Jiří Jarušek

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SOLVABILITY OF A DYNAMIC RATIONAL CONTACT WITH LIMITED INTERPENETRATION FOR VISCOELASTIC PLATES

Jiří Jarušek, Praha

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Abstract. Solvability of the rational contact with limited interpenetration of different kind of viscolastic plates is proved. The biharmonic plates, von Kármán plates, Reissner-Mindlin plates, and full von Kármán systems are treated. The viscoelasticity can have the classical ("short memory") form or the form of a certain singular memory. For all models some convergence of the solutions to the solutions of the Signorini contact is proved provided the thickness of the interpenetration tends to zero.

Keywords: dynamic contact problem; limited interpenetration; viscoelastic plate; existence of solution

MSC 2010: 35Q74, 74D10, 74H20, 74K20, 74M15

1. Introduction and notation

Despite a great amount of actual and/or possible applications, the theory of contact problems remains still underdeveloped. The study of contact problems started by Signorini [12], [13]. His model describing a contact of a deformable body with a rigid foundation respects the impenetrability of mass. It was extended to dynamic problems by Amerio, Prouse, Schatzman and further authors in late seventies and early eighties of the last century. The monograph [6] summed up the development in this field till its publication. The highly nonlinear Signorini model is complex. Therefore, a bit later the so-called normal compliance approach was introduced. This approach is nothing else than replacement of the original Signorini contact model by some kind of its penalization. Although such kind of approximation is a suitable auxiliary tool in the numerical investigation of contact problems, this approach has brought no deep results to their theory. It is usually easy to derive properties of

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solutions of such approximate problems and the real hard work starts by the limit process to the original problem.

However, the normal compliance approach has drawn the attention to the fact that the complete impenetrability of mass need not be completely physically realistic, because from the microscopical point of view no material is flat or smooth enough. Just in the medium advanced microscopes the seemingly perfectly flat or smoothly curved surfaces are seen as a huge collections of asperities and small holes or cavities. The asperities may be deformed or may fill the holes of the counterpart partially or completely. Hence, it has some good sense to study models where some interpenetration between body and the foundation is allowed to describe macroscopically those phenomena. However, to remain physically realistic, this interpenetration model must include a certain bound after which the further penetration is not possible. And, as well, it is realistic to assume that such a bound cannot be reached.

These are the premises of the rational contact model with a limited interpenetration which was introduced by [7] and [8], where the solvability of its static version has been proved (it was then still called the normal compliance model with a limited interpenetration). The first such dynamic (frictionless) rational contact has been investigated in [9]. It concerns a boundary contact of a body with a rigid foundation. I am convinced that this contact model is obviously physically well based, i.e. rational, if some interpenetration between the body and the foundation is admitted.

Since 2006 a series of papers about the solvability of dynamic Signorini contact problems for different models of plates [1]–[4] has been published. The purpose of this paper is to extend these results to the rational contact with limited interpenetration. Unlike [9] we face here a domain contact.

Since we have no knowledge that anybody else has studied the presented rational contact model or anything similar, there are no other relevant references to be cited than those listed at the end of this paper.

In the sequel by $W_p^k(\Omega)$ the (Sobolev-Slobodetskii if k is not entire) spaces of functions on a domain Ω having their (possibly fractional) derivatives up to order k integrable in the pth power are denoted. The symbol $\mathring{W}_p^k$ denotes the spaces of functions from $W_p^k(\Omega)$ which have zero traces. If $p = 2$ those spaces are denoted by $H^k(\Omega)$, $\mathring{H}^k(\Omega)$. Vectors are denoted by bold symbols; the same rule is used for spaces of vectors. The symbol $\hookrightarrow$ denotes an embedding, while $\hookrightarrow\hookrightarrow$ an embedding which is compact. For a function $u\colon \Omega \to \mathbb{R}$ we denote by $u_+ = \max\{u, 0\}$ its "positive part" and $u_- = \max\{-u, 0\}$ its "negative part".

Here we collect the results from the theory of function spaces needed in the sequel. Here and in the sequel $I = [0, T]$ is a nonempty time interval. The standard tool,

Lemma 1 (Aubin). *Let $\mathfrak{B}_0 \hookrightarrow\hookrightarrow \mathfrak{B} \hookrightarrow \mathfrak{B}_1$ be Banach spaces, the first reflexive and separable. Let $1 < p < \infty,\ 1 \leqslant q < \infty$. Then*

$$W \equiv \{v;\ v \in L_p(I; \mathfrak{B}_0),\ \dot{v} \in L_q(I; \mathfrak{B}_1)\} \hookrightarrow\hookrightarrow L_p(I; B),$$

is not sufficient for our purpose. Hence, we will use the following facts, the proofs of which follow from Chapter 2 of the monograph [6]:

Theorem 2 (Embedding theorem). *Let $M \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Let $p, q \in (1, \infty)$, $\gamma \in [0, 1]$, and $\alpha \in (\gamma, 1]$ be numbers such that the inequality*

$$\frac{1}{\alpha}\Big(\frac{N}{p} - \frac{N}{q} + \gamma\Big) \leqslant 1 \tag{0}$$

holds. Then the Sobolev-Slobodetskii space $W_p^\alpha(M)$ is continuously embedded into $W_q^\gamma(M)$. If the above inequality is strict, then the embedding is compact for any real $q \geqslant 1$. For $q = \infty$ this is true under the convention $1/q = 0$.

Corollary 3. *Let M and I be as above. Let p_i, q_i belong to $(1, \infty)$, α_i belong to $(0, 1]$ and γ_i to $[0, \alpha_i)$, $i = 1, 2$. Assume that (0) holds with $i = 1$ and N replaced by 1 and that it simultaneously holds for $i = 2$. Then $W_{p_1}^{\alpha_1}(I; W_{p_2}^{\alpha_2}(M))$ can be imbedded into $W_{q_1}^{\gamma_1}(I; W_{q_2}^{\gamma_2}(M))$. If both the inequalities are strict, the imbedding is compact. The last assertion still holds if q_i is infinite, provided we use the convention $1/q_i = 0$, $i = 1, 2$.*

Theorem 4 (Interpolation theorem). *Let M be as above, let k_1, k_2 belong to $[0, 1]$, let p_1, p_2 belong to $(1, \infty)$ and Θ_λ to $[0, 1]$. Then there exists a constant c such that for all $u \in W_{p_1}^{k_1}(M) \cap W_{p_2}^{k_2}(M)$ the following estimate holds:*

$$\|u\|_{W_p^k(M)} \leqslant c\|u\|^{\Theta_\lambda}_{W_{p_1}^{k_1}(M)} \|u\|^{1-\Theta_\lambda}_{W_{p_2}^{k_2}(M)}$$

with $k = \Theta_\lambda k_1 + (1 - \Theta_\lambda)k_2$ and $1/p = \Theta_\lambda/p_1 + (1 - \Theta_\lambda)/p_2$. The assertion remains true if $k_1 = k_2 = 0$ and p_1, p_2 belong to $[1, \infty]$.

Corollary 5 (Generalization). *Let M, k_1, k_2, p_1, p_2 be as above. Let I be a bounded interval in $\mathbb{R}$, let κ_1, κ_2 belong to $[0,1]$, let q_1, q_2 belong to $(1,\infty)$ and Θ_λ to $[0,1]$. Then there exists a constant c such that for all $u \in W^{\kappa_1}_{q_1}(I; W^{k_1}_{p_1}(M)) \cap W^{\kappa_2}_{q_2}(I; W^{k_2}_{p_2}(M))$ we have*

$$\|u\|_{W^{\kappa}_{q}(I;W^{k}_{p}(M))} \leqslant c\|u\|^{\Theta_\lambda}_{W^{\kappa_1}_{q_1}(I;W^{k_1}_{p_1}(M))}\|u\|^{1-\Theta_\lambda}_{W^{\kappa_2}_{q_2}(I;W^{k_2}_{p_2}(M))},$$

where $k = \Theta_\lambda k_1 + (1-\Theta_\lambda)k_2$, $\kappa = \Theta_\lambda \kappa_1 + (1-\Theta_\lambda)\kappa_2$, $1/q = \Theta_\lambda/q_1 + (1-\Theta_\lambda)/q_2$ and $1/p = \Theta_\lambda/p_1 + (1-\Theta_\lambda)/p_2$. If $\kappa_1 = \kappa_2 = 0$ and q_1, q_2 belong to $[1,\infty]$, the assertion still holds.

3. Abstract formulation of the problem for the clamped or simply supported viscoelastic plate and the scheme of its solution

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with sufficiently smooth boundary Γ. Let X be a Sobolev-type Hilbert space defined on Ω, let Y be the space of traces of elements from X on Γ. Let $A, B\colon X \to X^*$ be two linear symmetric strongly elliptic operators in the form $\mathscr{D}^* a \mathscr{D}$, $\mathscr{D}^* b \mathscr{D}$, respectively, where $\mathscr{D}$ is a differential operator and a, b are positively definite matrices or tensors of time constant but possibly space-dependent elements. Here the dual space X^* is defined via the suitable generalization of the $L_2(\Omega)$ scalar product. Let $\mathscr{X} \equiv L_2(I;X)$. We introduce the bilinear forms $\mathscr{A}\colon \{u,v\} \mapsto \langle a\mathscr{D}u, \mathscr{D}v\rangle_Q$, $\mathscr{B}\colon \{u,v\} \mapsto \langle b\mathscr{D}u, \mathscr{D}v\rangle_Q$, where $\langle\cdot,\cdot\rangle_Q$ is the $L_2(Q)$ scalar product and $Q \equiv I \times \Omega$. Let $S \equiv I \times \Gamma$. Let $E(t)\colon X^2 \to X^*$ be another operator $E(t) \equiv E(t)(u,\dot u)$, $u, \dot u \in X$, let $\mathscr{E}(u,\dot u)\colon v \mapsto \langle E(\cdot)(u,\dot u), v\rangle_Q$, $u, \dot u, v \in \mathscr{X}$.

We will call the elements of $v \in X$ or $v\colon I \to X$ such that $v \in \mathscr{X}$ displacements, and their first time derivatives (denoted by dots) velocities. Let γ be a negative real number. Let $p\colon \mathbb{R} \to \overline{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\}$ be a continuous and nonincreasing function such that $p(x) = 0$ for $x \geqslant 0$, $p(x) \in \mathbb{R}$ for $x > \gamma$, and $\lim_{x \searrow \gamma} p(x) = \infty$, where $\gamma \in \mathbb{R}$ is a given bound of the interpenetration. Our problem is to find $u \in \mathscr{X}$ such that $\dot u \in \mathscr{X}$ for which the following set of relations holds:

$$\begin{aligned} (1)\qquad \ddot u &= A\dot u + Bu - E(u.\dot u) + p(u+g) + f \text{ in } L_2(\Omega) \quad \text{on } I,\\ D(u) &= 0 \in Y,\\ u(0) &= u_0 \in X,\ \dot u(0) = u_1 \in X. \end{aligned}$$

Here D is a general differential operator of a Dirichlet or somewhat combined type. If $X = H^2(\Omega)$, the space of square integrable functions having the first and second generalized derivatives square integrable as well and A, B are differential operators of the fourth order then $D(u) \equiv \{D_1(u), D_2(u)\}$, $D_1(u) = u - u_0$ for both cases, $D_2(u) = \partial_{\tilde{n}}(u - u_0)$ (the outer co-normal derivative) or $D_2(u) = M(u)$ a Neumann-type operator, which ensures that after the integration by parts in the space variable in the variational formulation of the problem no additional boundary term occurs. The first couple describes a clamped plate while the second a simply supported plate. Let us mention that $p(u+g)$ stands there for the contact force, where $g \geqslant 0$ is the time-independent and bounded gap function.

We will define a sequence of auxiliary approximate problems to (1) by adding the following additional assumption on p: We assume the existence of a sequence $\{\delta_k\} \subset \mathbb{R}_+$ such that $\delta_k \searrow 0$ and for each $k \in \mathbb{N}$ there is a left derivative $\partial^l p$ at the points $\gamma + \delta_k$, $k \in \mathbb{N}$ such that $\partial^l p(\gamma + \delta_k) \geqslant \partial^l p(\gamma + \delta_{k+1})$, $k \in \mathbb{N}$, and $\lim\limits_{k\to\infty} \partial^l p(\gamma + \delta_k) = -\infty$. Then we define $p_k\colon\ y \mapsto \min\{p, p(\gamma + \delta_k) + \partial^l p(\gamma + \delta_k) \times (y - \gamma - \delta_k)\}$ for $y \leqslant \gamma + \delta_k$, $p_k = p$ elsewhere and the auxiliary problem is defined by the replacement of p by p_k in (1).

Let us denote by $\langle\cdot,\cdot\rangle_\Omega$ the duality pairing between X and X^* derived from the $L_2(\Omega)$ scalar product and by $\langle\cdot,\cdot\rangle_Q$ the duality pairing between $\mathscr{X}$ and $\mathscr{X}^*$ derived from the $L_2(Q)$ scalar product. Let $\mathscr{X}_0$ be a subspace of elements of $\mathscr{X}$ satisfying the appropriate homogeneous Dirichlet boundary condition in (1). Moreover, let $\mathscr{X}_1 \equiv \{v \in \mathscr{X}_0;\ \dot{v} \in L_2(Q)\}$.

Multiplying the first row of (1) by a test function $v \in \mathscr{X}_0$ and performing the integration by parts both in space variables and in time, we get the variational formulation of the problem (1): *Find $u \in u_0 + \mathscr{X}_0$ such that $\dot{u} \in \mathscr{X}$, the initial condition for u from* (1) *holds and for every $v \in \mathscr{X}_1$ the equation*

$$
\begin{aligned}
(2) \qquad -\langle\dot{u},\dot{v}\rangle_Q + \mathscr{A}(\dot{u},v) + \mathscr{B}(u,v) + \langle\mathscr{E}(u,\dot{u}),v\rangle_Q - \langle p(u+g),v\rangle_Q \\
+ \langle\dot{u}(T,\cdot),v(T,\cdot)\rangle_\Omega = \langle f,v\rangle_Q + \langle u_1,v(0,\cdot)\rangle_\Omega
\end{aligned}
$$

holds. For an approximate problem p is replaced by p_k and the integration by parts in time for the acceleration term is omitted:

$$
(3) \qquad \langle\ddot{u},v\rangle_Q + \mathscr{A}(\dot{u},v) + \mathscr{B}(u,v) + \langle\mathscr{E}(u,\dot{u}),v\rangle_Q - \langle p_k(u+g),v\rangle_Q = \langle f,v\rangle_Q,
$$

hence, it is sufficient to take the test functions from $\mathscr{X}_0$ here. Of course, we have to require additionally that the initial condition for the velocity in (1) is satisfied.

In the sequel, we will assume that the operator $\mathscr{E}(v.\dot{v}) \equiv \{E(t)(v,\dot{v});\ t \in I\}\colon \mathscr{X}^2 \to \mathscr{X}^*$ is completely continuous, or such that $v \mapsto \langle\mathscr{E}(v.\dot{v}),v\rangle_Q$ is weakly lower semicontinuous on $\mathscr{X}$. Moreover, we assume that the initial conditions in (1) are satisfied. Denoting $I_t \equiv [0,t]$, $Q_t \equiv I_t \times \Omega$, $t \in (0,T]$, we assume that $\langle\mathscr{E}(v,\dot{v}),\dot{v}\rangle_{Q_t} \geqslant -\text{const.}\ (u_0,u_1) - k_1\|\dot{v}\|_{\mathscr{X}} - k_0|v|_{L_\infty(I_t,X)}$ for $v \in \mathscr{X}$ such that $\dot{v} \in \mathscr{X}$ and $t \in I$

with all constants nonnegative (observe that an easy inequality $g_1 g_2 \leqslant g_1^2/\varepsilon + \varepsilon g_2^2$ for $g_i \in \mathbb{R}$, $i = 1,2$, and $\varepsilon > 0$ arbitrarily small as well as the Gronwall lemma are needed here to derive an *a priori* estimate if the constants are not zero). Further, we assume that

$$(4) \qquad u_0 \in H^2(Q) \quad \text{such that} \quad u_0 \geqslant c_0 \quad \text{on } \overline{Q}.$$

Here c_0 is a positive constant.

The proof of solvability of the auxiliary problem (3) under the assumption (4) does not differ from the proof of a penalized problem to the appropriate Signorini contact. It is solved via the Galerkin approximation using just identical arguments, because in this case the auxiliary contact term represents a completely continuous perturbation of the appropriate problem without contact. By putting $v = (\dot{u}_k - \dot{u}_0)\chi_{Q_t}$ in (2) with p_k, where χ_M is the characteristic function of a set M (equal 1 on M and vanishing elsewhere), $t \in (0,T]$, we get (after a certain small and obvious calculation) the *a priori* estimate of the respective solutions u_k to the approximate problems with p_k

$$(5) \quad \|\dot{u}_k\|^2_{L_\infty(I;L_2(\Omega))} + \|u_k\|^2_{L_\infty(I;X)} + \|\dot{u}_k\|^2_{\mathscr{X}} + \|P_k(u_k+g)\|_{L_\infty(I;L_1\Omega)} \leqslant \text{const.},$$

where $P_k\colon\ s \mapsto \int_s^\infty p_k(z)\,\mathrm{d}z$, $s \in \mathbb{R}$. Let us take in mind that $L_1(\Omega) \subset L_\infty(\Omega)^* \hookrightarrow X^*$, because for the primal spaces the compact reverse embeddings hold. Since

$$\|p_k(u_k+g)\|_{L_1(Q)} \leqslant c_0^{-1}\langle p_k(u_k+g), u_0 - u_k\rangle_Q$$

(observe that $xp_k(x) \leqslant 0,\ x \in \mathbb{R}$), the use of (2) for $p = p_k$ and $v = u_0 - u_k$ and the estimate (5) yield that the sequence $\{\|p_k(u_k+g)\|_{L_1(Q)}\}$ is bounded. Then we derive from this and (1) the dual estimate

$$(6) \qquad \|\ddot{u}_k\|_{L_1(I;X^*)} \leqslant \text{const.}$$

with the constant independent of k. Hence $\{\dot{u}_k\}$ is bounded in $W^{1-\varepsilon_1}_{1+\varepsilon_2}(I;H^{-2-\varepsilon_3}(\Omega))$ for any $\varepsilon_2 > 0$, $\varepsilon_3 > 0$ and for $\varepsilon_1 \equiv \varepsilon_1(\varepsilon_2) \searrow 0$ if $\varepsilon_2 \searrow 0$. Interpolating this space with the space $L_q(I;L_2(\Omega))$ for $q = 1 + 1/\varepsilon_2$, we get that

$$(7) \qquad \|\dot{u}_k\|_{H^{1/2}(I;H^{-1-\theta}(\Omega))} \leqslant C, \quad \text{i.e. } \|u_k\|_{H^{3/2}(I;H^{-1-\theta}(\Omega))} \leqslant C \quad \text{with } 0 < \theta \text{ arbitrarily small.}$$

Interpolating this result with the third term in (5), we get

$$(8) \qquad \{\dot{u}_k\} \text{ is bounded in } H^\theta(I;L_2(\Omega)) \quad \text{for any } \theta \in (0,1/3),$$

i.e. $\{u_k\}$ is bounded in the anisotropic space $H^{1+\theta,2}(Q)$, and with help of the extension of the functions from Ω to $\mathbb{R}^2$, the Fourier transform, and the Hölder inequality we can find for any $\theta \in (0, \frac{1}{3})$ an index $\theta_0 > 0$ such that

$$\{u_k\} \quad \text{is bounded in} \quad H^{(1+\theta)/2}(I; H^{1+\theta_0}(\Omega)) \hookrightarrow\hookrightarrow C_0(\overline{Q}). \tag{9}$$

By virtue of (5), (6), (8), and (9) we get certain u and ϑ such that the convergences

$$\begin{aligned}
&u_k \rightharpoonup^* u \text{ in } L_\infty(I; X),\ \dot{u}_k \rightharpoonup^* \dot{u} \text{ in } L_\infty(I; L_2(\Omega)),\\
&\dot{u}_k \to \dot{u} \text{ in } L_2(Q),\ u_k \to u \text{ in } C_0(\overline{Q}),\\
&\langle \mathscr{E}(u_k, \dot{u}_k), u_k\rangle_Q \to \langle \mathscr{E}(u, \dot{u}), u\rangle_Q \quad \text{or}\\
&\liminf_{k\to\infty} \langle \mathscr{E}(u_k, \dot{u}_k), u_k\rangle_Q \geqslant \langle \mathscr{E}(u, \dot{u}), u\rangle_Q,\\
&p_k(u_k + g) \rightharpoonup^* \vartheta \text{ in } L_\infty^*(Q)
\end{aligned} \tag{10}$$

hold for a possible subsequence. Observe that if v is a function such that $p(v+g) \in L_1(Q)$ then $\int_Q p_k(v+g)w \,\mathrm{d}x \,\mathrm{d}t \to p(v+g)w \,\mathrm{d}x \,\mathrm{d}t$ e.g. for any function $w \in L_\infty(Q)$. Indeed, if $w \geqslant 0$ this follows from the monotone convergence theorem. For a general w we use its decomposition to the positive and negative parts.

Performing the integration by parts in time for the acceleration term and putting $v = u_k - u_0$ in (2) with p_k and $v = u - u_0$ in the original (2), using the weak lower semicontinuity of the elliptic operators and the strong convergence of the others, we get $\langle \vartheta, u\rangle_Q \geqslant \limsup\limits_{k\to\infty} \langle p_k(u_k+g), u_k\rangle_Q$. In the sequel, we will denote the last fact as *the upper semicontinuity of the contact term* $\langle p_k(u_k+g), u_k\rangle_Q$ (i.e. with respect to k) and we will see it is common for all problems treated in this paper.

Since p_k are monotone, $k \in \mathbb{N}$, this yields $\langle \vartheta - p(v+g), u - v\rangle_Q \geqslant 0$ for every $v \in \mathscr{X}_0$ such that $p(v) \in L_1(Q)$. Hence $\{\vartheta, u\}$ may be added to the graph of p so that the extended graph remains monotone. Observe that $u + g \geqslant u_0$ on S, hence its continuity ensures the existence of some neighbourhood $\mathscr{U}$ of S, where $u+g$ is bounded away from γ and the Fatou Lemma ensures $\langle \vartheta, v\rangle_Q \geqslant \langle p(u+g), v\rangle_Q$ for every nonnegative $v \in \mathscr{X}_0$. Since it is surely possible to construct such nonnegative $v \in \mathscr{X}_0$ that $v = 1$ on $Q\setminus\mathscr{U}$, it is evident that $p(u+g) \in L_1(Q)$. Moreover, the superposition operator p is the derivative of the superposition operator P which is convex and lower semicontinuous. Since every Hilbert space satisfies the corresponding requirements, Theorem 5.1.7 of the monograph [5] ensures its maximal monotonicity taken as an operator from $\mathscr{X}_1$ into $\mathscr{X}_1^*$. This yields that $\vartheta = p(u+g)$, hence u is a solution of the variational equation (2) and we are done. We have proved:

Theorem 6. *Under the above mentioned assumptions for the employed operators and the function p there exists a solution to the problem* (2).

E x a m p l e 1. A biharmonic plate. Here $\mathscr{D} = \Delta$, a, b are positive constants and $\mathscr{E} = 0$.

E x a m p l e 2. A von Kármán plate without rotation inertia. First we introduce for two functions u, v

$$[u,v] = \partial_{11}\partial_{22}v + \partial_{22}u\partial_{11}v - 2\partial_{12}u\partial_{12}v, \tag{11}$$

where here and in the sequel $\partial_i \equiv \partial/\partial_{x_i}$, $i = 1,2$, $\partial_t \equiv \partial/\partial t$ and $\partial_{ij} \equiv \partial_i\partial_j$, $i,j = 1,2$. Then we define the bilinear operator $\Phi\colon H^2(\Omega)^2 \to \mathring{H}^2(\Omega)$ by means of the variational equation

$$\int_\Omega \Delta\Phi(u,v)\Delta\varphi \,\mathrm{d}x = \int_\Omega [u,v]\varphi \,\mathrm{d}x,\ u,v,\varphi \in \mathring{H}^2(\Omega). \tag{12}$$

The equation (12) has a unique solution, because $[u,v] \in L_1(\Omega) \hookrightarrow H^2(\Omega)^*$. The well-defined operator Φ is compact and symmetric. Let us recall Lemma 1 from [10] due to which $\Phi\colon H^2(\Omega)^2 \to W^2_p(\Omega)$ for any $p \in (2,\infty)$, and

$$\|\Phi(u,v)\|_{W^2_p(\Omega)} \leqslant c\|u\|_{H^2(\Omega)}\|v\|_{W^1_p(\Omega)} \quad \forall\, u,v \in H^2(\Omega)^2, \tag{13}$$

i.e. $w \mapsto \Phi(w,w)$ is completely continuous from $H^\delta(Q) \cap \mathscr{X}$ to $\mathscr{X}$ for any $\delta > 0$.

To avoid the introduction of the Airy stress function, we introduce directly the variational formulation. For this we introduce

$$\begin{aligned} A_0\colon \{u,y\} \mapsto b_0(&\partial_{ll}u\partial_{ll}y + \nu(\partial_{11}u\partial_{22}y + \partial_{22}u\partial_{11}y) \\ &+ 2(1-\nu)\partial_{12}u\partial_{12}y), \quad b_0 = \text{const.} > 0, \end{aligned} \tag{14}$$

where $\nu \in (-\frac{1}{2}, 1)$ is a material constant (the Poisson ratio) and the standard summation convention for the repeating index l is applied. Then we define $\langle \mathscr{A}\dot{u}, v\rangle_Q$ as $e_1 \int_Q A_0(\dot{u},v)\,\mathrm{d}x\,\mathrm{d}t$, $\langle \mathscr{B}u, v\rangle_Q$ as $e_0 \int_Q A_0(u,v)\,\mathrm{d}x\,\mathrm{d}t$, $\mathscr{E}\colon\ u \mapsto b([u, e_1\partial_t\Delta\Phi(u,u) + e_0\Delta\Phi(u,u)])$, where e_1, e_0 are other material constants (the Young moduli) which are positive. With such defined mappings the variational formulation of the problem has exactly the form of (2). It is easy to derive that

$$\langle \mathscr{E}(u_k,\dot{u}_k), u_k\rangle_Q = \frac{1}{2}\int_Q (e_1/2\ \partial_t(\Delta\Phi(u_k,u_k))^2 + e_0(\Delta\Phi(u_k,u_k))^2)\,\mathrm{d}x\,\mathrm{d}t \tag{15}$$

(cf. [1]). Hence it satisfies the corresponding requirements and the quadratic forms generated by so defined $\mathscr{A}$, $\mathscr{B}$, $\langle \mathscr{E}\cdot,\cdot\rangle_Q$ are weakly lower semicontinuous and we are done. We remark that $M(u) = b(e_1 m(\dot{u}) + e_0 m(u))$, where $m(u) = \Delta u + (1-\nu)(2n_1n_2\partial_{12}u - n_1^2\partial_{22}u - n_2^2\partial_{11}u)$.

E x a m p l e 3. A simply supported von Kármán plate with the rotation inertia. Here the original structure (1) is enriched by the additional term $Gu = g_0\Delta\ddot{u}$ on the right-hand side of the first row of (1). If g_0 is just a positive constant, then this term contributes (after the obvious integration by parts) to the extension of the a priori estimate (5) by the term $\|\nabla\dot{u}_k\|^2_{L_2(I;L_2(\Omega))}$. The dual estimate $\|g_0\Delta\ddot{u}_k - \ddot{u}_k\|_{L_2(I,X^*)} \leqslant$ const. is here k-dependent. After integration by parts this gives $\sup\limits_{v\in L_2(I;X),\,\|v\|\leqslant 1} \langle\ddot{u}, g_0\Delta v - v\rangle_Q \leqslant$ const. The operator $g_0\Delta - I$, where I is the identity, is an isometry between the space $X = H^2(\Omega)\cap\mathring{H}^1(\Omega)$ and $L_2(Q)$, hence the dual estimate yields $\ddot{u}_k \in L_2(Q)$. In further treatment an additional lower semicontinuous term $\langle g_0\nabla\dot{u}, \nabla\dot{u}\rangle_Q$ occurs, which does not change the treatment of the limit process from the approximate to the original problem. In fact, from the k-independent $L_1(Q)$ estimate of the approximate contact term we get (using again the properties of the operator $g_0\Delta - I$) the k-independent dual estimate $\|\ddot{u}_k\|_{L_1(I;L_2(\Omega))} \leqslant$ const. The strong convergences of u_k and $\dot{u}_k$ from (10) hold as well as the upper semicontinuity of the contact term with respect to k. Hence, we are in the same situation as above, and via the maximal monotonicity argument we are done.

4. Von Kármán model with a singular memory

Let us introduce the kernel K of the singular memory term which is assumed to be integrable over $\mathbb{R}_+$ and to have the form

$$
\begin{aligned}
&K\colon\ t\mapsto t^{-2\alpha}q(t)+r(t),\quad t\in\mathbb{R}_+\equiv(0,\infty)\text{ with }\alpha\in(0,\tfrac12),\\
&K\colon\ t\mapsto 0,\quad t\leqslant 0.
\end{aligned}
\tag{16}
$$

Both q and r belong to $C^1(\mathbb{R}_+)$, they are nonnegative and nonincreasing functions. Moreover, we assume that $q(t) > 0$ for t on an nonempty interval $[0, t_0]$. Let us denote $d_m\colon v\mapsto \int_0^t K(t-s)(v(t,\cdot)-v(s,\cdot))\,\mathrm{d}s$ for a function v on Q. Let us point out that

$$
\begin{aligned}
\langle d_m v, \dot{v}\rangle_Q &= \int_Q\int_s^T \frac12\big(\partial_t(K(t-s)(v(t)-v(s))^2)\\
&\quad - (v(t)-v(s))^2\partial_t K(t-s)\big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s\\
&= \int_Q \frac12 K(T-s)(v(T)-v(s))^2\,\mathrm{d}x\,\mathrm{d}s\\
&\quad - \int_Q\int_0^t \frac12 K'(t-s)(v(t)-v(s))^2\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}s
\end{aligned}
\tag{17}
$$

holds and the second term in this formula leads to the fractional time-derivative norm of v. We recall that such a norm used in the sequel is for a Banach space X defined as follows:

$$\|v\|^2_{H^\alpha(I;X)} \equiv \int_I \|v\|^2_X \, \mathrm{d}t + \int_I \int_I \frac{\|v(t) - v(s)\|^2_X}{|t-s|^{1+2\alpha}} \, \mathrm{d}s \, \mathrm{d}t.$$

We solve the problem

$$\begin{gathered} \ddot{u} - e_1 d_m A u - e_0 A u + \mathscr{E}_0 u = f + p(u+g) \quad \text{on } Q, \\ u = 0, \quad M(u) = 0 \text{ on } S,\ u = u_0,\ \dot{u} = u_1 \text{ on } \Omega. \end{gathered} \tag{18}$$

Here A is the fourth order elliptic operator of the form similar to that of A_0 defined in (14), i.e.

$$b_0(\partial_{ll}\partial_{ll} + \nu(\partial_{11}\partial_{22} + \partial_{22}\partial_{11}) + 2(1-\nu)\partial_{12}\partial_{12}) \tag{19}$$

and

$$\begin{aligned} &\mathscr{E}_0\colon\ u \mapsto [u, e_1 d_m \Delta\Phi(u,u) + e_0 \Delta\Phi(u,u)], \\ &M(u) = e_0 d_m m(u) + e_1 m(u), \\ &\qquad\qquad \text{where } m(u) = \Delta u + (1-\nu)(2n_1 n_2 \partial_{1,2} u - n_1^2 \partial_{2,2} u - n_2^2 \partial_{1,1} u) \\ &\qquad\qquad \text{for a simply supported plate,} \\ &M(u) = \partial u/\partial n \text{ for the clamped plate.} \end{aligned}$$

To be able to handle the singular memory term it is necessary to assume its smallness as follows

$$\int_0^\infty K(s)\, \mathrm{d}s < \frac{e_0}{2e_1}, \tag{20}$$

which ensures that the quadratic form

$$Z\colon\ V \mapsto \int_Q (e_1 d_m V + e_0 V) V \, \mathrm{d}x \, \mathrm{d}t, \quad V \in L_2(Q), \tag{21}$$

is strongly monotone.

We introduce the variational formulation of the problem. Let $X = H^2(\Omega) \cap \mathring{H}^1(\Omega)$ for the simply supported plate and $X = \mathring{H}^2(\Omega)$ for the clamped plate. The formulation reads: *Find* $u \in L_2(I;X) \cap H^1(I;L_2(\Omega))$ *such that* $u(0,\cdot) = u_0$ *is satisfied and for every* $y \in L_2(I;X_0) \cap H^1(I;L_2(\Omega))$ *the equation*

$$\begin{aligned}(22)\qquad \int_Q (e_1 d_m A_0(u,y) + e_0 A_0(u,y) - \dot{u}\dot{y} + \mathscr{E}_0 uy - p(u+g)y - fy)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_\Omega (-(\dot{u}y)(T,\cdot) + u_1(y(0,\cdot)))\,\mathrm{d}x = 0\end{aligned}$$

holds with A_0 *from* (14). We will solve this problem assuming that (4) holds.

We formulate the approximate problem again by replacing p by p_k, but unlike (22) no integration by parts in time for the acceleration term is applied, hence the test function may be taken just from $\mathscr{X}_0$. However, the initial condition for $\dot{u}$ from (18) must be added, cf. the difference between (2) and (3). It is solved again by the standard Galerkin procedure, for details cf. [2]. To get the k-independent *a priori* estimate for their solution, $y = \dot{u}_k - \dot{u}_0$ must be taken. After some calculation we finally obtain

$$\begin{aligned}(23)\qquad \|u_k\|^2_{H^\alpha(I;H^2(\Omega))} + \|\dot{u}_k\|^2_{L_\infty(I;L_2(\Omega))} + \|u_k\|^2_{L_\infty(I;H^2(\Omega))} \\ + \|\Phi(u_k,u_k)\|^2_{H^\alpha(I;H^2(\Omega))} \\ + \|P_k(u+g)\|_{L_\infty(I;L_1(\Omega))} \leqslant c \equiv c(f,u_0,u_1).\end{aligned}$$

The assumed smallness of the memory term yields again the uniform estimate of $\{\|p_k(u+g)\|_{L_1(Q)}\}$ which leads to the dual estimate $\|\ddot{u}_k\|_{L_1(I;X^*)} \leqslant$ const. Hence, the estimate (7) is valid also for this problem. Interpolating it with the fact that $\{u_k\}$ is bounded in $H^\alpha(I;H^2(\Omega))$, we get that $\{\dot{u}_k\}$ is bounded in $H^{\theta_1}(I;L_2(\Omega))$ for $\theta_1 \in (0,\alpha/3)$. Interpolation of this space with the time-fractional derivative space from (23) gives the space $L_2(I;H^{\theta_2}(\Omega))$ with $\theta_2 \in (0, 2\alpha/(3-2\alpha))$, hence $\{\dot{u}_k\}$ is bounded in the anisotropic space $H^{\theta_1,\theta_2}(Q)$. This space is compactly imbedded into $L_2(Q)$, which ensures that $\langle \dot{u}_k, \dot{u}_k\rangle_Q$ tends strongly to the limit $\langle \dot{u}, \dot{u}\rangle_Q$ even for the weak convergence of u_k in the employed spaces. Moreover, we are able to prove the relation (9) for $\theta \in (0,\alpha/3)$, hence $u_k \to u$ in $C_0(\overline{Q})$. Similarly to (15) we can derive that

$$\langle \mathscr{E}_0 u_k, u_k\rangle_Q = \int_Q (e_1 \Delta d_m \Phi(u_k,u_k)\Delta\Phi(u_k,u_k) + e_0(\Delta\Phi(u_k,u_k))^2)\,\mathrm{d}x\,\mathrm{d}t.$$

The compactness of Φ based on (13) and the fractional time-derivative norm in (23) yield the needed strong convergence of this term. Hence, we are ready for the limit procedure $k \to \infty$ to prove again the upper semicontinuity of $\langle p_k(u_k+g), u_k\rangle_Q$ and with the maximal monotonicity argument to prove $p_k(u_k+g) \rightharpoonup p(u+g)$. Thus u

is a solution of (22) and with the additional assumption (20) the existence theorem is proved also for this problem.

5. The problem for more complex viscoelastic plate models

In this section we will treat the Reissner-Mindlin plate model as well as the full von Kármán system. The plates are again in contact with the limited interpenetration with the foundation.

5.1. Contact of Reissner-Mindlin plates. This 2nd order model besides the vertical deflection u involves the 2D-vector $\boldsymbol{\varphi}$ of angles of rotations of the cross sections of the plate. We denote by $\mathbb{S}$ the set of symmetric 2×2 tensors with the product $\boldsymbol{\kappa} \odot \boldsymbol{\lambda} = \kappa_{ij}\lambda_{ij}$, where the Einstein summation convention (summing over repeated indices) is employed. Moreover, for $\omega \equiv \{\omega_{ij},\ i,j = 1,2\} \in \mathbb{S}$ we denote $\operatorname{Div}\omega \equiv (\partial_i\omega_{1i}, \partial_i\omega_{2i})$ and $\operatorname{tr}\omega = \omega_{11} + \omega_{22}$.

With the notation

$$
\begin{aligned}
J(u,\boldsymbol{\varphi}) &= e_1(\nabla\dot{u} + \dot{\boldsymbol{\varphi}}) + e_0(\nabla u + \boldsymbol{\varphi}),\\
\mathscr{C}_i(\omega) &= \frac{\tilde{c}}{1-\nu_i^2}(\nu_i(\operatorname{tr}\omega)I_{\mathbb{S}} + (1-\nu_i)\omega), \quad \omega \in \mathbb{S},\ i = 0,1,
\end{aligned}
\tag{24}
$$

where $I_{\mathbb{S}}$ is the unit matrix in $\mathbb{S}$, $\tilde{c}$, e_0, e_1 are given positive constants, and the Poisson ratio $\nu_i \in (-1, 1/2)$, $i = 0,1$, *the classical formulation of the viscoelastic ("short memory") problem* is as follows: We look for $(u,\boldsymbol{\varphi})$ such that the system

$$
\left.
\begin{aligned}
&\ddot{u} - \operatorname{div} J(u,\boldsymbol{\varphi}) = f + p(u+g),\\
&\ddot{\boldsymbol{\varphi}} - \operatorname{Div}(\mathscr{C}_1(\boldsymbol{\varepsilon}_0(\dot{\boldsymbol{\varphi}})) + \mathscr{C}_0(\boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}))) + J(u,\boldsymbol{\varphi}) = \boldsymbol{M}
\end{aligned}
\right\} \quad \text{on } Q,
\tag{25}
$$

the boundary value conditions

$$
\left.
\begin{aligned}
&u = u_0,\ \boldsymbol{\varphi} = \mathbf{0} \text{ for a clamped plate},\\
&u = u_0,\ (\mathscr{C}_1(\boldsymbol{\varepsilon}_0(\dot{\boldsymbol{\varphi}})) + \mathscr{C}_0\boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}))\cdot\boldsymbol{n} = \mathbf{0} \text{ for a simply supported one}
\end{aligned}
\right\} \quad \text{on } S,
\tag{26}
$$

and the initial conditions

$$
\left.
\begin{aligned}
&u(0,\cdot) = u_0,\ \dot{u}(0,\cdot) = u_1,\\
&\boldsymbol{\varphi}(0,\cdot) = \boldsymbol{\varphi}^{(0)},\ \dot{\boldsymbol{\varphi}}(0,\cdot) = \boldsymbol{\varphi}^{(1)}
\end{aligned}
\right\} \quad \text{on } \Omega
\tag{27}
$$

are satisfied. Here $\boldsymbol{\varepsilon}_0$ is the standard 2D linearized strain tensor and $\boldsymbol{n}$ is the unit outer normal vector. We assume that the function p satisfies all the assumptions listed at the beginning of Section 3, we assume that (4) still holds, in particular the positive function u_0 is again bounded away from 0. Moreover, we assume that $\boldsymbol{\varphi}^{(1)} \in \boldsymbol{L}_2(\Omega)$, $\boldsymbol{\varphi}^{(0)} \in \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{M} \in \boldsymbol{L}_2(Q)$.

The variational formulation of the problem based on appropriate integrations by parts has the following form: *Look for* $\{u, \boldsymbol{\varphi}\} \in (u_0 + L_2(I; \mathring{H}^1(\Omega)) \times \boldsymbol{X}(Q)$ *such that* $\dot{u} \in L_2(I; H^1(\Omega))$, $\dot{\boldsymbol{\varphi}} \in L_2(I; \boldsymbol{X}(\Omega))$, $\ddot{\boldsymbol{\varphi}} \in \boldsymbol{L}^2(Q)$, *the first condition in the first row and the second row of the initial conditions* (27) *are satisfied and the system*

$$\begin{aligned}
(28)\qquad &\int_Q (J(u, \boldsymbol{\varphi}) \cdot \nabla y - \dot{u}\dot{y} - p(u+g)y) \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad = \int_\Omega (u_1 y(0, \cdot) - \dot{u}(T, \cdot) y(T, \cdot)) \,\mathrm{d}x + \int_Q f y \,\mathrm{d}x \,\mathrm{d}t, \\
&\int_Q (\ddot{\boldsymbol{\varphi}} \cdot \boldsymbol{\psi} + (\mathscr{C}_1(\boldsymbol{\varepsilon}_0(\dot{\boldsymbol{\varphi}})) + \mathscr{C}_0(\boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}))) \odot \boldsymbol{\varepsilon}_0(\boldsymbol{\psi}) + J(u, \boldsymbol{\varphi}) \cdot \boldsymbol{\psi}) \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad = \int_Q \boldsymbol{M} \cdot \boldsymbol{\psi} \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}$$

holds for any $\{y, \boldsymbol{\psi}\} \in \mathring{H}^1(Q) \times L_2(I, \boldsymbol{X}(\Omega))$. Here $\boldsymbol{X}$ stands for $\mathring{\boldsymbol{H}}^1$ and $\boldsymbol{H}^1$ for clamped and simply supported plates, respectively.

As in the previous cases we introduce the approximate problems by replacing the original function p by the approximate function p_k, by omitting the integration by parts in time for the acceleration term in the first row of (28) and by adding the initial conditions for $\dot{u}$ from (27). Hence, it has the form

$$(29)\qquad \int_Q (J(u_k, \boldsymbol{\varphi}_k) \cdot \nabla y + \ddot{u}_k y) \,\mathrm{d}x \,\mathrm{d}t = \int_Q (f + p_k(u_k + g)) y \,\mathrm{d}x \,\mathrm{d}t.$$

We put $\{y, \boldsymbol{\psi}\} = \{\dot{u}_k - \dot{u}_0, \dot{\boldsymbol{\varphi}}_k\}$ as the test function of the approximate system and integrate on the interval $[0, s]$, $s \leqslant T$. Adding both lines of (28) and using the standard integration by parts, we get

$$\begin{aligned}
(30)\quad \int_{Q_s} \Big(&\frac{1}{2}\partial_t(\dot{u}_k^2 + e_0|\nabla u_k + \boldsymbol{\varphi}_k|^2 + |\dot{\boldsymbol{\varphi}}_k|^2 + \mathscr{C}_0(\boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k)) \odot \boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k) + P_k(u_k + g)) \\
&+ e_1|\nabla \dot{u}_k + \dot{\boldsymbol{\varphi}}_k|^2 + \mathscr{C}_1(\boldsymbol{\varepsilon}_0(\dot{\boldsymbol{\varphi}}_k)) \odot \boldsymbol{\varepsilon}_0(\dot{\boldsymbol{\varphi}}_k)\Big) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_{Q_s} (f\dot{u}_k + \boldsymbol{M}\dot{\boldsymbol{\varphi}}_k) \,\mathrm{d}x \,\mathrm{d}t + \int_{Q_s} R(\dot{u}_0) \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned}$$

where in $R(u_0)$ we sum up all the terms containing $\dot{u}_0$, or its derivatives. From the positive definiteness of the tensors $\mathscr{C}_i$ and the last identity we derive after some calculation the *a priori* estimate

$$
\begin{aligned}
(31)\qquad \|\dot u_k\|^2_{L_\infty(I;L_2(\Omega))} &+ \|\dot{\boldsymbol{\varphi}}_k\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))} + \|\dot u_k\|^2_{L_2(I;H^1(\Omega))}\\
&+ \|\dot{\boldsymbol{\varphi}}_k\|^2_{L_2(I;\boldsymbol{H}^1(\Omega))} + \|u_k\|^2_{C(\overline{I};H^1(\Omega))}\\
&+ \|\boldsymbol{\varphi}_k\|^2_{C(\overline{I};\boldsymbol{H}^1(\Omega))} + \|P_k(u_k+g)\|_{L_\infty(I;L_1(\Omega))}\\
&\leqslant c \equiv c(f,\boldsymbol{M},u_0,u_1,\boldsymbol{\varphi}^{(0)},\boldsymbol{\varphi}^{(1)}).
\end{aligned}
$$

Observe that this estimate is k-independent.

We continue with the estimates of the acceleration terms. After using $\{\ddot u_k - \ddot u_0, \ddot{\boldsymbol{\varphi}}_k\}$ as the test function, we obtain

$$
(32)\qquad \|\ddot{\boldsymbol{\varphi}}_k\|^2_{\boldsymbol{L}_2(Q)} \leqslant c,
$$

$$
(33)\qquad \|\ddot u_k\|^2_{L_2(Q)} \leqslant c_k, \quad k\in\mathbb{N}.
$$

From (31) it is easy to see that (32) is again k-independent. However, (33) depends on k and for the limit process $k\to\infty$ it has to be replaced by the dual estimate of $\ddot u_k$ based on the uniform estimate of $\|p_k(u_k+g)\|_{L_1(Q)}$ which is obtained in the same way as in Section 3.

These approximate problems are solved by means of the Galerkin approximation. Since they do not structurally differ from the penalized problems for the Signorini contact (in both described cases the approximate contact term represents a compact perturbation of the noncontact problems) and we are focused here on the difference between the rational contact with limited interpenetration and the Signorini contact, we omit details of this well-known process here and refer the readers to [4] for them.

To derive the crucial dual estimate of $\ddot u_k$ we can use the general abstract approach of Section 2. However, the space $H^1(\Omega)$ is not imbedded into $L_\infty(\Omega)$, hence we must use $X = H^1(\Omega)\cap L_\infty(\Omega)$ here. The resulting estimate (6) yields the required strong convergence of $\dot u_k$ in $L_2(Q)$ in the process $k\to\infty$ via the Aubin Lemma. We put $\{y,\boldsymbol{\psi}\} = \{u_k - u_0, \boldsymbol{\varphi}_k\}$ in (28) and add both equations. We get

(34)

$$
\begin{aligned}
\int_{Q_T} &(-\dot u_k^2 + e_0|\nabla u_k + \boldsymbol{\varphi}_k|^2 + |\dot{\boldsymbol{\varphi}}_k|^2 + \mathscr{C}_0(\boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k))\odot\boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k) + p_k(u_k+g)(u_k-u_0))\\
&+ \partial_t(e_1|\nabla u_k + \boldsymbol{\varphi}_k|^2 + \mathscr{C}_1(\boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k))\odot\boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k))\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_{Q_T}(f\dot u_k + \boldsymbol{M}\dot{\boldsymbol{\varphi}}_k)\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} R_1(u_0,u_1)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where R_1 contains all the remaining terms. Obviously they contain u_0 or u_1 or their derivatives. This identity shows again that it belongs to the abstract structure described in Section 3. Besides weakly lower semicontinuous elliptic terms and weakly continuous terms as $\dot{u}_k^2$ and R_1 the only remaining term, the contact one, must be upper semicontinuous. Let $\{u, \boldsymbol{\varphi}\}$ be the weak limit of $\{u_k, \boldsymbol{\varphi}_k\}$ such that all their derivatives mentioned in (31) tend weakly or weakly* to the derivatives of u in their respective spaces.

We have not the strong convergence of u_k in $C_0(\overline{Q})$, but we can reiterate from the form of (28) with no integration by parts in time performed and with test functions from $\mathscr{H} \equiv \{v \in H^1(Q); u|_S = 0\}$ so that the sequence $\{p(u_k + g)\}$ is bounded in $\mathscr{H}^*$ and we can assume that $p(u_k + g) \rightharpoonup \vartheta$ in $\mathscr{H}^*$. We take $v \in \mathscr{H}$ such that $p(v + g) \in L_1(Q)$. Since $u_k \rightharpoonup u$ in $\mathscr{H}$, $u_k \to u$ in $L_2(Q)$. Simultaneously, both sequences of their respective positive and negative parts are bounded in $\mathscr{H}$ and they have some accumulation points. However, their strong $L_2(Q)$ convergence shows that the only accumulation points can be $u_\pm$. We have $0 \leqslant \langle p_k(v+g), (u_k - u)_\pm \rangle_Q \leqslant \langle p(v+g), (u_k - u)_\pm \rangle_Q \to 0$, hence $\langle p_k(v+g), (u_k - u)\rangle_Q \to 0$. and $\langle p_k(v+g), u_k \rangle_Q \to \langle p(v+g), u \rangle_Q$. Moreover, $p(u + g) \in \mathscr{H}^*$, because for a nonnegative $v \in \mathscr{H}$ the inequality $\langle \vartheta, v \rangle_Q \geqslant \langle p(u+g), v \rangle_Q \geqslant 0$ follows from the Fatou Lemma, therefore the last duality is finite for every $v \in \mathscr{H}$. So we can use again the maximal monotonicity argument on $\mathscr{H}$ to prove that $\vartheta = p(u + g)$. and u satisfies (28) and we are done.

In the classical formulation of the Reissner-Mindlin plate with singular memory we replace all the "short memory" terms in J and $\mathscr{C}_1$ (i.e. the terms containing the time derivatives) by the corresponding singular memory terms (the d_m versions of the elastic terms), where we use again the kernel K defined in (16). Therefore, $J(u, \boldsymbol{\varphi}) \equiv e_0(\nabla u + \boldsymbol{\varphi}) + e_1 d_m(\nabla u + \boldsymbol{\varphi})$. With this modification the structure of (25), (26), and (27) remains preserved. We assume again the sufficient smallness of the memory. To get it exactly in the form (20) we assume $\nu_1 = \nu_0$.

We present explicitly its variational formulation which reads: *Look for* $\{u, \boldsymbol{\varphi}\} \in (u_0 + \mathring{H}^1(Q)) \times L_2(I; \boldsymbol{X}(\Omega))$ *such that* $\ddot{\boldsymbol{\varphi}} \in \boldsymbol{L}^2(Q)$*, the first condition in the first row and the second row of* (27) *are satisfied and the system*

$$\begin{aligned}
(35)\qquad & \int_Q (J(u, \boldsymbol{\varphi}) \cdot \nabla y - \dot{u}\dot{y}) \,\mathrm{d}x\,\mathrm{d}t \\
& \qquad = \int_\Omega (u_1 y(0, \cdot) - \dot{u}(T, \cdot) y(T, \cdot)) \,\mathrm{d}x + \int_Q (f + p(u+g)) y \,\mathrm{d}x\,\mathrm{d}t, \\
& \int_Q (\ddot{\boldsymbol{\varphi}} \cdot \boldsymbol{\psi} + (\mathscr{C}_1(d_m \boldsymbol{\varphi}) + \mathscr{C}_0(\boldsymbol{\varphi})) \odot \boldsymbol{\varepsilon}_0(\boldsymbol{\psi}) + J(u, \boldsymbol{\varphi}) \cdot \boldsymbol{\psi}) \,\mathrm{d}x\,\mathrm{d}t \\
& \qquad = \int_Q \boldsymbol{M} \cdot \boldsymbol{\psi} \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

holds for any $\{y, \boldsymbol{\psi}\} \in L_2(I; H^1(\Omega)) \times L_2(I, \boldsymbol{X}(\Omega))$. Here again $\boldsymbol{X}$ stands for $\mathring{\boldsymbol{H}}^1$ and $\boldsymbol{H}^1$ for clamped and simply supported plates, respectively.

We formulate again the approximate problems by replacing the function p by p_k, by omitting the integration by parts at the acceleration term and by adding the initial conditions for $\dot{u}$ from (27). We solve this problem via the Galerkin method as usual. We again omit here the details referring the readers to the paper [4]. Since it is not clear at the beginning whether the velocity $\dot{u}_k$ possesses the required qualities of the test function, the a priori estimates have been derived there for the finite-dimensional space approximations and then the limit process to the original infinite-dimensional space has been performed. However, the result is the same as if we put formally $\{\dot{u}_k - \dot{u}_0, \dot{\boldsymbol{\varphi}}_k\}$ as the test function.

Summing up the two equations and limiting the integration to the cylinder Q_s for $s \leqslant T$, we obtain using the properties of the kernel function K the identity

$$
\begin{aligned}
(36)\quad & \int_{Q_s} \Big(\frac{1}{2} \partial_t (\dot{u}_k^2 + e_0 |\nabla u_k + \boldsymbol{\varphi}_k|^2 + |\dot{\boldsymbol{\varphi}}_k|^2 + \mathscr{C}(\boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k)) \odot \boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k) + P_k(u_k + g)) \\
&\quad + \frac{e_1}{2} K(s-t) |\nabla (u_k(s) - u_k(t)) + \boldsymbol{\varphi}_k(s) - \boldsymbol{\varphi}_k(t)|^2 \\
&\quad + \frac{1}{2} K(s-t) \mathscr{C}(\boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k(s) - \boldsymbol{\varphi}_k(t))) \odot \boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k(s) - \boldsymbol{\varphi}_k(t)) \Big) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \frac{e_1}{2} \int_{Q_s} \int_0^t K_t'(t-\tau) |\nabla (u_k(t) - u_k(\tau)) + \boldsymbol{\varphi}_k(t) - \boldsymbol{\varphi}_k(\tau)|^2 \, \mathrm{d}\tau \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \frac{1}{2} \int_{Q_s} \int_0^t K_t'(t-\tau) \mathscr{C}(\boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k(t) - \boldsymbol{\varphi}_k(\tau)) \odot \boldsymbol{\varepsilon}_0(\boldsymbol{\varphi}_k(t) - \boldsymbol{\varphi}_k(\tau)) \, \mathrm{d}\tau \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_{Q_s} (f \dot{u}_k + \boldsymbol{M} \dot{\boldsymbol{\varphi}}_k) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

By virtue of (16), (20) the identity (36) leads to the *a priori* estimates independent of $k \in \mathbb{N}$:

$$
\begin{aligned}
(37)\quad \|\dot{u}_k\|^2_{L_\infty(I;L_2(\Omega))} &+ \|\dot{\boldsymbol{\varphi}}_k\|^2_{L_\infty(I;\boldsymbol{L}_2(\Omega))} + \|u_k\|^2_{H^\alpha(I;H^1(\Omega))} \\
&+ \|\boldsymbol{\varphi}_k\|^2_{H^\alpha(I;\boldsymbol{H}^1(\Omega))} + \|u_k\|^2_{L_\infty(I;H^1(\Omega))} \\
&+ \|\boldsymbol{\varphi}_k\|^2_{L_\infty(I;\boldsymbol{H}^1(\Omega))} + \|P_k(u_k + g)\|_{L_\infty(I;L_1(\Omega))} \\
&\leqslant c \equiv c(f, \boldsymbol{M}, u^{(0)}, u^{(1)}, \boldsymbol{\varphi}^{(0)}, \boldsymbol{\varphi}^{(1)}).
\end{aligned}
$$

The estimate of the accelerations is a straightforward consequence of the *a priori* estimate (37) and the approximate system to (35) and has the form:

$$(38)\qquad \|\ddot{\boldsymbol{\varphi}}_k\|^2_{\boldsymbol{L}_2(I;(H^1(\Omega))^*)} \leqslant c,$$

$$(39)\qquad \|\ddot{u}_k\|^2_{L_2(I;(H^1(\Omega))^*)} \leqslant c_k$$

(the first of them is again k-independent).

For the limit process $k \to \infty$ we can get the dual estimate $\|\ddot{u}_k\|_{L_1(I;H^{-1-\tilde{\varepsilon}}(\Omega))} \leqslant$ const. via the L_1 estimate of the approximate contact term (cf. (6)) if we employ the dual embedding $H^{1+\tilde{\varepsilon}}(\Omega) \hookrightarrow L_\infty(\Omega)$ for any $\tilde{\varepsilon} > 0$. Then the sequence $\{\dot{u}_k\}$ is bounded in $W^{1-\varepsilon_1}_{1+\varepsilon_2}(I; H^{-1-\tilde{\varepsilon}}(\Omega))$ for any $\varepsilon_2 > 0$ and $\varepsilon_1 \equiv \varepsilon_1(\varepsilon_2) \searrow 0$ if $\varepsilon_2 \searrow 0$. Simultaneously it is bounded in $L_q(I; L_2(\Omega))$ for any $q \geqslant 2$. After interpolating the spaces $W^{1-\varepsilon_1}_{1+\varepsilon_2}(I; H^{-1-\varepsilon_0}(\Omega))$ and $L_q(I; L_2(\Omega))$ for $q \geqslant 1 + 1/\varepsilon_2$ we have

$$\|\dot{u}_k\|_{H^{1/2}(I;H^{-1/2-\varepsilon_3}(\Omega))} \leqslant \text{const.}, \quad \text{i.e. } \|u_k\|_{H^{3/2}(I;H^{-1/2-\varepsilon_3}(\Omega))} \leqslant \text{const.}, \quad k \in \mathbb{N},$$

where $\varepsilon_3 > 0$ is arbitrarily small. Interpolating this result with the fact that $\{u_k\}$ is bounded in $H^\alpha(I; H^1(\Omega))$ for the given $\alpha \in (0, \frac{1}{2})$, we obtain that $\{u_k\}$ is bounded in the space $H^{1+\theta}(I; L_2(\Omega))$ for any $\theta \in (0, \alpha/3)$. Interpolating this result with the same space, we get the boundedness of $\{u_k\}$ in $H^1(I; H^\delta(\Omega))$ for any $\delta \in (0, \alpha/(3-2\alpha))$. The intersection of both resulting spaces is obviously compactly imbedded into $H^1(I; L_2(\Omega))$, hence the strong convergence of the velocities is proved. As earlier, the resulting upper semicontinuity of the contact term and the maximal monotonicity argument for p on the space $\mathscr{H}$ leads to the fact that $p_k(u_k + g) \rightharpoonup p(u + g)$ and the existence of a solution to the system (35) is thus ensured.

5.2. Contact of viscoelastic plates described by full von Kármán system. This model of plates describes the vertical deflection u as well as the horizontal ones denoted by $\boldsymbol{u} \equiv \{u_1, u_2\}$. We assume that the potential contact is both with the foundation of the plate and on the boundary Γ. We preserve the notation of $\mathscr{C}_i$ from (24), but the physical meaning of some terms may differ here from the previous parts. Denoting

$$\begin{aligned} &\mathfrak{C}_0 = e_0 \mathscr{C}_0(\varepsilon(\boldsymbol{u}) + \Psi(\nabla u)), \\ &\mathfrak{C}_1 = e_1 \mathscr{C}_1(\varepsilon(\dot{\boldsymbol{u}}) + \partial_t \Psi(\nabla u)), \quad \Psi(\boldsymbol{a}) = \frac{1}{2} \boldsymbol{a} \otimes \boldsymbol{a}, \ \boldsymbol{a} \in \mathbb{R}^2, \end{aligned} \tag{40}$$

we state the classical formulation of the problem:

We look for $\{\boldsymbol{u}, u\}$ such that the system

$$\left.\begin{aligned} &\ddot{\boldsymbol{u}} - \operatorname{Div}(\mathfrak{C}_1 + \mathfrak{C}_0) = \boldsymbol{F}, \\ &\ddot{u} - a\Delta\ddot{u} + b(e_1\Delta^2\dot{u} + e_0\Delta^2 u) - \operatorname{div}((\mathfrak{C}_1 + \mathfrak{C}_0)\nabla u) = f + p(u+g) \end{aligned}\right\} \text{ on } Q \tag{41}$$

holds, the boundary value conditions

$$\left.\begin{aligned} &(\mathfrak{C}_1 + \mathfrak{C}_0)\boldsymbol{n} \cdot \boldsymbol{n} = \tilde{q}(\widetilde{u_n}), \quad (\mathfrak{C}_1 + \mathfrak{C}_0)\boldsymbol{n} \cdot \boldsymbol{\tau} = 0, \ u = u^{(0)} \\ &e_1(\Delta\dot{u} + (1-\nu_1)B\dot{u}) + e_0(\Delta u + (1-\nu_0)Bu) = 0 \end{aligned}\right\} \text{ on } S \tag{42}$$

with

$$\widetilde{u_n} \equiv \boldsymbol{u}\cdot\boldsymbol{n},\ Bw = 2n_1n_2\partial_{12}w - n_1^2\partial_{22}w - n_2^2\partial_{11}w$$

are satisfied, and the initial conditions

$$\boldsymbol{u}(0,\cdot) = \boldsymbol{u}^{(0)},\ \ \dot{\boldsymbol{u}}(0,\cdot) = \boldsymbol{u}^{(1)},\ \ u(0,\cdot) = u^{(0)},\ \ \dot{u}(0,\cdot) = u^{(1)} \quad \text{on } \Omega \tag{43}$$

are valid. Both functions p and $\tilde{q}(-\cdot)$ are assumed to satisfy all the conditions for p in Section 2. We note that we assume our plate to be simply supported, because it does not seem physically reasonable to consider the clamped plate with the possible limited interpenetration on the boundary. The constants a, b, e_0, e_1 are positive, the nonnegative function g is again the gap function. Of course, we can introduce another gap function to the boundary contact, but it seems to have little use in practical applications. Let us point out that the problem defined in this way describes the behaviour of a cover of a fully recessed stack.

For $z, y \in L_2(I; H^2(\Omega))$ we define the following bilinear forms:

$$\begin{aligned} A_i\colon\ (z,y) \mapsto be_i(&\partial_{kk}z\partial_{kk}y + \nu_i(\partial_{11}z\partial_{22}y + \partial_{22}z\partial_{11}y) \\ &+ 2(1-\nu_i)\partial_{12}z\partial_{12}y), \quad i = 0,1. \end{aligned} \tag{44}$$

Then our problem has the following variational formulation:

Look for $\{\boldsymbol{u}, u\} \in \boldsymbol{H}^1(Q) \times (L_2(I; H^2(\Omega)) \cap (u^{(0)} + L_2(I; \mathring{H}^1(\Omega))))$ *such that* $\dot{\boldsymbol{u}} \in L_2(I; \boldsymbol{H}^1(\Omega))$, $\dot{u} \in L_2(I; H^2(\Omega))$, *the initial condition* (43) *holds for* u *and* $\boldsymbol{u}$ *and the system*

$$\begin{aligned} &\int_Q ((\mathfrak{C}_1 + \mathfrak{C}_0)\varepsilon(\boldsymbol{y}) - \dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{y}})\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega ((\dot{\boldsymbol{u}}\cdot\boldsymbol{y})(T,\cdot) - \boldsymbol{u}^1\cdot\boldsymbol{y}(0,\cdot))\,\mathrm{d}x \\ &\qquad = \int_Q \boldsymbol{F}\cdot\boldsymbol{y}\,\mathrm{d}x\,\mathrm{d}t - \int_S q(\widetilde{u_n})y_n\,\mathrm{d}x_s\,\mathrm{d}t, \\ &\int_Q (A_1(\dot{u},z) + A_0(u,z) + [(\mathfrak{C}_1 + \mathfrak{C}_0)\nabla u]\cdot\nabla z) - \dot{u}\dot{z} - a\nabla\dot{u}\cdot\nabla\dot{z})\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_\Omega ((\dot{u}z + a\nabla\dot{u}\cdot\nabla z)(T,\cdot) - u^{(1)}z(0,\cdot) - a\nabla u^{(1)}\cdot\nabla z(0,\cdot))\,\mathrm{d}x \\ &\qquad = \int_Q (f + p(u+g))z\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{45}$$

is satisfied for every $\{\boldsymbol{y}, z\} \in \boldsymbol{Y}$ with

$$\begin{aligned} &\boldsymbol{Y} \equiv \boldsymbol{Y}_0 \cap \boldsymbol{Y}_d \text{ with } \boldsymbol{Y}_0 \equiv \{L_2(I; \boldsymbol{H}^1(\Omega)) \times (L_2(I; \mathring{H}^1(\Omega)) \cap L_2(I; H^2(\Omega)))\}, \\ &\boldsymbol{Y}_d \equiv \{\boldsymbol{z} \in \boldsymbol{Y}_0\,;\ \dot{\boldsymbol{z}} \in L_2(Q; \mathbb{R}^3)\}. \end{aligned} \tag{46}$$

The approximate problems are defined as usual by replacing p, q by p_k, q_k, respectively, and keeping acceleration terms in such modified system (45) in their original form. Since this problem is remarkably more complex that all the previous ones and leads to more complex formulae, we will denote the solution of this problem also by $\{\boldsymbol{u}, u\}$. Similarly to the previous problems this approximate problem is again solved with help of the Galerkin approximation. Since there is no substantial difference between it and the penalized problem treated in [3], we refer the readers to that paper for details. To derive *a priori* estimates for the solutions of the approximate problem we put $\chi_{Q_s}\{\dot{\boldsymbol{u}}, \dot{u} - \dot{u}^{(0)}\}$ for $s \in (0, T]$ as a test function of the appropriate variant of (45). We obtain after integration and summation

(47)

$$\int_{Q_s} \Big(\frac{1}{2}\partial_t(\dot{u}^2 + a|\nabla\dot{u}|^2 + |\dot{\boldsymbol{u}}|^2 + \mathscr{C}_0(\varepsilon(\boldsymbol{u}) + \Psi(\nabla u))\cdot(\varepsilon(\boldsymbol{u}) + \Psi(\nabla u)) + A_0(u, u))$$
$$+ A_1(\dot{u}, \dot{u}) + \mathscr{C}_1(\varepsilon(\dot{\boldsymbol{u}}) + \partial_t\Psi(\nabla u))\cdot(\varepsilon(\dot{\boldsymbol{u}}) + \partial_t\Psi(\nabla u)) + \partial_t P_k(u + g)\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_S \partial_t\widetilde{Q}_k(\widetilde{u}_n)\,\mathrm{d}x_s\,\mathrm{d}t = \int_{Q_s}(\boldsymbol{F}\cdot\dot{\boldsymbol{u}} + f\dot{w})\,\mathrm{d}x\,\mathrm{d}t + R(u^{(0)}),$$

where $\widetilde{Q}\colon\ r \mapsto \int_{-\infty}^{r} q(\zeta)\,\mathrm{d}\zeta$ and $R(u^{(0)})$ sums up all the terms containing $u^{(0)}$ or its derivatives. Using the coercivity of the form A_i and the form of the operators $\mathscr{C}_i$, we obtain the estimate

$$\begin{aligned}(48)\qquad &\|\dot{u}(s)\|^2_{H^1(\Omega)} + \|u(s)\|^2_{H^2(\Omega)} + \|\dot{\boldsymbol{u}}(s)\|^2_{\boldsymbol{L}_2(\Omega)} + \|\varepsilon(\boldsymbol{u})(s) + \Psi(\nabla u)(s)\|^2_{L_2(\Omega;\mathbb{S})}\\ &+ \|\dot{u}\|^2_{L_2(I_s;H^2(\Omega))} + \|\varepsilon(\dot{\boldsymbol{u}}) + \partial_t\Psi(\nabla u)\|^2_{L_2(Q_s;\mathbb{S})}\\ &+ \|P_k(u(s) + g)\|_{L_1(Q)} + \|\widetilde{Q}_k(\widetilde{u}_n(s))\|_{L_1(S)}\\ &\leqslant C(\boldsymbol{u}^{(0)}, \boldsymbol{u}^{(1)}, u^{(0)}, u^{(1)}, \boldsymbol{F}, f)\quad \forall\, s \in (0, T].\end{aligned}$$

Applying the continuous imbedding $H^2(\Omega) \hookrightarrow W^1_4(\Omega)$, we obtain the estimate

$$\|\Psi(\nabla u)(s)\|_{L_2(\Omega;\mathbb{S})} + \|\partial_t\Psi(\nabla u)\|_{L_2(I_s;L_2(\Omega;\mathbb{S}))} \leqslant C(\boldsymbol{u}^{(0)}, \boldsymbol{u}^{(1)}, u^{(0)}, u^{(1)}, \boldsymbol{F}, f)\quad \forall\, s \in (0, T]$$

which implies

$$\|\varepsilon(\boldsymbol{u})(s)\|_{L_2(\Omega;\mathbb{S})} + \|\dot{\varepsilon}(\boldsymbol{u})\|_{L_2(I_s;L_2(\Omega;\mathbb{S}))} \leqslant C(\boldsymbol{u}^{(0)}, \boldsymbol{u}^{(1)}, u^{(0)}, u^{(1)}, \boldsymbol{F}, f)\quad \forall\, s \in (0, T].$$

Using the coerciveness of strains (see e.g. [6], Theorem 1.2.3) we obtain

$$\|\boldsymbol{u}(s)\|_{\boldsymbol{H}^1(\Omega)} + \|\dot{\boldsymbol{u}}\|_{L_2(I_s;\boldsymbol{H}^1(\Omega))} \leqslant C(\boldsymbol{u}^{(0)}, \boldsymbol{u}^{(1)}, u^{(0)}, u^{(1)}, \boldsymbol{F}, f))\quad \forall\, s \in (0, T],$$

which together with (48) implies the *a priori* estimate

(49)
$$\begin{aligned}\|\dot{\boldsymbol{u}}\|_{L_\infty(I;\boldsymbol{L}_2(\Omega))} + \|\dot{\boldsymbol{u}}\|_{L_2(I;\boldsymbol{H}^1(\Omega))} + \|\boldsymbol{u}\|_{L_\infty(I;\boldsymbol{H}^1(\Omega))} + \|\dot{u}\|_{L_\infty(I;L_2(\Omega))} \\ + \|\dot{u}\|_{L_2(I;H^2(\Omega))} + \|u\|_{L_\infty(I;H^2(\Omega))} \\ + \|P_k(u+g)\|_{L_\infty(I;L_1(Q))} + \|\widetilde{Q}_k(\widetilde{u}_n)\|_{L_\infty(I;L_1(S)]} \\ \leqslant C(\boldsymbol{u}^{(0)}, \boldsymbol{u}^{(1)}, u^{(0)}, u^{(1)}, \boldsymbol{F}, f).\end{aligned}$$

Since there is no substantial difference between the proof of the solvability of our approximate problem and that of the penalized problem treated in [3], we refer the readers to that paper for details. Via the standard method it is proved that such a solution is unique.

As in all previous problems the main task is to perform the limit process $k \to \infty$ for which the k-independent estimates of the acceleration terms are needed. To estimate $\ddot{\boldsymbol{u}}_k \in L_2(I; \boldsymbol{H}^{-1}(\Omega))$ we put an arbitrary $\boldsymbol{w} \in L_2(I; \mathring{\boldsymbol{H}}^1(\Omega))$ in the approximate variant of (45) and use (49). To get the estimate $\ddot{u} \in L_1(I; H^2(\Omega)^*)$ we have to assume (4) which yields the uniform estimate for $\|p_k(u_k+g)\|_{L_1(Q)}$ as in Section 3. Then we are in the same situation as in Example 3, the Aubin Lemma gives us the crucial strong $L_2(Q)$-convergence of all components of velocities.

We take the space $\mathscr{X}_0 \equiv L_2(I; H^2(\Omega)) \cap L_2(I; \mathring{H}^1(\Omega))$ for the operator p and the trace space $\mathscr{Y} \equiv H^{1/4,1/2}(S)$ for the operator q. Let ϑ be the weak limit of $p_k(u_k+g)$ both in $\mathscr{X}_0^*$ and in $L_\infty^*(Q)$ and ω the weak limit of $q_k(\widetilde{(u_k)_n})$ both in $\mathscr{Y}^*$ and in $L_\infty^*(S)$. As earlier the upper semicontinuity of $\langle p_k(u_k)u_k\rangle_Q$ and $\langle q_k(\widetilde{(u_k)_n})\widetilde{(u_k)_n}\rangle_S$ has been proved with ϑ and ω, respectively. Using the same consideration as in Section 3, we prove the maximal monotonicity of p on $\mathscr{X}_0$. The operator q is evidently maximal monotone, too, cf. [9]. This yields that in fact $\vartheta = p(u+g)$ and $\omega = q(\widetilde{u_n})$. The existence of solutions to (45) is proved.

Similarly to the previous sections we can formulate the full von Kármán system with the singular memory replacing all "short memory" terms in (41) and (45) by the corresponding singular memory ones. As in the previous cases under the assumption (20) it is possible to pass from the appropriate approximate problem to the original one in such a way that the crucial strong convergence of velocities holds, which leads to the same conclusion as mentioned in the previous paragraph. We take the liberty of leaving this case to kind readers as an exercise.

6. Relation to the Signorini contact

In this section we will prove that for a sequence of the problems with the thickness of the interpenetration

$$\gamma_l \nearrow 0, \quad l \in \mathbb{N}, \tag{50}$$

there is a subsequence of their solutions called u_l tending to a limit u which is a solution of a problem without interpenetration, i.e. of the appropriate Signorini version of the problem. Since there is the well-known generic nonuniqueness of the solutions to the dynamic contact of the Signorini type related to the lack of information about the amount of the energy conservation in the contact and thus about the development of the solution after the contact, probably nothing more can be proved in general.

The common feature of the problems treated in the previous sections is that the estimates performed there as k-independent are also γ independent. Hence, if we have a sequence u_l tending weakly or weakly* to u in the spaces for which the a priori and dual estimates have been derived, we have the strong L_2 convergence $\dot{u}_l \to \dot{u}$. Obviously, for the full von Kámán system $\dot{\boldsymbol{u}}_l \to \dot{\boldsymbol{u}}$ in $\boldsymbol{L}_2(Q)$ holds as well (cf. [7]). Since $u_l + g > \gamma_l$ a.e. in Q, we have $u + g \geqslant 0$ a.e. there. Moreover, for the full von Kármán system we get similarly $\widetilde{u_n} \leqslant 0$. We define

$$\begin{aligned}
&\mathscr{K} \equiv \{v \in \mathscr{X}_0;\ v \geqslant -g \text{ a.e. in } Q\} \text{ for problems in Sections 2, 3,}\\
&\mathscr{K} \equiv \{\{v, \boldsymbol{\omega}\} \in (u_0 + \mathring{H}^1(Q)) \times \boldsymbol{X}(Q);\ v \geqslant -g \text{ a.e. in } Q\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for Reissner-Mindlin plates,}\\
&\mathscr{K} \equiv \{\{\boldsymbol{w}, v\} \in \boldsymbol{Y}_0;\ \boldsymbol{w}_n \leqslant 0 \text{ a.e. in } S,\ v \geqslant -g \text{ a.e. in } Q\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for the full von Kármán system,}
\end{aligned} \tag{51}$$

where $\boldsymbol{X}(Q) = L_2(I; \boldsymbol{X}(\Omega))$, cf. (2), (28), and (46). Obviously in all cases $p(v+g) = 0$ if v is (possibly a component) from $\mathscr{K}$ and, moreover, $q(w_n) = 0$ in the last case. Denoting $\Theta \equiv \lim_{l\to\infty} \langle p_l(u_l + g), u_l\rangle_Q$ and $\vartheta \equiv \lim_{l\to\infty} p_l(u_l + g)$, the monotonicity of p_l used for the couple $\{u_l, u\}$ yields $\Theta \geqslant \vartheta u$. On the other hand, we can derive from (2) (the solution there has to be denoted by u_l) with $v = u_l - y$, $y \in \mathscr{K}$ the opposite inequality, because in general for $y \in \mathscr{K}$ the lower semicontinuity of $\mathscr{A}$ and $\mathscr{B}$ in the limit process $l \to \infty$ yields

$$\begin{aligned}
-\langle \dot{u}, \dot{y} - \dot{u}\rangle_Q + \langle \mathscr{A}u, y - u\rangle_Q + \langle \mathscr{B}u, y - u\rangle_Q + \langle \mathscr{E}u, y - u\rangle_Q + \langle \vartheta, y\rangle_Q - \Theta\\
+ \langle \dot{u}(T,\cdot), (y-u)(T,\cdot)\rangle_\Omega \geqslant \langle f, y - u\rangle_Q + \langle u_1, y(0,\cdot) - u_1\rangle_\Omega,
\end{aligned} \tag{52}$$

and putting $y = u$ we are done. Hence, (52) holds just without the terms with ϑ and Θ and this is the exact formulation of the corresponding Signorini problem with u being its solution. This pattern can be exactly followed also in all the other cases, because their variational formulations contain only some lower semicontinuous parts, strongly converging terms and linear terms for which the weak convergence is sufficient. Hence, we prove everytimes $\Theta = \langle\vartheta, u\rangle_Q$ and then we can see that the resulting limit variational inequality is the variational formulation of the corresponding Signorini problem indeed. Of course, for Reissner-Mindlin plates we keep the second equation of (28) in the original form observing that the convergences, which remain weak there, are sufficient.

We only mention the full von Kármán system more in detail. We denote

$$\Lambda \equiv \lim_{l\to\infty} \langle q(\widetilde{(u_l)_n}), \widetilde{(u_l)_n}\rangle_S \quad \text{and} \quad \lambda \equiv \lim_{l\to\infty} q(\widetilde{(u_l)_n}).$$

and derive immediately that $\Lambda \geqslant \langle \lambda, \widetilde{u_n}\rangle_S$. Denoting the solution of (45) by $\{\boldsymbol{u}_l, u_l\}$, we put $\{\boldsymbol{v} - \boldsymbol{u}_l, w - u_l\}$ as a test function in (45) for an arbitrary $\{\boldsymbol{v}, w\} \in \mathscr{K}$. Then we perform the limit process $l \to \infty$, denote by $\{\boldsymbol{u}, u\}$ the limit of $\{\boldsymbol{u}_l, u_l\}$ and put $\{\boldsymbol{w}, v\} = \{\boldsymbol{u}, u\}$ as a test function into the resulting inequality. Thus we find $\Lambda = \langle \lambda, \widetilde{u_n}\rangle_S$ from the first row of it while $\Theta = \langle \vartheta, u\rangle_Q$. From this we get that the resulting inequality is in fact the variational formulation of the Signorini contact for the full von Kármán system and $\{\boldsymbol{u}, u\}$ is its solution.

7. Conclusion

The existence of solutions has been proved for the dynamic contact problems with limited interpenetration for viscoelastic variants of several classical models of plates. These results are now available for technical practice. Probably the most challenging task for the application is the determination of the function p which describes the interpenetration. Performing some sensitivity analysis with respect to its choice may help here.

References

[1] *I. Bock, J. Jarušek*: Unilateral dynamic contact of viscoelastic von Kármán plates. Adv. Math. Sci. Appl. *16* (2006), 175–187. zbl MR

[2] *I. Bock, J. Jarušek*: Unilateral dynamic contact of von Kármán plates with singular memory. Appl. Math., Praha *52* (2007), 515–527. zbl MR doi

[3] *I. Bock, J. Jarušek*: Dynamic contact problem for a bridge modeled by a viscoelastic full von Kármán system. Z. Angew. Math. Phys. *61* (2010), 865–876. zbl MR doi

[4] *I. Bock, J. Jarušek*: Unilateral dynamic contact problem for viscoelastic Reissner-Mindlin plates. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *74* (2011), 4192–4202. zbl MR doi

[5] *J. M. Borwein, Q. J. Zhu*: Techniques of Variational Analysis. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 20, Springer, New York, 2005. zbl MR doi
[6] *C. Eck, J. Jarušek, M. Krbec*: Unilateral Contact Problems. Variational Methods and Existence Theorems. Pure and Applied Mathematics (Boca Raton) 270, Chapman & Hall/CRC, Boca Raton, 2005. zbl MR doi
[7] *C. Eck, J. Jarušek, J. Stará*: Normal compliance contact models with finite interpenetration. Arch. Ration. Mech. Anal. *208* (2013), 25–57. zbl MR doi
[8] *J. Jarušek*: Static semicoercive normal compliance contact problem with limited interpenetration. Z. Angew. Math. Phys. *66* (2015), 2161–2172. zbl MR doi
[9] *J. Jarušek, J. Stará*: Solvability of a rational contact model with limited interpenetration in viscoelastodynamics. Math. Mech. Solids *23* (2018), 1040–1048. zbl MR doi
[10] *H. Koch, A. Stachel*: Global existence of classical solutions to the dynamical von Kármán equations. Math. Methods Appl. Sci. *16* (1993), 581–586. zbl MR doi
[11] *J. E. Lagnese*: Boundary Stabilization of Thin Plates. SIAM Studies in Applied Mathematics 10, Society for Industrial and Applied Mathematics, Philadelphia, 1989. zbl MR doi
[12] *A. Signorini*: Sopra alcune questioni di statica dei sistemi continui. Ann. Sc. Norm. Super. Pisa, II. Ser. *2* (1933), 231–251. (In Italian.) zbl MR
[13] *A. Signorini*: Questioni di elasticità non linearizzata e semilinearizzata. Rend. Mat. Appl., V. Ser. *18* (1959), 95–139. (In Italian.) zbl MR

Author's address: *Jiří Jarušek*, Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: jarusek@math.cas.cz.