}u\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(u),v\rangle_{L',L} + \langle a(v),u\rangle_{L',L}).$$

For any $(f,\boldsymbol{F})\in L'\times\boldsymbol{L}'$, the abstract linear elliptic problem reads:

$$(5.4)\qquad \begin{aligned}&\text{Find } \bar u\in W_{\mathrm{G}} \text{ such that,}\\ &\forall v\in W_{\mathrm{G}},\quad \langle \boldsymbol{a}(\mathrm{G}\bar u),\mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\bar u),v\rangle_{L',L} = \langle f,v\rangle_{L',L} - \langle \boldsymbol{F},\mathrm{G}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}\end{aligned}$$

or, by Lemma 4.10:

$$(5.5)\qquad \begin{aligned}&\text{Find } \bar u\in W_{\mathrm{G}} \text{ such that}\\ &\boldsymbol{a}(\mathrm{G}\bar u)+\boldsymbol{F}\in \boldsymbol{W}_{\mathrm{D}} \quad\text{and}\quad -\mathrm{D}(\boldsymbol{a}(\mathrm{G}\bar u)+\boldsymbol{F}) + a(\bar u) = f.\end{aligned}$$

Theorem 5.1 (Existence and uniqueness of a solution to (5.4))**.** *Under Hypothesis* (5.1a)–(5.1d)*, there exists one and only one solution to Problem* (5.4)*.*

Proof. This is an immediate consequence of Lax-Milgram theorem, on the Hilbert space W_{G} endowed with the inner product defined by (5.3). □

Table 5 presents the links between this abstract linear elliptic setting and the standard elliptic PDE, for all BCs proposed in the introduction of this paper.

B.C.	homogeneous Dirichlet	homogeneous Neumann	nonhomogeneous Neumann	Fourier
L	$L^2(\Omega)$	$L^2(\Omega)$	$L^2(\Omega)\times L^2(\partial\Omega)$	$L^2(\Omega)\times L^2(\partial\Omega)$
$\boldsymbol{L}$	$L^2(\Omega)^d$	$L^2(\Omega)^d$	$L^2(\Omega)^d$	$L^2(\Omega)^d$
$\boldsymbol{a}$:	$\boldsymbol{v}\mapsto\Lambda\boldsymbol{v}$	$\boldsymbol{v}\mapsto\Lambda\boldsymbol{v}$	$\boldsymbol{v}\mapsto\Lambda\boldsymbol{v}$	$\boldsymbol{v}\mapsto\Lambda\boldsymbol{v}$
a:	$u\mapsto 0$	$u\mapsto(\int_\Omega u)1_\Omega$	$(u,w)\mapsto(\int_\Omega u)(1_\Omega,0)$	$(u,w)\mapsto(0,bw)$

Table 5. Link between the abstract linear elliptic problem and the usual elliptic PDE $-\operatorname{div}(\Lambda\nabla\bar u) = f + \operatorname{div}(\mathbf{F})$, for various boundary conditions.

Given a gradient discretisation $\mathcal{D}$ in the sense of Definition 3.6, we consider the following scheme for the approximation of Problem (5.4): Find $u\in X_{\mathcal{D}}$ such that

$$(5.6)\qquad \forall v\in X_{\mathcal{D}},\quad \langle \boldsymbol{a}(\mathrm{G}_{\mathcal{D}}u),\mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\mathrm{P}_{\mathcal{D}}u),\mathrm{P}_{\mathcal{D}}v\rangle_{L',L} = \langle f,\mathrm{P}_{\mathcal{D}}v\rangle_{L',L} - \langle \boldsymbol{F},\mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}.$$

For a given basis $(\xi^{(i)})_{i=1,\ldots,N}$ of $X_{\mathcal{D}}$, the scheme (5.6) is equivalent to solving the linear square system $AU = B$, where

$$
\begin{aligned}
(5.7)\qquad\qquad u &= \sum_{j=1}^{N} U_j \xi^{(j)},\\
A_{ij} &= \langle \boldsymbol{a}(\mathrm{G}_{\mathcal{D}}\xi^{(j)}), \mathrm{G}_{\mathcal{D}}\xi^{(i)}\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\mathrm{P}_{\mathcal{D}}\xi^{(j)}), \mathrm{P}_{\mathcal{D}}\xi^{(i)}\rangle_{L',L},\\
B_i &= \langle f, \mathrm{P}_{\mathcal{D}}\xi^{(i)}\rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}_{\mathcal{D}}\xi^{(i)}\rangle_{\boldsymbol{L}',\boldsymbol{L}}.
\end{aligned}
$$

The next theorem gives an error estimate for the gradient scheme (5.6).

Theorem 5.2 (Error estimate, abstract linear elliptic problem)**.** *Under Assumptions* (5.1a)–(5.1d)*, let $\bar u \in W_{\mathrm{G}}$ be the solution to Problem* (5.4) *and let $\mathcal{D}$ be a GD in the sense of Definition* 3.6*. Then there exists one and only one $u_{\mathcal{D}} \in X_{\mathcal{D}}$ solution to the GS* (5.6)*. This solution satisfies the inequalities*

$$
(5.8)\quad \|\mathrm{G}\bar u - \mathrm{G}_{\mathcal{D}}u_{\mathcal{D}}\|_{\boldsymbol{L}} \leqslant \frac{1}{\underline{\alpha}}(W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar u) + \boldsymbol{F}) + (\overline{\alpha}(1 + C_{\mathcal{D}}) + \underline{\alpha})S_{\mathcal{D}}(\bar u)),
$$

$$
(5.9)\quad \|\bar u - \mathrm{P}_{\mathcal{D}}u_{\mathcal{D}}\|_{L} \leqslant \frac{1}{\underline{\alpha}}(C_{\mathcal{D}}W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar u) + \boldsymbol{F}) + (C_{\mathcal{D}}(1 + C_{\mathcal{D}})\overline{\alpha} + \underline{\alpha})S_{\mathcal{D}}(\bar u)),
$$

where $C_{\mathcal{D}}$, $S_{\mathcal{D}}$ and $W_{\mathcal{D}}$ are respectively the norm of the reconstruction operator $\mathrm{P}_{\mathcal{D}}$, the consistency measure and the conformity defect, defined by (3.9), (3.10) *and* (3.12)*.*

Moreover, we also have the reverse inequalities

$$
(5.10)\qquad W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar u) + \boldsymbol{F}) \leqslant \overline{\alpha}\|\mathrm{G}\bar u - \mathrm{G}_{\mathcal{D}}u_{\mathcal{D}}\|_{\boldsymbol{L}},
$$

$$
(5.11)\qquad S_{\mathcal{D}}(\bar u) \leqslant \|\bar u - \mathrm{P}_{\mathcal{D}}u_{\mathcal{D}}\|_{L} + \|\mathrm{G}\bar u - \mathrm{G}_{\mathcal{D}}u_{\mathcal{D}}\|_{\boldsymbol{L}},
$$

which shows the existence of $C_2 > 0$ and $C_3 > 0$, only depending on $\overline{\alpha}$ and $\underline{\alpha}$, such that

$$
\begin{aligned}
(5.12)\quad \frac{C_2}{1 + C_{\mathcal{D}}}(S_{\mathcal{D}}(\bar u) + W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar u) + \boldsymbol{F})) &\leqslant \|\bar u - \mathrm{P}_{\mathcal{D}}u_{\mathcal{D}}\|_{L} + \|\mathrm{G}\bar u - \mathrm{G}_{\mathcal{D}}u_{\mathcal{D}}\|_{\boldsymbol{L}}\\
&\leqslant C_3(1 + C_{\mathcal{D}})^2(S_{\mathcal{D}}(\bar u) + W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar u) + \boldsymbol{F})).
\end{aligned}
$$

P r o o f. Let us first prove that, if (5.8)–(5.9) holds for any solution $u_{\mathcal{D}} \in X_{\mathcal{D}}$ to Scheme (5.6), then the solution to this scheme exists and is unique. To this purpose, we prove that if (5.8) holds then the matrix A of the linear system (5.7) is nonsingular, i.e. that if $AU = 0$ then $U = 0$. Thus, we consider the particular case, where $f = 0$ and $\boldsymbol{F} = 0$, which gives a zero right-hand side. In this case the solution $\bar u$

of (5.4) is equal to zero. Then from (5.8)–(5.9), any solution to the scheme satisfies $\|u_{\mathcal{D}}\|_{\mathcal{D}} = 0$. Since $\|\cdot\|_{\mathcal{D}}$ is a norm on $X_{\mathcal{D}}$ this leads to $u_{\mathcal{D}} = 0$. Therefore (5.7) (as well as (5.6)) has a unique solution for any right-hand side f and $\boldsymbol{F}$.

Let us now prove that any solution $u_{\mathcal{D}} \in X_{\mathcal{D}}$ to Scheme (5.6) satisfies (5.8) and (5.9). Let $\boldsymbol{\varphi} = \boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F}$; then $\boldsymbol{\varphi}$ belongs to $\boldsymbol{W}_{\mathrm{D}}$ and can thus be considered in the definition (3.10) of $W_{\mathcal{D}}$. This gives, for any $v \in X_{\mathcal{D}}$,

$$|\langle \boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F}, \mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle \mathrm{D}(\boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F}), \mathrm{P}_{\mathcal{D}}v\rangle_{L',L}| \leqslant \|v\|_{\mathcal{D}}\, W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F}).$$

Since $-f + a(\bar{u}) = \mathrm{D}(\boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F})$, this yields

$$|\langle \boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F}, \mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle -f + a(\bar{u}), \mathrm{P}_{\mathcal{D}}v\rangle_{L',L}| \leqslant \|v\|_{\mathcal{D}}\, W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F}). \tag{5.13}$$

Using the gradient scheme (5.6) to replace the terms involving f and $\mathbf{F}$ on the left-hand side, we infer

$$\begin{aligned}|\langle \boldsymbol{a}(\mathrm{G}\bar{u} - \mathrm{G}_{\mathcal{D}}u_{\mathcal{D}}), \mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\bar{u} - \mathrm{P}_{\mathcal{D}}u_{\mathcal{D}}), \mathrm{P}_{\mathcal{D}}v\rangle_{L',L}|& \\ \leqslant \|v\|_{\mathcal{D}}\, W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F}).&\end{aligned} \tag{5.14}$$

Define $I_{\mathcal{D}}\bar{u} = \underset{w \in X_{\mathcal{D}}}{\arg\min}(\|\mathrm{P}_{\mathcal{D}}w - \bar{u}\|_L + \|\mathrm{G}_{\mathcal{D}}w - \mathrm{G}\bar{u}\|_{\boldsymbol{L}})$ and notice that, by definition (3.12) of $S_{\mathcal{D}}$,

$$\|\mathrm{P}_{\mathcal{D}}I_{\mathcal{D}}\bar{u} - \bar{u}\|_L + \|\mathrm{G}_{\mathcal{D}}I_{\mathcal{D}}\bar{u} - \mathrm{G}\bar{u}\|_{\boldsymbol{L}} = S_{\mathcal{D}}(\bar{u}). \tag{5.15}$$

Recalling the definition of $\|\cdot\|_{\mathcal{D}}$ in Definition 3.6, introducing $\mathrm{G}\bar{u}$ and $\mathrm{P}\bar{u}$ and using (5.14) gives

$$\begin{aligned}\langle \boldsymbol{a}(\mathrm{G}_{\mathcal{D}}I_{\mathcal{D}}\bar{u} - \mathrm{G}_{\mathcal{D}}u_{\mathcal{D}}), \mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} &+ \langle a(\mathrm{P}_{\mathcal{D}}I_{\mathcal{D}}\bar{u} - \mathrm{P}_{\mathcal{D}}u_{\mathcal{D}}), \mathrm{P}_{\mathcal{D}}v\rangle_{L',L}\\ &\leqslant \|v\|_{\mathcal{D}}\, W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F})\\ &\quad + |\langle \boldsymbol{a}(\mathrm{G}_{\mathcal{D}}I_{\mathcal{D}}\bar{u} - \mathrm{G}\bar{u}), \mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\mathrm{P}_{\mathcal{D}}I_{\mathcal{D}}\bar{u} - \bar{u}), \mathrm{P}_{\mathcal{D}}v\rangle_{L',L}|\\ &\leqslant \|v\|_{\mathcal{D}}[W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F}) + \overline{\alpha}(\|\mathrm{G}_{\mathcal{D}}I_{\mathcal{D}}\bar{u} - \mathrm{G}\bar{u}\|_{\boldsymbol{L}} + C_{\mathcal{D}}\|\mathrm{P}_{\mathcal{D}}I_{\mathcal{D}}\bar{u} - \bar{u}\|_L)]\\ &\leqslant \|v\|_{\mathcal{D}}[W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F}) + \overline{\alpha}(1 + C_{\mathcal{D}})S_{\mathcal{D}}(\bar{u})].\end{aligned}$$

Choose $v = I_{\mathcal{D}}\bar{u} - u_{\mathcal{D}}$ and apply Hypothesis (5.1a)–(5.1d):

$$\underline{\alpha}\|I_{\mathcal{D}}\bar{u} - u_{\mathcal{D}}\|_{\mathcal{D}} \leqslant W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F}) + \overline{\alpha}(1 + C_{\mathcal{D}})S_{\mathcal{D}}(\bar{u}). \tag{5.16}$$

Estimate (5.8) follows by using the triangle inequality:

$$\begin{aligned}\|\mathrm{G}\bar{u} - \mathrm{G}_{\mathcal{D}}u_{\mathcal{D}}\|_{\boldsymbol{L}} &\leqslant \|\mathrm{G}\bar{u} - \mathrm{G}_{\mathcal{D}}I_{\mathcal{D}}\bar{u}\|_{\boldsymbol{L}} + \|\mathrm{G}_{\mathcal{D}}(I_{\mathcal{D}}\bar{u} - u_{\mathcal{D}})\|_{\boldsymbol{L}}\\ &\leqslant \|\mathrm{G}\bar{u} - \mathrm{G}_{\mathcal{D}}I_{\mathcal{D}}\bar{u}\|_{\boldsymbol{L}} + \|I_{\mathcal{D}}\bar{u} - u_{\mathcal{D}}\|_{\mathcal{D}}\\ &\leqslant S_{\mathcal{D}}(\bar{u}) + \frac{1}{\underline{\alpha}}(W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar{u}) + \boldsymbol{F}) + \overline{\alpha}(1 + C_{\mathcal{D}})S_{\mathcal{D}}(\bar{u})).\end{aligned} \tag{5.17}$$

Using (3.9) and (5.16), we get

$$\underline{\alpha}\|\mathrm{P}_{\mathcal{D}}I_{\mathcal{D}}\bar{u}-\mathrm{P}_{\mathcal{D}}u_{\mathcal{D}}\|_{L}\leqslant C_{\mathcal{D}}(W_{\mathcal{D}}(\boldsymbol{a}(\mathrm{G}\bar{u})+\boldsymbol{F})+\overline{\alpha}(1+C_{\mathcal{D}})S_{\mathcal{D}}(\bar{u})), \tag{5.18}$$

which yields (5.9) by invoking, as in (5.17), the triangle inequality and the estimate $\|\bar{u}-\mathrm{P}_{\mathcal{D}}I_{\mathcal{D}}\bar{u}\|_{\boldsymbol{L}}\leqslant S_{\mathcal{D}}(\bar{u})$.

Let us now turn to the proof of (5.10). The gradient scheme (5.6) gives for any $v\in X_{\mathcal{D}}\setminus\{0\}$,

$$\begin{aligned}\langle f-a(\bar{u}),\mathrm{P}_{\mathcal{D}}v\rangle_{L',L}&-\langle\boldsymbol{a}(\mathrm{G}\bar{u})+\boldsymbol{F},\mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}\\&=\langle\boldsymbol{a}(\mathrm{G}_{\mathcal{D}}u-\mathrm{G}\bar{u}),\mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}+\langle a(\mathrm{P}_{\mathcal{D}}u-\bar{u}),\mathrm{P}_{\mathcal{D}}v\rangle_{L',L}\end{aligned}$$

and thus

$$\frac{|\langle f-a(\bar{u}),\mathrm{P}_{\mathcal{D}}v\rangle_{L',L}-\langle\boldsymbol{a}(\mathrm{G}\bar{u})+\boldsymbol{F},\mathrm{G}_{\mathcal{D}}v\rangle_{\boldsymbol{L}',\boldsymbol{L}}|}{\|v\|_{\mathcal{D}}}\leqslant\overline{\alpha}(\|\mathrm{G}_{\mathcal{D}}u-\mathrm{G}\bar{u}\|_{\boldsymbol{L}}+C_{\mathcal{D}}\|\mathrm{P}_{\mathcal{D}}u-\bar{u}\|_{L}).$$

Taking the supremum over v on the left hand side yields (5.10) since (5.5) holds. Inequality (5.11) is an immediate consequence of the definition of $S_{\mathcal{D}}(\bar{u})$. ☐

Remark 5.3 (On the compactness assumption)**.** Note that, in the linear case, the compactness of the sequence of GDs is not required to obtain the convergence. This compactness assumption is in general only needed for some nonlinear problems.

Remark 5.4 (Consistency and limit-conformity are necessary conditions)**.** We state here a kind of reciprocal property to the convergence property. Let us assume that, under Hypothesis (5.1a)–(5.1d), a sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ of GDs is such that for all $f\in L$ and $\boldsymbol{F}\in\boldsymbol{L}$ and for all $m\in\mathbb{N}$, there exists $u_m\in X_{\mathcal{D}_m}$ which is a solution to the gradient scheme (5.6) and such that $\mathrm{P}_{\mathcal{D}_m}u_m$ and $\mathrm{G}_{\mathcal{D}_m}u_m$ converge respectively in L to the solution $\bar{u}$ of (5.4) and in $\boldsymbol{L}$ to $\mathrm{G}\bar{u}$. Then $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is consistent and limit-conforming in the sense of Definitions 3.8 and 3.9.

Indeed, for $\varphi\in W_{\mathrm{G}}$, let us consider $f=a(\varphi)$ and $\mathbf{F}=-\boldsymbol{a}(\mathrm{G}\varphi)$ in (5.4). Since in this case $\bar{u}=\varphi$, the assumption that $\mathrm{P}_{\mathcal{D}_m}u_m$ and $\mathrm{G}_{\mathcal{D}_m}u_m$ converge respectively in L to the solution φ of (5.4) and in $\boldsymbol{L}$ to $\mathrm{G}\varphi$ and inequality (5.11) proves that $S_{\mathcal{D}_m}(\varphi)$ tends to 0 as $m\to\infty$, and therefore the sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is consistent.

For $\boldsymbol{\varphi}\in\boldsymbol{W}_{\mathrm{D}}$, let us set $f=\mathrm{D}\boldsymbol{\varphi}$ and $\boldsymbol{F}=-\boldsymbol{\varphi}$ in (5.4). In this case, the solution $\bar{u}$ is equal to 0, since the right-hand side of (5.4) vanishes for any $v\in W_{\mathrm{G}}$. Then inequality (5.10) implies

$$W_{\mathcal{D}_m}(\boldsymbol{\varphi})\leqslant\overline{\alpha}\|\mathrm{G}_{\mathcal{D}_m}u_m\|_{\boldsymbol{L}}\to0\quad\text{as }m\to0,$$

hence concluding that the sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is limit-conforming.

Note that, if we now assume that $\mathrm{G}_{\mathcal{D}_m} u_m$ converges only weakly in $\boldsymbol{L}$ to $\mathrm{G}\bar{u}$, the same conclusion holds. Indeed, the other hypotheses on $(\mathcal{D}_m)_{m\in\mathbb{N}}$ are sufficient to prove that $\mathrm{G}_{\mathcal{D}_m} u_m$ actually converges strongly in $\boldsymbol{L}$ to $\mathrm{G}\bar{u}$. Indeed,

$$\lim_{m\to\infty} (\langle f, \mathrm{P}_{\mathcal{D}_m} u_m\rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}_{\mathcal{D}_m} u_m\rangle_{\boldsymbol{L}',\boldsymbol{L}}) = \langle f, \bar{u}\rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}\bar{u}\rangle_{\boldsymbol{L}',\boldsymbol{L}}.$$

Then we take $v = \bar{u}$ in (5.4) and $v = u_m$ in (5.6), this leads to

$$\begin{aligned}\lim_{m\to\infty} (\langle \boldsymbol{a}(\mathrm{G}_{\mathcal{D}_m} u_m), \mathrm{G}_{\mathcal{D}_m} u_m\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\mathrm{P}_{\mathcal{D}_m} u_m), \mathrm{P}_{\mathcal{D}_m} u_m\rangle_{L',L})\\ = \langle f, \bar{u}\rangle_{L',L} - \langle \boldsymbol{F}, \mathrm{G}\bar{u}\rangle_{\boldsymbol{L}',\boldsymbol{L}} = \langle \boldsymbol{a}(\mathrm{G}\bar{u}), \mathrm{G}\bar{u}\rangle_{\boldsymbol{L}',\boldsymbol{L}} + \langle a(\bar{u}), \bar{u}\rangle_{L',L}.\end{aligned}$$

In addition to the assumed weak convergence property of $\mathrm{G}_{\mathcal{D}_m} u_m$, this proves

$$\lim_{m\to\infty} \langle \boldsymbol{a}(\mathrm{G}_{\mathcal{D}_m} u_m - \mathrm{G}\bar{u}), \mathrm{G}_{\mathcal{D}_m} u_m - \mathrm{G}\bar{u}\rangle_{\boldsymbol{L}',\boldsymbol{L}} = 0,$$

and the convergence of $\mathrm{G}_{\mathcal{D}_m} u_m$ to $\mathrm{G}\bar{u}$ in $\boldsymbol{L}$ follows from the coercivity of $\boldsymbol{a}$ assumed in (5.1a)–(5.1d).

6. Other applications of the unified discretisation setting

We briefly present here other PDE models that can be analysed using the unified setting presented in this paper.

6.1. A hybrid-dimensional problem. We consider a simplified model for a Darcy flow in a convex domain $\Omega \subset \mathbb{R}^3$, in which a fracture Γ splits the domain Ω into two subdomains, Ω_1 and Ω_2. This fracture is defined by $\Gamma = \Omega \cap P$, where P is a plane. We assume that $\boldsymbol{n}_{12}$ is the unit vector normal to Γ, oriented from Ω_1 to Ω_2. The model reads

$$(6.1)\qquad \begin{cases} -\operatorname{div}(\Lambda\nabla u) = r & \text{in } \Omega_i, \quad i = 1, 2,\\ u = 0 & \text{on } \partial\Omega,\\ -\operatorname{div}_\Gamma(\Lambda_\Gamma \nabla_\Gamma u) + (\Lambda\nabla u_{|\Omega_1} - \Lambda\nabla u_{|\Omega_2})\cdot \boldsymbol{n}_{12} = r_\Gamma & \text{on } \Gamma,\end{cases}$$

where ∇_Γ or $\operatorname{div}_\Gamma$ is respectively the $2D$ gradient or divergence along Γ, $r \in L^2(\Omega)$, $r_\Gamma \in L^2(\Gamma)$.

Defining the space

$$H = \{v \in H^1_0(\Omega)\colon\ \gamma_\Gamma v \in H^1(\Gamma)\},$$

the weak formulation of Problem (6.1) is given by: find $\bar{u} \in V$ such that

$$(6.2)\quad \forall v \in H, \quad \int_\Omega \Lambda\nabla\bar{u}\cdot\nabla v\,\mathrm{d}\boldsymbol{x} + \int_\Gamma \Lambda_\Gamma\nabla_\Gamma\gamma_\Gamma\bar{u}\cdot\nabla_\Gamma\gamma_\Gamma v\,\mathrm{ds} = \int_\Omega rv\,\mathrm{d}\boldsymbol{x} + \int_\Gamma r_\Gamma\gamma_\Gamma v\,\mathrm{ds}.$$

This weak formulation is then identical to (5.4) by letting:

▷ $L = L^2(\Omega) \times L^2(\Gamma)$, $\boldsymbol{L} = L^2(\Omega)^3 \times L^2(\Gamma)^2$,

▷ $W_{\mathrm{G}} = \{(v, \gamma_\Gamma v), v \in H\}$ and $\mathrm{G}(v, \gamma_\Gamma v) = (\nabla v, \nabla_\Gamma\gamma_\Gamma v)$,

▷ $V = \{0\}$, $\boldsymbol{a}(\boldsymbol{v}, \boldsymbol{w}) = (\Lambda\boldsymbol{v}, \Lambda_\Gamma\boldsymbol{w})$, $f = (r, r_\Gamma)$, $\boldsymbol{F} = 0$.

Then, in this very simple case of fracture, the abstract Gradient Discretisation Method defined here applied to this problem is identical to that of [10]. It is expected that the general case of fractured domain studied in [10] could enter into this framework as well; this however does not avoid the tricky proof of the density results in [10]. Note that an interesting problem would be to check whether the abstract Gradient Discretisation Method could be also applied to a similar hybrid-dimensional problem studied in [22], which includes several types of parabolic degeneracies (this problem can modelize for example the interaction between surface and ground water flows).

6.2. Linear elasticity in solid continuum mechanics. Consider now the following spaces:

▷ $\Omega \subset \mathbb{R}^3$,

▷ $L = L^2(\Omega)^3$, so that $L' = L^2(\Omega)^3 = L$,

▷ $\boldsymbol{L} = \text{Ł}^2(\Omega)^{3\times 3}$, so that $\boldsymbol{L}' = L^2(\Omega)^{3\times 3}$,

▷ $\boldsymbol{W}_{\mathrm{D}} = H_{\mathrm{div}}(\Omega)^3$, and $V = \{0\}$,

▷ $W_{\mathrm{G}} = H_0^1(\Omega)^3$.

The operators $\mathrm{G}\colon H_0^1(\Omega)^3 \to L^2(\Omega)^{3\times 3}$ and $\mathrm{D}\colon H_{\mathrm{div}}(\Omega)^3 \to L^2(\Omega)^3$ are defined for $u \in H_0^1(\Omega)^3$ (the "displacement field") by

$$(\mathrm{G}u)_{i,j} = \tfrac{1}{2}(\partial_i u^{(j)} + \partial_j u^{(i)}),$$

and, for $\sigma \in H_{\mathrm{div}}(\Omega)^3$ (the "stress field"), by

$$(\mathrm{D}\sigma)_i = \sum_{j=1}^{3} \partial_j \sigma^{(i,j)}.$$

Then, the construction in Section 5 handles the case of the linear elasticity theory in solid continuum mechanics. Indeed, a strong formulation of the equilibrium of a solid under internal forces is Problem (5.5) where the linear operator $\boldsymbol{a}$ expresses

Hooke's law, that is: $\boldsymbol{a}(\mathrm{G}u)_{i,j} = \lambda \sum_{k=1}^{3} (\mathrm{G}u)_{k,k}\delta_{i,j} + 2\mu(\mathrm{G}u)_{i,j}$ with $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise, the Lamé coefficients $\lambda \geqslant 0$, $\mu > 0$ are given. Equation (5.4) is the so-called "virtual displacement" formulation, that is the weak formulation of (5.5).

6.3. Riemannian geometry. Let (M, g) be a compact orientable Riemannian manifold of dimension d without boundary, and with the corresponding measure μ_g. We denote by $TM = \bigcup_{x\in M} (\{x\} \times T_xM)$ the tangent bundle to M, and define the operators and spaces

- ▹ $L = L^2(M)$, so that $L' = L^2(M) = L$,
- ▹ $\boldsymbol{L} = L^2(TM) := \{\boldsymbol{v}\colon \boldsymbol{v}(x) \in T_xM$ for all $x \in M$ and $x \mapsto g_x(\boldsymbol{v}(x), \boldsymbol{v}(x))^{1/2} \in L^2(M)\}$; we have $\boldsymbol{L}' = \boldsymbol{L}$,
- ▹ $\mathrm{G}\colon C^1(M) \to L^2(TM)$ the standard gradient, that is $\mathrm{G}u = \nabla_g u$ such that, for any smooth vector field X and any $x \in M$, $\nabla_g u(x) \in T_xM$ and $g_x(X(x), \nabla_g u(x)) = \mathrm{d}u_x(X(x))$, where $\mathrm{d}u_x$ is the differential of u at x,
- ▹ W_G is the closure in $L^2(M)$ of $C^1(M)$ for the norm

$$u \mapsto \left(\int_M |u(x)|^2 \,\mathrm{d}\mu_g(x) + \int_M g_x(\nabla_g u(x), \nabla_g u(x)) \,\mathrm{d}\mu_g(x)\right)^{1/2}.$$

Then G is naturally extended, by density, to W_G.

Then, following the construction in Section 3.1, D is the standard divergence div_g on M and $\boldsymbol{W}_\mathrm{D} = \{\boldsymbol{v} \in L^2(TM)\colon \mathrm{div}_g\, \boldsymbol{v} \in L^2(M)\}$. We can then take $V = \mathrm{span}\{1\}$ and see that (3.7) holds by the Poincaré-Wirtinger inequality in W_G (this inequality follows as in bounded open sets of $\mathbb{R}^d$ by using the compact embedding $W_\mathrm{G} \hookrightarrow L^2(M)$).

In the setting described by (5.1), Problem (5.5) contains as a particular case the Poisson equation $-\Delta_g \bar{u} = f$ on M (with selection of the unique solution having zero average on the manifold), obtained by letting $\boldsymbol{a}(\nabla_g u) = \nabla_g u$ and $a(u) = \int_M u(x) \,\mathrm{d}\mu_g(x)$. In its generic form, (4.4) is an extension of the Leray-Lions equations to M.

Acknowledgements. The authors warmly thank Wolfgang Arendt and Isabelle Chalendar for inspiring discussions.

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