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Gerardo R. Chacón; Gerardo A. Chacón
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VARIABLE EXPONENT FOCK SPACES

Gerardo R. Chacón, Wahington D.C., Gerardo A. Chacón, Bogotá

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Abstract. We introduce variable exponent Fock spaces and study some of their basic properties such as boundedness of evaluation functionals, density of polynomials, boundedness of a Bergman-type projection and duality. We also prove that under the global log-Hölder condition, the variable exponent Fock spaces coincide with the classical ones.

Keywords: Fock space; variable exponent Lebesgue space; Bergman projection

MSC 2010: 30H20, 46E30

1. Introduction

Variable exponent Lebesgue spaces are a generalization of classical Lebesgue spaces L^p in which the exponent p is a measurable function. Such spaces were introduced by Orlicz in [17] and developed by Kováčik and Rákosník in [13]. Although such spaces have received a considerable amount of attention, little is known about their analytic version.

Recently, the research subject has received increasing interest and some progress has been made. For example, in [11] and [12] variable exponent Hardy spaces of analytic functions in the unit disk are considered. In [8] a version of BMO spaces with variable exponents is considered. Bergman spaces with variable exponents have been studied in [1], [2], [3] and a different approach has been taken in [9] and [10], much of the research done in the area assumes the log-Hölder condition on the exponent, which is a growth condition that usually guarantees boundedness of a Hardy-Littlewood maximal operator in a related space. One case of function spaces on unbounded domains have been studied in [16]. The theory of Orlicz spaces

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of analytic functions has also received recent interest, see for example [14] and [15]. In this article, we study variable exponent Fock spaces, which can be seen as generalized Orlicz spaces that are not rearrangement invariant.

Variable exponent Fock spaces $\mathcal{F}^{p(\cdot)}$ are spaces of entire functions that belong to a weighted $L^{p(\cdot)}$ space with respect to the Gaussian measure. The classical case has received a great interest in the last years as can be seen in [19]. One property of variable exponent Fock spaces that makes its study relevant is that the Gaussian measure is not a Muckenhoupt weight; this makes it difficult to use one important resource in the theory of variable exponent Lebesgue spaces: the boundedness of the maximal function. In this article, we will introduce variable exponent Fock spaces and study some of their basic properties such as the boundedness of evaluation functionals, density of polynomials, boundedness of a Bergman-type projection and duality. We also prove that under the global log-Hölder condition, the variable exponent Fock spaces coincide with the classical ones, this is a phenomenon that singles out Fock spaces from other spaces of analytic functions with variable exponents previously investigated.

The article is distributed as follows. In the next section we will present some preliminary concepts and results as well as the notation that will be used throughout the rest of the article. In Section 3, we will first prove a version of Hölder's inequality to obtain an equivalent norm in variable exponent Fock spaces. Then we will show that evaluation functionals are bounded and prove an inclusion relation between variable exponent Fock spaces with different exponents. We also prove that when restricted to exponents that satisfy the global log-Hölder condition, the variable exponent Fock spaces coincide with the classical ones. An example of an exponent for which the variable exponent space differ from all the classical ones is also given, and as a consequence, it is shown that if $1 < p < \infty$, then $\mathcal{F}^p \neq \bigcap\limits_{q>p} \mathcal{F}^q$ and $\mathcal{F}^p \neq \bigcup\limits_{q<p} \mathcal{F}^q$. This is an instance in which variable exponent techniques lead to results in the classical case. Finally, in Section 4 we study a Bergman-type projection and characterize the dual space of variable exponent Fock spaces under the assumption of boundedness of the Hardy-Littlewood maximal operator.

For the rest of the paper, we will use the notation $a \lesssim b$ if there exists a constant $C > 0$, independent of a and b, such that $a \leqslant Cb$. Similarly, we use $a \asymp b$ if we have $a \lesssim b \lesssim a$.

2. Preliminaries

2.1. Fock spaces. We will need some properties of the classical Fock spaces that we include here for the sake of completeness. These results will be taken from [19].

Definition 2.1. Let A denote the Lebesgue area measure defined on the complex plane $\mathbb{C}$ and let $1 \leqslant p < \infty$. Denote by $L^p = L^p(\mathbb{C})$ the Lebesgue space of p-integrable functions with respect to the measure A. We will denote by $\mathcal{L}^p = \mathcal{L}^p(\mathbb{C})$ the Banach space of all measurable functions $f\colon \mathbb{C} \to \mathbb{C}$ such that

$$\|f\|_{L^p}^p = \frac{p}{\pi} \int_{\mathbb{C}} |f(z)\mathrm{e}^{-|z|^2}|^p \,\mathrm{d}A(z) < \infty.$$

In other words, $\mathcal{L}^p(\mathbb{C})$ consists of all functions f such that $f\mathrm{e}^{-|\cdot|^2} \in L^p(\mathbb{C})$. The *Fock space* $\mathcal{F}^p$ is defined as the space of all entire functions that belong to $\mathcal{L}^p$.

Remark 2.2. In the literature, the $\mathcal{L}^p$ spaces just defined are known as *weighted* L^p *spaces* in which the weight (in this case the Gaussian weight) acts as a *multiplier*.

For every $z \in \mathbb{C}$, the *evaluation functional* $\gamma_z\colon \mathcal{F}^p \to \mathbb{C}$, defined as

$$\gamma_z(f) := f(z), \tag{2.1}$$

is bounded since the following inequality holds for every $f \in \mathcal{F}^p$:

$$|f(z)| \leqslant \|f\|_{\mathcal{F}^p} \mathrm{e}^{|z|^2}. \tag{2.2}$$

The space $\mathcal{F}^p$ is a closed subspace of $\mathcal{L}^p$ and consequently it is a Banach space.

In the case when $p = 2$, the Fock space $\mathcal{F}^2$ is a Hilbert space with inner product

$$\langle f, g \rangle = \frac{2}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)}\mathrm{e}^{-2|z|^2} \,\mathrm{d}A(z). \tag{2.3}$$

As a consequence of Riesz representation theorem for every $z \in \mathbb{C}$ there exists an element $K_z \in \mathcal{F}^2$ such that for every $f \in \mathcal{F}^2$,

$$\langle f, K_z \rangle = f(z). \tag{2.4}$$

The functions K_z are called *reproducing kernels* and are given by

$$K_z(w) = \mathrm{e}^{2w\overline{z}}. \tag{2.5}$$

It is shown in [18] that the reproducing formula (2.4) holds for general functions $f \in \mathcal{F}^p$ in the sense that

$$f(z) = \frac{2}{\pi} \int_{\mathbb{C}} f(w)\mathrm{e}^{2\overline{w}z}\mathrm{e}^{-2|w|^2} \,\mathrm{d}A(w). \tag{2.6}$$

2.2. Generalized Orlicz spaces. We will also need some results from the theory of generalized Orlicz spaces that will be presented as in [6].

Definition 2.3. A function $\varphi\colon \mathbb{C}\times[0,\infty)\to[0,\infty)$ is said to belong to the class Φ if for every $t\in[0,\infty)$ the function $\varphi(\cdot,t)$ is measurable and for every $z\in\mathbb{C}$ the function $\varphi(z,\cdot)$ satisfies the following conditions:

(i) $\varphi(z,\cdot)$ is increasing;
(ii) $\varphi(z,\cdot)$ is left-continuous;
(iii) $\varphi(z,\cdot)$ is convex;
(iv) $\varphi(z,0)=\lim\limits_{t\to0^+}\varphi(z,t)=0$ and $\lim\limits_{t\to\infty}\varphi(z,t)=\infty$.

Some examples of functions of the class Φ are:

$$\varphi_1(z,t)=t^{p(z)},\quad \varphi_2(z,t)=\frac{1}{p(z)}t^{p(z)},\quad \varphi_3(z,t)=t^{p(z)}\mathrm{e}^{-|z|^2p(z)},$$

where $p\colon \mathbb{C}\to[1,\infty)$ is a measurable function.

Definition 2.4. Given a function φ in the class Φ, define for every measurable function $f\colon \mathbb{C}\to\mathbb{C}$ the modular

$$\varrho_\varphi(f)=\int_{\mathbb{C}}\varphi(z,|f(z)|)\,\mathrm{d}A(z). \tag{2.7}$$

The *Generalized Orlicz space* L^φ is defined as the set of all measurable functions f such that $\varrho_\varphi(\lambda f)<\infty$ for some $\lambda>0$. L^φ is a Banach space when equiped with the *Luxemburg-Nakano norm:*

$$\|f\|_{L^\varphi}=\inf\Big\{\lambda>0\colon\ \varrho_\varphi\Big(\frac{f}{\lambda}\Big)\leqslant1\Big\}. \tag{2.8}$$

In the case in which $\varphi=\varphi_1$, we call $L^\varphi=L^{p(\cdot)}$ a *variable exponent Lebesgue space.* The basics on the subject may be found in the monographs (see [4], [5]). Denote $p^+=\operatorname*{ess\,sup}\limits_{z\in\mathbb{C}}p(z)$ and $p^-=\operatorname*{ess\,inf}\limits_{z\in\mathbb{C}}p(z)$. The measurable function $p\colon \mathbb{C}\to[1,\infty)$ is called a *variable exponent*, and the set of all variable exponents with $p^+<\infty$ is denoted as $\mathscr{P}(\mathbb{C})$.

We will be particularly interested in the case in which $\varphi=\varphi_3$. We will denote such generalized Orlicz space as $\mathcal{L}^{p(\cdot)}$ to stress the dependence of the exponent and we use the calligraphic $\mathcal{L}$ in order to differentiate from the unweighted Lebesgue space.

2.3. Variable exponent Fock spaces. We are ready to introduce the main concept of this article.

Definition 2.5. Let $p\colon \mathbb{C}\to[1,\infty)$ be a measurable function in $\mathscr{P}(\mathbb{C})$. The *variable exponent Fock space* $\mathcal{F}^{p(\cdot)}$ is defined as the set of all entire functions that belong to the generalized Orlicz space $\mathcal{L}^{p(\cdot)}$.

In other words, $\mathcal{F}^{p(\cdot)}$ is the set of entire functions such that

$$\int_{\mathbb{C}} |f(z)|^{p(z)} \mathrm{e}^{-|z|^2 p(z)} \, \mathrm{d}A(z) < \infty.$$

In order to have a better correspondence with the definition of Fock spaces for constant exponents, a modification will be made to the modular defined in (2.7). We will denote

$$C_{p(\cdot)} := \int_{\mathbb{C}} \mathrm{e}^{-|z|^2 p(z)} \, \mathrm{d}A(z)$$

and define the modular

$$\varrho_{p(\cdot)}(f) = C_{p(\cdot)}^{-1} \int_{\mathbb{C}} |f(z)|^{p(z)} \mathrm{e}^{-|z|^2 p(z)} \, \mathrm{d}A(z).$$

The choice of the constant is made so that $\varrho_{p(\cdot)}(1) = 1$.

Remark 2.6. An entire function f belongs to $\mathcal{F}^{p(\cdot)}$ if and only if the function $z \mapsto f(z)\mathrm{e}^{-|z|^2}$ belongs to $L^{p(\cdot)}$.

2.4. Weighted Fock spaces.

Definition 2.7. Fix $r > 0$ and $1 < p < \infty$. Let $A_{p,r}$ denote the class of weights $w \colon \mathbb{C} \to [0, \infty)$ such that

$$\sup_{z \in \mathbb{C}} \left(\frac{1}{|B(z,r)|} \int_{B(z,r)} w \, \mathrm{d}A \right) \left(\frac{1}{|B(z,r)|} \int_{B(z,r)} w^{-1/(p-1)} \, \mathrm{d}A \right)^{p-1} \leqslant C_r$$

for some $0 < C_r < \infty$.

We will say that w belongs to the Muckenhoupt class A_1 if

$$\operatorname*{ess\,sup}_{z \in \mathbb{C}} \frac{Mw(z)}{w(z)} < \infty,$$

where M denotes the Hardy-Littlewood maximal operator.

It is shown in [4], Section 4.2 that $A_1 \subset A_{p,r}$ for every $1 < p < \infty$ and $r > 0$.

Given a weight w, we will denote by $L^p(w)$ the weighted Lebesgue space whereas $\mathcal{L}^p(w)$ will denote the weighted $\mathcal{L}^p$ space. The following theorem is shown in [7].

Theorem 2.8. *The following are equivalent for any weight w on $\mathbb{C}$:*

(i) $w \in A_{p,r}$ *for some* $r > 0$.

(ii) *The operator* $H\colon L^p(w)\to L^p(w)$ *defined as*

$$Hf(z)=\int_{\mathbb{C}} \mathrm{e}^{-|z-u|^2} f(u)\,\mathrm{d}A(u)$$

is bounded.

(iii) *The operator* $P\colon \mathcal{L}^p(w)\to \mathcal{F}^p(w)$ *defined as*

$$Pg(z)=\frac{2}{\pi}\int_{\mathbb{C}} g(w)\mathrm{e}^{2\overline{w}z}\mathrm{e}^{-2|w|^2}\,\mathrm{d}A(w)$$

is bounded.

(iv) $w\in A_{p,r}$ *for all* $r>0$.

Remark 2.9. The operator P is bounded on $\mathcal{L}^p(w)$ if and only if the operator

$$\begin{aligned} J\colon \mathcal{L}^p(w) &\longrightarrow \mathcal{L}^p(w),\\ g &\longmapsto \int_{\mathbb{C}} |g(z)\mathrm{e}^{2\overline{z}w}\mathrm{e}^{-2|z|^2}|\,\mathrm{d}A(z)\end{aligned}$$

is bounded.

The following proposition is a version of Rubio de Francia extrapolation result in the framework of variable Lebesgue spaces.

Theorem 2.10 ([4], Theorem 5.24)**.** *Given a set* Ω, *suppose that for some* $p_0\geqslant 1$ *there exists a family* $\mathcal{D}$ *of pairs of functions such that for all* $w\in A_1$,

$$\int_\Omega F(x)^{p_0}w(x)\,\mathrm{d}x\leqslant C_0\int_\Omega G(x)^{p_0}w(x)\,\mathrm{d}x,\quad (F,G)\in\mathcal{D}. \tag{2.9}$$

Given $p\in\mathscr{P}(\Omega)$, *if* $p_0\leqslant p_-\leqslant p_+<\infty$ *and suppose that the Hardy-Littlewood maximal operator is bounded on* $L^{(p(\cdot)/p_0)'}(\Omega)$, *then*

$$\|F\|_{L^{p(\cdot)}(\Omega)}\leqslant C_{p(\cdot)}\|G\|_{L^{p(\cdot)}(\Omega)} \tag{2.10}$$

for every $(F,G)\in\mathcal{D}$.

3. Some properties of variable exponent Fock spaces

In this section we will start by showing a version of Hölder inequality for $\mathcal{L}^{p(\cdot)}$ spaces that will be useful in this specific context. It is important to notice that, as generalized Orlicz spaces, a Hölder inequality and a duality result already exists, however here we prove a different version that better suits our purposes.

Theorem 3.1 (Hölder's inequality)**.** *Suppose $p \in \mathscr{P}(\mathbb{C})$ and let $p'\colon \mathbb{C} \to (1,\infty)$ be such that $1/p(z) + 1/p'(z) = 1$ for all $z \in \mathbb{C}$. Then if $f \in \mathcal{L}^{p(\cdot)}$ and $g \in \mathcal{L}^{p'(\cdot)}$, we have that*

$$\left|\int_{\mathbb{C}} f(z)\overline{g(z)}\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z)\right| \leqslant 2\|f\|_{\mathcal{L}^{p(\cdot)}}\|g\|_{\mathcal{L}^{p'(\cdot)}}.$$

Proof. It is clear that if any of the norms on the right-hand side is equal to zero, then the inequality holds. So suppose both norms are nonzero, and use Young's inequality to obtain

$$\frac{|f(z)|}{\|f\|_{\mathcal{L}^{p(\cdot)}}}\frac{|g(z)|}{\|g\|_{\mathcal{L}^{p'(\cdot)}}}\mathrm{e}^{-2|z|^2} \leqslant \frac{|f(z)|^{p(z)}\mathrm{e}^{-p(z)|z|^2}}{p(z)\|f\|_{\mathcal{L}^{p(\cdot)}}} + \frac{|g(z)|^{\mathcal{L}^{p'(\cdot)}}\mathrm{e}^{-p'(z)|z|^2}}{p'(z)\|g\|_{p'(\cdot)}},$$

and the inequality follows. □

With the previous inequality in hand, we can define an alternative norm on $\mathcal{L}^{p(\cdot)}$ as

$$|||f|||_{\mathcal{L}^{p(\cdot)}} = \sup\left\{\frac{2}{\pi}\left|\int_{\mathbb{C}} f(z)\overline{g(z)}\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z)\right|\colon\ g \in \mathcal{L}^{p'(\cdot)},\ \varrho_{p'(\cdot)}(g) \leqslant 1\right\}.$$

Notice that

$$\begin{aligned}|||f|||_{\mathcal{L}^{p(\cdot)}} &= \sup\left\{\frac{2}{\pi}\int_{\mathbb{C}} |f(z)||g(z)|\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z)\colon\ g \in \mathcal{L}^{p'(\cdot)},\ \varrho_{p'(\cdot)}(g) \leqslant 1\right\}\\ &= \sup\left\{\frac{2}{\pi}\int_{\mathbb{C}} |f(z)||g(z)|\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z)\colon\ g \in \mathcal{L}^{p'(\cdot)},\ \varrho_{p'(\cdot)}(g) < \infty\right\}.\end{aligned}$$

Theorem 3.2. *Suppose that $p \in \mathcal{P}(\mathbb{C})$ and let $p'\colon \mathbb{C} \to (1,\infty)$ be such that $1/p(z) + 1/p'(z) = 1$ for all $z \in \mathbb{C}$. Then there exist constants $c > 0$ and $C > 0$ such that*

$$c\|f\|_{\mathcal{L}^{p(\cdot)}} \leqslant |||f|||_{\mathcal{L}^{p(\cdot)}} \leqslant C\|f\|_{\mathcal{L}^{p(\cdot)}}.$$

Proof. Suppose $f \in \mathcal{L}^{p(\cdot)}$. Then by Hölder's inequality we have

$$\begin{aligned}|||f|||_{\mathcal{L}^{p(\cdot)}} &\leqslant \sup\left\{\frac{2}{\pi}\int_{\mathbb{C}} |f(z)||h(z)|\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z)\colon\ h \in \mathcal{L}^{p'(\cdot)},\ \varrho_{p'(\cdot)}(h) \leqslant 1\right\}\\ &\leqslant \frac{4}{\pi}\sup\{\|f\|_{\mathcal{L}^{p(\cdot)}}\|h\|_{\mathcal{L}^{p'(\cdot)}}\colon\ h \in \mathcal{L}^{p'(\cdot)},\ \varrho_{p'(\cdot)}(h) \leqslant 1\} \leqslant \frac{4}{\pi}\|f\|_{p(\cdot)}.\end{aligned}$$

On the other hand, suppose that $|||f|||_{\mathcal{L}^{p(\cdot)}} = 1$. Define function g as

$$g(z) = \frac{\pi}{2C_{p(\cdot)}}|f(z)|^{p(z)-1}\mathrm{e}^{-|z|^2(p(z)-2)}.$$

Then

$$\begin{aligned}\varrho_{p'(\cdot)}(g) &= \frac{\pi}{2C_{p'(\cdot)}C_{p(\cdot)}}\int_{\mathbb{C}} |f(z)|^{(p(z)-1)p'(z)}\mathrm{e}^{-|z|^2(p(z)-2)p'(z)}\mathrm{e}^{-|z|^2p'(z)}\,\mathrm{d}A(z)\\ &= \frac{\pi}{2C_{p'(\cdot)}C_{p(\cdot)}}\int_{\mathbb{C}} |f(z)|^{p(z)}\mathrm{e}^{-|z|^2p(z)}\,\mathrm{d}A(z) \leqslant \frac{\pi}{2C_{p'(\cdot)}}\varrho_{p(\cdot)}(f).\end{aligned}$$

Thus, $g \in \mathcal{L}^{p'(\cdot)}$ and consequently,

$$\begin{aligned}\varrho_{p(\cdot)}(f) &= C_{p(\cdot)}^{-1}\int_{\mathbb{C}} |f(z)||f(z)|^{p(z)-1}\mathrm{e}^{-|z|^2(p(z)-2)}\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z)\\ &= \frac{2}{\pi}\int_{\mathbb{C}} |f(z)||g(z)|\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z) \leqslant |||f|||_{\mathcal{L}^{p(\cdot)}} = 1.\end{aligned}$$

Thus

$$\varrho_{p(\cdot)}\Big(\frac{f}{|||f|||_{\mathcal{L}^{p(\cdot)}}}\Big) = \varrho_{p(\cdot)}(f) \leqslant 1,$$

which implies that

$$\|f\|_{\mathcal{L}^{p(\cdot)}} \leqslant |||f|||_{\mathcal{L}^{p(\cdot)}}.$$

The general case $|||f|||_{\mathcal{L}^{p(\cdot)}} \neq 1$ follows from the homogeneity of the norm. $\square$

We pass now to prove that evaluation functionals are continuous. Given $z \in \mathbb{C}$ the evaluation functional is defined as

$$\begin{aligned}\gamma_z\colon\ \mathcal{F}^{p(\cdot)} &\longrightarrow \mathbb{C},\\ f &\longmapsto f(z).\end{aligned}$$

We will need the next lemma.

Lemma 3.3. *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function and let $R > 0$. Then for every $z \in \mathbb{C}$,*

$$|f(z)|\mathrm{e}^{-|z|^2} \leqslant \frac{\mathrm{e}^{R^2}}{R^2\pi}\int_{B(z,R)} |f(w)|\mathrm{e}^{-|w|^2}\,\mathrm{d}A(w).$$

Proof. Fix $z \in \mathbb{C}$ and define an entire function

$$\begin{aligned}g\colon\ \mathbb{C} &\longrightarrow \mathbb{C},\\ w &\longmapsto f(w+z)\mathrm{e}^{-2\overline{z}w}.\end{aligned}$$

By the mean value theorem we have that

$$|g(0)| \leqslant \frac{1}{R^2\pi}\int_{B(0,R)} |g(w)|\,\mathrm{d}A(w).$$

Writing the equation in terms of f and multiplying it by $\mathrm{e}^{-|z|^2}$ we get

$$\begin{aligned}|f(z)|\mathrm{e}^{-|z|^2} &\leqslant \frac{1}{R^2\pi}\int_{B(0,R)}|f(w+z)||\mathrm{e}^{-2\overline{z}w}|\mathrm{e}^{-|z|^2}\,\mathrm{d}A(w)\\ &\leqslant \frac{\mathrm{e}^{R^2}}{R^2\pi}\int_{B(z,R)}|f(w)|\mathrm{e}^{-|w|^2}\,\mathrm{d}A(w).\end{aligned}$$

□

Theorem 3.4. *Suppose that $p\in\mathscr{P}(\mathbb{C})$. Then there exists a constant $C>0$ such that for every $f\in\mathcal{F}^{p(\cdot)}$*

$$|f(z)|\leqslant C\mathrm{e}^{|z|^2}\|f\|_{\mathcal{F}^{p(\cdot)}}.$$

P r o o f. Take $R=1$ in the previous lemma to get

$$|f(z)|\mathrm{e}^{-|z|^2}\leqslant\frac{\mathrm{e}}{\pi}\int_{B(z,1)}|f(w)|\mathrm{e}^{-|w|^2}\,\mathrm{d}A(w).$$

Now, applying Theorem 3.1 we get that if $1/p(w)+1/p'(w)=1$ for all $w\in\mathbb{C}$, then

$$\begin{aligned}\int_{B(z,1)}|f(w)|\mathrm{e}^{-|w|^2}\,\mathrm{d}A(w) &= \int_{\mathbb{C}}|f(w)|\mathrm{e}^{|w|^2}\chi_{B(z,1)}(w)\mathrm{e}^{-2|w|^2}\,\mathrm{d}A(w)\\ &\leqslant 2\|f\|_{\mathcal{F}^{p(\cdot)}}\|\mathrm{e}^{|\cdot|^2}\chi_{B(z,1)}\|_{\mathcal{L}^{p'(\cdot)}}\end{aligned}$$

and the result follows. □

Corollary 3.5. *Suppose that $p\in\mathscr{P}(\mathbb{C})$. Then $\mathcal{F}^{p(\cdot)}$ is a closed subspace of $\mathcal{L}^{p(\cdot)}$ and hence it is a Banach space.*

As another consequence of the previous theorem, we can now show a relation between $\mathcal{F}^{p(\cdot)}$ spaces.

Theorem 3.6. *Suppose $p,q\in\mathscr{P}(\mathbb{C})$ and there exists $R>0$ such that $p(z)\leqslant q(z)$ for every $z\in\mathbb{C}$, $|z|\geqslant R$. Then $\mathcal{F}^{p(\cdot)}\subset\mathcal{F}^{q(\cdot)}$ and moreover for every $f\in\mathcal{F}^{p(\cdot)}$,*

$$\|f\|_{\mathcal{F}^{q(\cdot)}}\lesssim\|f\|_{\mathcal{F}^{p(\cdot)}}.$$

P r o o f. Suppose $f\in\mathcal{F}^{p(\cdot)}$ and $\|f\|_{\mathcal{F}^{q(\cdot)}}\leqslant 1$. We write

$$\varrho_{q(\cdot)}(f)=I_1+I_2,$$

where

$$I_1 = C_{q(\cdot)}^{-1} \int_{|z|<R} |f(z)|^{q(z)} \mathrm{e}^{-|z|^2 q(z)} \, \mathrm{d}A(z)$$

and

$$I_2 = C_{q(\cdot)}^{-1} \int_{|z|\geqslant R} |f(z)|^{q(z)} \mathrm{e}^{-|z|^2 q(z)} \, \mathrm{d}A(z).$$

By Theorem 3.4, there exists $C > 0$ such that

$$I_2 \leqslant C_{q(\cdot)}^{-1} \int_{|z|\geqslant R} |f(z)|^{p(z)} (C\mathrm{e}^{|z|^2})^{q(z)-p(z)} \mathrm{e}^{-|z|^2 q(z)} \, \mathrm{d}A(z) \lesssim \varrho_{p(\cdot)}(f).$$

On the other hand, using again Theorem 3.4, we obtain that $I_1 \lesssim R^2$. Thus,

$$\varrho_{q(\cdot)}(f) \lesssim \varrho_{p(\cdot)}(f) + 1,$$

and consequently $f \in \mathcal{F}^{q(\cdot)}$.

For a general f, notice that

$$\varrho_{q(\cdot)}\Big(\frac{f}{\|f\|_{\mathcal{F}^{p(\cdot)}}}\Big) \lesssim \varrho_{p(\cdot)}\Big(\frac{f}{\|f\|_{\mathcal{F}^{p(\cdot)}}}\Big) + 1 \leqslant 2.$$

$\square$

A natural question emerges. Does there exist $p(\cdot)$ such that the space $\mathcal{F}^{p(\cdot)}$ does not coincide with the classical Fock space $\mathcal{F}^p$ with a constant exponent p? We will exhibit a family of such spaces.

Example 3.7. For $0 < a < 1$ let $p\colon \mathbb{C} \to (1, \infty)$ be defined as $p(z) = 2 + (\log(\mathrm{e} + |z|))^{-a}$. Notice that by the previous theorem, $\mathcal{F}^2 \subset \mathcal{F}^{p(\cdot)}$.

Aditionally, it is shown in [19] that the inclusion is strict in the case of constant exponents. Consequently, for a fixed $q > 2$ we can always find $q > s > 2$ and $R > 0$ such that $p(z) \leqslant s$ if $|z| \geqslant R$. Thus, $\mathcal{F}^{p(\cdot)} \subset \mathcal{F}^s \subset \mathcal{F}^q$, but since $\mathcal{F}^s \neq \mathcal{F}^q$, then we have that $\mathcal{F}^{p(\cdot)} \neq \mathcal{F}^q$.

Now we will show that $\mathcal{F}^2 \neq \mathcal{F}^{p(\cdot)}$. Suppose the contrary, then the inclusion operator is invertible and by the open mapping theorem there exists $C > 1$ such that for every $f \in \mathcal{F}^2$,

$$\|f\|_{\mathcal{F}^2} \leqslant C\|f\|_{\mathcal{F}^{p(\cdot)}}.$$

Consequently,

$$\varrho_{p(\cdot)}\Big(\frac{Cf}{\|f\|_{\mathcal{F}^2}}\Big) \geqslant 1. \tag{3.1}$$

Now for each integer $n \geqslant 2$ let $f_n\colon \mathbb{C} \to \mathbb{C}$ by defined as $f_n(z) = z^n$. It has been noticed in [19], using Stirling's formula, that

$$\|f\|^2_{\mathcal{F}^2} = \frac{(n!)^{1/2}}{2^{n/2}} \sim \frac{n^{n/2}n^{1/4}}{(2\mathrm{e})^{n/2}}.$$

Then

$$\varrho_{p(\cdot)}\Big(\frac{f_n}{\|f_n\|_{\mathcal{F}^2}}\Big) \sim \int_0^\infty \Big(\frac{r^n \mathrm{e}^{-r^2}(2\mathrm{e})^{n/2}}{n^{n/2}n^{1/4}}\Big)^{2+(\log(\mathrm{e}+r))^{-a}} r\,\mathrm{d}r. \tag{3.2}$$

Define an auxiliary function $g_n\colon\ [0,\infty] \to [0,\infty]$ as

$$g_n(r) := \frac{r^n \mathrm{e}^{-3r^2/4}(2\mathrm{e})^{n/2}}{n^{n/2}}.$$

Function g_n is decreasing on the interval $((\frac{1}{3}2n)^{1/2},\infty)$ and consequently for $r > \sqrt{2n}$ we have that

$$g_n(r) \leqslant g_n(\sqrt{2n}) = \Big(\frac{4}{\mathrm{e}^2}\Big)^{n/2} \leqslant 1.$$

Thus,

$$\frac{r^n \mathrm{e}^{-r^2}(2\mathrm{e})^{n/2}}{n^{n/2}n^{1/4}} \leqslant \frac{\mathrm{e}^{-r^2/4}}{n^{1/4}}.$$

Moreover, for $r > \sqrt{2n} \geqslant 2$ we have that $(\log(\mathrm{e}+r))^a < r$ and $\log(n) < r$, which implies that

$$r^2 \geqslant \frac{\log(n)(\log(\mathrm{e}+r))^a}{(\log(\mathrm{e}+\sqrt{2n}))^a},$$

and consequently,

$$\frac{-r^2}{4} \leqslant \frac{\log(n)}{4}\Big(1 - \frac{(\log(\mathrm{e}+r))^a}{(\log(\mathrm{e}+\sqrt{2n}))^a}\Big).$$

Rearranging the previous inequality we get

$$\Big(\frac{\mathrm{e}^{-r^2/4}}{n^{1/4}}\Big)^{(\log(\mathrm{e}+r))^{-a}} \leqslant n^{-1/(4(\log(\mathrm{e}+\sqrt{2n}))^a)}. \tag{3.3}$$

On the other hand, since for every positive integer n and $r > 0$ we have that

$$\frac{r^n \mathrm{e}^{-r^2}(2\mathrm{e})^{n/2}}{n^{n/2}} \leqslant 1,$$

then if $r < \sqrt{2n}$, the following inequality holds:

$$\Big(\frac{r^n \mathrm{e}^{-r^2}(2\mathrm{e})^{n/2}}{n^{n/2}n^{1/4}}\Big)^{1/(\log(\mathrm{e}+r))^a} \leqslant n^{-1/(4(\log(\mathrm{e}+r))^a)} \leqslant n^{-1/(4(\log(\mathrm{e}+\sqrt{2n}))^a)}. \tag{3.4}$$

Putting estimates (3.2), (3.3) and (3.4) together we get

$$\begin{aligned}\varrho_{p(\cdot)}\Big(\frac{f_n}{\|f_n\|_{\mathcal{F}^2}}\Big) &\lesssim n^{-1/(4(\log(\mathrm{e}+\sqrt{2n}))^a)}\|f_n\|_{\mathcal{F}^2}^{-2}\int_0^\infty r^{2n}\mathrm{e}^{-2r^2}r\,\mathrm{d}r\\ &\sim n^{-1/(4(\log(\mathrm{e}+\sqrt{2n}))^a)} \to 0 \quad \text{as } n\to\infty,\end{aligned}$$

contradicting inequality (3.1).

The previous example can be generalized to conclude the following result about classical Fock spaces.

Proposition 3.8. *Let* $1 < p < \infty$. *Then*

$$\mathcal{F}^p \neq \bigcap_{q>p} \mathcal{F}^q.$$

A similar example as Example 3.7 can be used to show that if $1 < p < \infty$, then

$$\mathcal{F}^p \neq \bigcup_{q<p} \mathcal{F}^q.$$

The exponents used in Example 3.7 were chosen for their *s*low convergence to zero as $|z| \to \infty$. Such behavior is somewhat unwanted. As a matter of fact, there exists a common condition that is used to study variable exponent spaces that discards such example. In what follows, we will study such condition and see its implications in our context.

Definition 3.9. A function $p\colon \mathbb{C} \to [1,\infty)$ is said to be *log-Hölder continuous* or to satisfy the *Dini-Lipschitz condition* on $\mathbb{C}$ if there exists a positive constant $C_{\log}$ such that

$$|p(z) - p(w)| \leqslant \frac{C_{\log}}{\log(1/|z-w|)} \tag{3.5}$$

for all $z, w \in \mathbb{C}$ such that $|z-w| < \frac{1}{2}$. The function p is said to satisfy the *log-Hölder decay condition* if there exists $p_\infty \in [1,\infty)$ and a positive constant C such that

$$|p(z) - p_\infty| \leqslant \frac{C}{\log(\mathrm{e}+|x|)} \tag{3.6}$$

for all $z \in \mathbb{C}$. We say that the function p is *globally log-Hölder continuous in* $\mathbb{C}$ if it satisfies both (3.5) and (3.6). We denote by $\mathscr{P}^{\log}(\mathbb{C})$ the set of all globally *log-Hölder continuous functions in* $\mathbb{C}$ for which

$$1 < p_- \leqslant p_+ < \infty.$$

It is known that the condition $p \in \mathscr{P}^{\log}(\mathbb{C})$ implies boundedness of the Hardy-Littlewood maximal operator on $L^{p(\cdot)}(\mathbb{C})$. This however is not a characterization, there are exponents $p \notin \mathscr{P}^{\log}(\mathbb{C})$ such that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{C})$. An example of a family of such exponent is given in Example 3.7 (see [4], Example 4.13).

The following theorem shows that under the $\mathscr{P}^{\log}(\mathbb{C})$ condition, Fock spaces and variable exponent Fock spaces coincide. We will need the following result.

Lemma 3.10 ([4], Lemma 3.26). *Suppose that $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{C})$ and $1 < p_\infty < \infty$. There exists a constant $C > 0$ such that given any set $E \subset \mathbb{C}$ and any function F on $\mathbb{C}$ with $0 \leqslant F(z) \leqslant 1$, for all $z \in E$ we have that*

$$\int_E F(z)^{p(z)}\,\mathrm{d}A(z) \leqslant C\int_E F(z)^{p_\infty}\,\mathrm{d}A(z) + \int_E R(z)^{p^-}\,\mathrm{d}A(z) \tag{3.7}$$

and

$$\int_E F(z)^{p_\infty}\,\mathrm{d}A(z) \leqslant C\int_E F(z)^{p(z)}\,\mathrm{d}A(z) + \int_E R(z)^{p^-}\,\mathrm{d}A(z), \tag{3.8}$$

where

$$R(z) = \frac{1}{(\mathrm{e} + |z|)^k}, \quad k \geqslant 2.$$

Theorem 3.11. *Suppose that $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{C})$ and that $1 < p_\infty < \infty$. Then $\mathcal{F}^{p(\cdot)} = \mathcal{F}^{p_\infty}$, and for every $f \in \mathcal{F}^{p_\infty}$ we have that $\|f\|_{\mathcal{F}^{p(\cdot)}} \sim \|f\|_{\mathcal{F}^{p_\infty}}$.*

Proof. Let $f \in \mathcal{F}^{p_\infty}$, then by Theorem 3.4 there exists a constant $C > 0$ such that for every $z \in \mathbb{C}$,

$$g(z) := \frac{|f(z)|\mathrm{e}^{-|z|^2}}{C\|f\|_{\mathcal{F}^{p_\infty}}} \leqslant 1.$$

By inequality (3.7) we conclude that

$$\int_{\mathbb{C}} |g(z)|^p(z)\,\mathrm{d}A(z) < \infty,$$

and consequently $\|f\|_{\mathcal{F}^{p(\cdot)}} \lesssim \|f\|_{\mathcal{F}^{p_\infty}}$. The second part is proven in a similar way using inequality (3.8). □

In this section, we introduce a Bergman-type projection on variable exponent Fock spaces and we will obtain a duality result. Due to Theorem 3.11, the global log-Hölder condition is too restrictive to study variable exponent Fock spaces. In this section, we will use a less restrictive condition that the Hardy-Littlewood maximal operator is bounded on $L^{(p(\cdot)/p_0)'}(\mathbb{C})$ for some $p_0 < p^-$. Of course, this is not a condition simple to verify and the question remains open to find a more constructive condition.

Let us formally define a projection P of a function g as

$$Pg(z) = \frac{2}{\pi}\int_{\mathbb{C}} g(w)\mathrm{e}^{2\overline{w}z}\mathrm{e}^{-2|w|^2}\,\mathrm{d}A(w). \tag{4.1}$$

It is shown in [19], Section 2.2 that P is a linear operator that maps $\mathcal{L}^p$ onto $\mathcal{F}^p$. We will show an analogous result for the case of variable exponents by putting Theorems 2.8 and 2.10 together.

Theorem 4.1. *Suppose that p belongs to $\mathscr{P}(\mathbb{C})$ and suppose that the Hardy-Littlewood maximal operator is bounded on $L^{(p(\cdot)/p_0)'}(\mathbb{C})$ for some $p_0 < p^-$. Then the operator P defined in* (4.1) *is bounded from $\mathcal{L}^{p(\cdot)}$ onto $\mathcal{F}^{p(\cdot)}$.*

Proof. First notice that since $\mathcal{F}^{p(\cdot)} \subset \mathcal{F}^{p^+}$, representation (2.6) holds for every function $g \in \mathcal{F}^{p(\cdot)}$ and consequently $Pg = g$. This implies that P is surjective.

Now we will use Theorem 2.10. It was mentioned before that if w is any weight in A_1, then it belongs to $A_{p_0,r}$ for any $r > 0$.

In consequence, by Theorem 2.8 we have that there exists a constant $C > 0$ such that for any $g \in \mathcal{L}^{p_0}(w)$ the following inequality holds:

$$\int_{\mathbb{C}} |Pg(z)|^{p_0}\mathrm{e}^{-|z|^2 p_0}w(z)\,\mathrm{d}A(z) \leqslant C\int_{\mathbb{C}} |g(z)|^{p_0}\mathrm{e}^{-|z|^2 p_0}w(z)\,\mathrm{d}A(z).$$

Hence, if we define the family

$$\mathscr{D} = \{(\mathrm{e}^{-|\cdot|^2}Pg, \mathrm{e}^{-|\cdot|^2}g)\colon\ g \in \mathcal{L}^{p_0}(w)\},$$

then we have the hypotheses of Theorem 2.10. Thus, there exists a constant $C_{p(\cdot)} > 0$ such that

$$\|\mathrm{e}^{-|\cdot|^2}Pg\|_{L^{p(\cdot)}(\mathbb{C})} \leqslant C_{p(\cdot)}\|\mathrm{e}^{-|\cdot|^2}g\|_{L^{p(\cdot)}(\mathbb{C})}$$

for every $g \in \mathcal{L}^{p_0}(w)$, and therefore

$$\|Pg\|_{\mathcal{L}^{p(\cdot)}(\mathbb{C})} \leqslant C_{p(\cdot)}\|g\|_{\mathcal{L}^{p(\cdot)}(\mathbb{C})}.$$

Finally, by density of $\mathcal{L}^{p_0}$ in $\mathcal{L}^{p(\cdot)}$ we get that the inequality holds for every function in $\mathcal{L}^{p(\cdot)}$. $\square$

We are almost ready to show a duality result for variable exponent Fock spaces. This will come as a consequence of the corresponding duality result for $\mathcal{L}^{p(\cdot)}$ spaces.

Theorem 4.2. *Suppose that* $p \in \mathscr{P}(\mathbb{C})$ *and let* p' *be such that* $1/p(z) + 1/p'(z) = 1$ *for all* $z \in \mathbb{C}$. *Then the dual space of* $\mathcal{L}^{p(\cdot)}$ *is isomorphic to* $\mathcal{L}^{p'(\cdot)}$ *and every functional* $\Lambda \in (\mathcal{L}^{p(\cdot)})^*$ *is of the type*

$$f \longmapsto \frac{2}{\pi}\int_{\mathbb{C}} f(z)\overline{h(z)}\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z),$$

and

$$\|\Lambda\| \sim \|h\|_{\mathcal{L}^{p'(\cdot)}}.$$

P r o o f. First notice that the operator

$$\begin{aligned} Q_{p(\cdot)}\colon\ L^{p(\cdot)} &\longrightarrow \mathcal{L}^{p(\cdot)},\\ f &\longmapsto C_{p(\cdot)}\mathrm{e}^{|\cdot|^2} \end{aligned}$$

satisfies that for every $f \in L^{p(\cdot)}$, it holds that

$$\varrho_{p(\cdot)}\Big(\frac{Q_{p(\cdot)}f}{\lambda}\Big) = \varrho_{\varphi_3}\Big(\frac{f}{\lambda}\Big).$$

This implies that

$$\|Q_{p(\cdot)}f\|_{\mathcal{L}^{p(\cdot)}} = \|f\|_{L^{p(\cdot)}}.$$

Now suppose that $h \in \mathcal{L}^{p'(\cdot)}$ and define the linear functional

$$\begin{aligned} \Lambda_h\colon\ \mathcal{L}^{p(\cdot)} &\longrightarrow \mathbb{C},\\ f &\longmapsto \frac{2}{\pi}\int_{\mathbb{C}} f(z)\overline{h(z)}\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z). \end{aligned}$$

By Theorem 3.1 we have that

$$|\Lambda_h(f)| \leqslant \frac{4}{\pi}\|f\|_{\mathcal{L}^{p(\cdot)}}\|h\|_{\mathcal{L}^{p'(\cdot)}}$$

and consequently $\lambda_h \in (\mathcal{L}^{p(\cdot)})^*$. On the other hand, suppose that $\Lambda \in (\mathcal{L}^{p(\cdot)})^*$ and define

$$\begin{aligned} \Gamma\colon\ L^{p(\cdot)} &\longrightarrow \mathbb{C},\\ g &\longmapsto C_{p(\cdot)}^{-1}\Lambda Q_{p(\cdot)}g. \end{aligned}$$

Since Λ and $Q_{p(\cdot)}$ are bounded, $\Gamma \in (L^{p(\cdot)})^*$ and by the duality result for variable exponent Lebesgue spaces (see for Example [4], Section 2.8) there exists a function $u \in L^{p'(\cdot)}$ such that $\|u\|_{L^{p'(\cdot)}} \sim \|\Gamma\|$ and for every $g \in L^{p(\cdot)}$

$$\Gamma g = \int_{\mathbb{C}} g(z)\overline{u(z)}\,\mathrm{d}A(z).$$

Take

$$h = \frac{\pi}{2C_{p'(\cdot)}} Q_{p'(\cdot)} u \in \mathcal{L}^{p'(\cdot)}$$

and notice that if $f \in \mathcal{L}^{p(\cdot)}$, then

$$\Lambda f = C_{p(\cdot)} \Gamma Q_{p(\cdot)}^{-1} f = \frac{2}{\pi} \int_{\mathbb{C}} f(z) \overline{g(z)} \mathrm{e}^{-2|z|^2} \, \mathrm{d}A(z).$$

Moreover, $\|\Lambda\| = \|\Gamma\| \sim \|u\|_{L^{p'(\cdot)}} = \|h\|_{\mathcal{L}^{p(\cdot)}}$. □

Remark 4.3. In the previous theorem, the constant $2/\pi$ is being considered in order to keep the correspondence with the representation given by equation (2.6).

Theorem 4.4. *Suppose that $p \in \mathscr{P}(\mathbb{C})$ and suppose that the Hardy-Littlewood maximal operator is bounded on $L^{(p(\cdot)/p_0)'}(\Omega)$ for some $p_0 < p^-$. Let p' be such that $1/p(z) + 1/p'(z) = 1$ for all $z \in \mathbb{C}$. Then the dual space of $\mathcal{F}^{p(\cdot)}$ is isomorphic to $\mathcal{F}^{p'(\cdot)}$ and every functional $\Lambda \in (\mathcal{F}^{p(\cdot)})^*$ is of the type*

$$f \mapsto \langle f, h \rangle = \frac{2}{\pi} \int_{\mathbb{C}} f(z) \overline{h(z)} \mathrm{e}^{-2|z|^2} \, \mathrm{d}A(z)$$

and

$$\|\Lambda\| \sim \|h\|_{\mathcal{F}^{p'(\cdot)}}.$$

Proof. First, by a similar reasoning as in the previous theorem, for every function $h \in \mathcal{F}^{p'(\cdot)}$ the linear functional

$$\begin{aligned} \Lambda_h \colon \mathcal{F}^{p(\cdot)} &\longrightarrow \mathbb{C}, \\ f &\longmapsto \frac{2}{\pi} \int_{\mathbb{C}} f(z) \overline{h(z)} \mathrm{e}^{-2|z|^2} \, \mathrm{d}A(z) \end{aligned}$$

is bounded.

On the other hand, if $\Lambda \in (\mathcal{F}^{p(\cdot)})^*$, then since $\mathcal{F}^{p(\cdot)}$ is a closed subset of $\mathcal{L}^{p(\cdot)}$, we use Hahn-Banach theorem to extend Λ to a bounded linear functional $\widetilde{\Lambda} \in (\mathcal{L}^{p(\cdot)})^*$ with $\|\widetilde{\Lambda}\| = \|\Lambda\|$. By the previous theorem, there exists a function $\tilde{h} \in \mathcal{L}^{p'(\cdot)}$ such that $\|\tilde{h}\| = \|\widetilde{\Lambda}\|$ and for every $f \in \mathcal{L}^{p(\cdot)}$,

$$\Lambda(f) = \frac{2}{\pi} \int_{\mathbb{C}} f(z) \overline{\tilde{h}(z)} \mathrm{e}^{-2|z|^2} \, \mathrm{d}A(z).$$

Take $h = P\tilde{h}$, where P denotes the projection defined in equation (4.1). Since P is bounded, $\|h\|_{\mathcal{F}^{p(\cdot)}} \lesssim \|\tilde{h}\|_{\mathcal{L}^{p(\cdot)}}$. Moreover, if $f \in \mathcal{F}^{p(\cdot)}$, then we know that $Pf = f$

and consequently,

$$\begin{aligned}\Lambda f = \widetilde{\Lambda} f &= \frac{2}{\pi}\int_{\mathbb{C}} f(z)\overline{\tilde{h}(z)}\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z)\\ &= \frac{4}{\pi^2}\int_{\mathbb{C}} f(w)\mathrm{e}^{-2|w|^2}\int_{\mathbb{C}}\mathrm{e}^{2z\overline{w}}\overline{\tilde{h}(z)}\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z)\,\mathrm{d}A(w)\\ &= \frac{2}{\pi}\int_{\mathbb{C}} f(w)\overline{h(w)}\mathrm{e}^{-2|w|^2}\,\mathrm{d}A(w).\end{aligned}$$

The use of Fubini's theorem is justified since

$$\int_{\mathbb{C}}|f(w)\mathrm{e}^{-2|w|^2}\int_{\mathbb{C}}\mathrm{e}^{2z\overline{w}}\overline{\tilde{h}(z)}\mathrm{e}^{-2|z|^2}\,\mathrm{d}A(z)|\,\mathrm{d}A(w) \lesssim \|f\|_{\mathcal{L}^{p(\cdot)}}\|J\tilde{h}\|_{\mathcal{L}^{p'(\cdot)}},$$

and using Corollary 2.9 in combination with 2.10 we conclude that

$$\|J\tilde{h}\|_{\mathcal{L}^{p'(\cdot)}} \lesssim \|\tilde{h}\|_{\mathcal{L}^{p'(\cdot)}}.$$

Finally, again by Hölder's inequality we have that $\|\Lambda\| \lesssim \|h\|_{\mathcal{F}^{p(\cdot)}}$ which implies that

$$\|\Lambda\| \sim \|h\|_{\mathcal{F}^{p(\cdot)}}.$$

□

We finish this article with one consequence of the previous duality. Recall that if $p \in \mathscr{P}(\mathbb{C})$, then $\mathcal{F}^{p^-} \subset \mathcal{F}^{p(\cdot)} \subset \mathcal{F}^{p^+}$ and as an immediate corollary, we get that $\{K_z\colon z \in \mathbb{C}\} \subset \mathcal{F}^{p(\cdot)}$. Consequently, every $f \in \mathcal{F}^{p(\cdot)}$ satisfies representation (2.6). Moreover, denote as V the closed vector subspace of $\mathcal{F}^{p(\cdot)}$ generated by the set $\{K_z\colon z \in \mathbb{C}\}$ and suppose that $f \in \mathcal{F}^{p(\cdot)} \setminus V$. Then there exists $h \in \mathcal{F}^{p'(\cdot)}$, $h \neq 0$ such that $\langle K_z, h\rangle = 0$ for every $z \in \mathbb{C}$, but then $h \equiv 0$, a contradiction. We record this in the following corollary.

Corollary 4.5. *Suppose $p \in \mathscr{P}(\mathbb{C})$. Then the set of all linear combinations of reproducing kernels is dense in $\mathcal{F}^{p(\cdot)}$.*

As a direct consequence, we have the density of the set of polynomials in $\mathcal{F}^{p(\cdot)}$.

Corollary 4.6. *Suppose $p \in \mathscr{P}(\mathbb{C})$. Then the set of polynomials is dense in $\mathcal{F}^{p(\cdot)}$.*

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Authors' addresses: Gerardo R. Chacón (corresponding author), Gallaudet University, Department of Science, Technology and Mathematics, 800 Florida Ave NE, Washington, DC 20002, USA, e-mail: `gerardo.chacon@gallaudet.edu`; Gerardo A. Chacón, Doctorado en Educación Matemática, Universidad Antonio Nariño, Bogotá, Colombia, e-mail: `gerardoachg@uan.edu.co`.