Rudolf Scitovski; Kristian Sabo

The adaptation of the k-means algorithm to solving the multiple ellipses detection problem by using an initial approximation obtained by the DIRECT global optimization algorithm

THE ADAPTATION OF THE k-MEANS ALGORITHM TO SOLVING THE MULTIPLE ELLIPSES DETECTION PROBLEM BY USING AN INITIAL APPROXIMATION OBTAINED BY THE DIRECT GLOBAL OPTIMIZATION ALGORITHM

Rudolf Scitovski, Kristian Sabo, Osijek

Abstract. We consider the multiple ellipses detection problem on the basis of a data points set coming from a number of ellipses in the plane not known in advance, whereby an ellipse E is viewed as a Mahalanobis circle with center S, radius r, and some positive definite matrix Σ. A very efficient method for solving this problem is proposed. The method uses a modification of the k-means algorithm for Mahalanobis-circle centers. The initial approximation consists of the set of circles whose centers are determined by means of a smaller number of iterations of the DIRECT global optimization algorithm. Unlike other methods known from the literature, our method recognizes well not only ellipses with clear edges, but also ellipses with noisy edges. CPU-time necessary for running the corresponding algorithm is very short and this raises hope that, with appropriate software optimization, the algorithm could be run in real time. The method is illustrated and tested on 100 randomly generated data sets.

Keywords: multiple ellipses detection problem; globally optimal k-partition; Lipschitz continuous function; DIRECT; k-means

MSC 2010: 65K05, 90C26, 90C27, 90C56, 90C57, 05E05

1. Introduction

Let $\mathcal{A} = \{a^i = (x_i, y_i)^\top \in \mathbb{R}^2 \colon \alpha_1 \leqslant x_i \leqslant \beta_1, \ \alpha_2 \leqslant y_i \leqslant \beta_2, \ i = 1, \ldots, m\}$ be a set of data points coming from a number of ellipses in the plane not known in advance that should be reconstructed or detected. Note that $\mathcal{A} \subset [\alpha, \beta] := [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \subset \mathbb{R}^2$, $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$. There are several different approaches to solving this

problem in the literature, but most of them cannot be used in real-time applications. Let us mention some of them. In [1], simple and robust nonparametric algorithms for the geometric fitting of ellipse is proposed. In [14], a novel ellipse fitting method which is selective for digital and noisy elliptic curves is proposed. In [6], an ellipse is viewed as a Mahalanobis circle with some positive definite matrix. Two methods for solving this problem are proposed in this paper. The former method very successfully combines the well-known direct least square method and the RANSAC algorithm with the realistic statistical model of multiple ellipses in the plane, while the latter, less efficient method, is based on incremental clustering algorithms. Akinlar and Topal [2] have proposed a real-time, parameter-free circle detection (Algorithm EDCircles) with high detection rates that can also be applied to near-circular ellipses.

Ellipses are the most common nonlinear geometric objects that appear in the images as an approximation of real objects. They can be fully or partially visible, and may appear as discrete random points. The multiple ellipses detection problem appears in different areas of applied research as e.g. in problems in pattern recognition and computer vision, pupil tracking, biological cell segmentation, agriculture, elliptical anomalies in wireless sensors, astronomical and geological shape segmentation, applications in medicine, robotics, object detection, and other image processing industrial applications, etc. [1], [2], [12], [14].

In our paper, we consider the multiple ellipses detection problem on the basis of a data points set coming from a number of ellipses in the plane not known in advance. For the purpose of solving the aforementioned problem, an efficient method is proposed which is in fact a combination of the well-known DIRECT algorithm for global optimization (see [4], [7], [8], [13], [17]) and a modification of the well-known k-means algorithm [3]. The partition with the most appropriate number of clusters with ellipses as centers is selected by using a modified Davies-Bouldin index. The method is tested on and illustrated by 100 randomly generated sets of data points.

The paper is organized as follows. The next section gives basic terms referring to the center-based clustering problem and ellipse interpretation as a Mahalanobis circle. In Section 3, a precise statement of the problem is given as well as a detailed description of the proposed method. The proposed method is tested and illustrated in Section 4. Appropriate algorithms are also given in that section. Finally, some conclusions are given in Section 5.

2. Multiple ellipses detection problem

In [18], a similar problem is considered as a center-based clustering problem, where centers of clusters are circles. Based on this idea, in our paper we propose a new, very efficient method for solving the multiple ellipses detection problem.

First, let us mention a few basic terms about the hard clustering problem.

2.1. Center-based clustering problem—basic terms. A hard partition of a set $\mathcal{A} = \{a^i \in \mathbb{R}^n \colon i = 1, \ldots, m\}$ into k nonempty disjoint subsets $\pi_1, \ldots, \pi_k$, $1 \leqslant k \leqslant m$ will be denoted by $\Pi(\mathcal{A}) = \{\pi_1, \ldots, \pi_k\}$ and the set of all such partitions will be denoted by $\mathcal{P}(\mathcal{A}; k)$. The elements $\pi_1, \ldots, \pi_k$ of the partition Π are called *clusters*.

If $d \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $\mathbb{R}_+ = [0, \infty)$, is some distance-like function (see e.g. [9]), then with each cluster $\pi_j \in \Pi$ we can associate its center c_j defined by

$$(2.1) \qquad c_j := \underset{x \in \mathrm{conv}(\mathcal{A})}{\arg\min} \sum_{a^i \in \pi_j} d(x, a^i).$$

After that, by introducing the objective function $\mathcal{F} \colon \mathcal{P}(\mathcal{A}; k) \to \mathbb{R}_+$, the quality of a partition can be defined, and searching for a *globally optimal k-partition* comes down to solving the optimization problem:

$$(2.2) \qquad \underset{\Pi \in \mathcal{P}(\mathcal{A}; k)}{\arg\min} \mathcal{F}(\Pi), \quad \mathcal{F}(\Pi) = \sum_{j=1}^{k} \sum_{a \in \pi_j} d(c_j, a),$$

where $c = (c_1, \ldots, c_k)^\top$.

Conversely, for a given set of points $c_1, \ldots, c_k \in \mathbb{R}^n$, by applying the minimal distance principle, we can define the partition $\Pi = \{\pi(c_1), \ldots, \pi(c_k)\}$ of the set $\mathcal{A}$ consisting of clusters

$$\pi(c_j) = \{a \in \mathcal{A} \colon d(c_j, a) \leqslant d(c_s, a) \ \forall \, s = 1, \ldots, k\}, \quad j = 1, \ldots, k.$$

Thereby, one has to take into account that every element of the set $\mathcal{A}$ occurs in one and only one cluster. Hence, the problem of finding an optimal partition of the set $\mathcal{A}$ can be reduced to the following *global optimization problem* (GOP) (see e.g. [21], [9]):

$$(2.3) \qquad \underset{c \in \mathrm{conv}(\mathcal{A})^k}{\arg\min} F(c), \quad F(c) = \sum_{i=1}^{m} \min_{1 \leqslant j \leqslant k} d(c_j, a^i).$$

The solutions of (2.2) and (2.3) coincide [20], [21].

2.2. An ellipse as a Mahalanobis circle. Generally, an ellipse in the plane $E(S, \xi, \eta, \vartheta)$ can be defined as a set of all points $(x, y) \in \mathbb{R}^2$ satisfying the equation (see [5], [10], [14])

$$(2.4) \qquad \frac{[(x-p)\cos\vartheta + (y-q)\sin\vartheta]^2}{\xi^2} + \frac{[(x-p)\sin\vartheta - (y-q)\cos\vartheta]^2}{\eta^2} = 1,$$

where $S = (p, q)^\top$ is the center of the ellipse, $\xi, \eta > 0$ are the lengths of semi-axes and ϑ is the angle between the semi-axis ξ and the positive direction of the abscissa. Note that ellipse (2.4) can be written in parametric form as

$$(2.5) \qquad \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S + U(\vartheta) \begin{bmatrix} \xi \cos t \\ \eta \sin t \end{bmatrix}, \quad U(\vartheta) = \begin{bmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{bmatrix},$$

where $t \in [0, 2\pi]$, or as

$$(2.6) \qquad d_m(u, S; \Sigma) = 1, \quad u = (x, y)^\top,$$

where $d_m \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ is a Mahalanobis distance-like function defined by

$$(2.7) \qquad d_m(u, v; \Sigma) = \|u - v\|_\Sigma^2 = (u - v)^\top \Sigma^{-1}(u - v),$$

and $\Sigma \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix of the form

$$(2.8) \qquad \Sigma = U \begin{bmatrix} \xi^2 & 0 \\ 0 & \eta^2 \end{bmatrix} U^\top.$$

Note that the ellipse semi-axes ξ, η correspond to the square roots of the eigenvalues of the matrix Σ, and $\vartheta = \arctan(u_{21}/u_{11})$.

To ensure monotone decreasing of the function value in the implementation of the k-means algorithm, in Subsection 3.1.1 we will define (see [11], [21]) the *normalized Mahalanobis distance-like function* $d_M \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$,

$$(2.9) \qquad d_M(u, v; \Sigma) := \sqrt{\det \Sigma}(u - v)^\top \Sigma^{-1}(u - v) = \|u - v\|_\Sigma^2.$$

Lemma 2.1. *Ellipse $E(S, \xi, \eta, \vartheta)$ given by (2.4), i.e. (2.6), can be presented as an M-circle*

$$(2.10) \qquad E(S, r, \Sigma) = \{x \in \mathbb{R}^2 \colon d_M(S, x; \Sigma) = r^2\},$$

where $r^2 = \sqrt{\det \Sigma} = \xi \eta$, (see [10]).

Conversely, the M-circle $E(S, r, \Sigma)$ corresponds to the ellipse $E(S, \xi, \eta, \vartheta)$, where the semi-axes ξ, η and the angle ϑ are determined from the eigenvalue decomposition

$$(2.11) \qquad \mathrm{diag}(\xi^2, \eta^2) = U\left(\frac{r^2}{\sqrt{\det \Sigma}} \Sigma\right) U^\top \quad \text{and} \quad \vartheta = \arctan \frac{u_{21}}{u_{11}}.$$

P r o o f. The first statement is verified directly.

In order to prove the second statement, ellipse $E(S, r, \Sigma)$ is written in the following form:

$$\frac{1}{r^2} d_M(S, x; \Sigma) = 1 \Rightarrow \frac{\sqrt{\det \Sigma}}{r^2}(S - x)^\top \Sigma^{-1}(S - x) = 1$$

$$\Rightarrow (S - x)^\top \left(\frac{r^2}{\sqrt{\det \Sigma}}\Sigma\right)^{-1}(S - x) = 1.$$

The required statement is obtained by eigenvalue decomposition of the matrix $\left(\frac{r^2}{\sqrt{\det \Sigma}}\Sigma\right)$. $\qquad\square$

R e m a r k 2.1. The set $\{u \in \mathbb{R}^2 : d_M(u, S; \Sigma) = 1\}$ represents a normalized ellipse with the center in S and semi-axes ξ', η' whose product is $\xi'\eta' = 1$.

Note also that the normalizing factor $\sqrt{\det \Sigma}$ is proportional to the area of the ellipse $(\xi\eta\pi)$, and the length of the "radius" r defined in this way is the geometric mean of the semi-axes ξ, η of the ellipse.

3. A NEW METHOD FOR SOLVING THE MULTIPLE ELLIPSES DETECTION PROBLEM

The method we propose in this paper is based on solving a special center-based clustering problem. Data set $\mathcal{A}$ coming from a number of ellipses in the plane will be grouped into k clusters $\pi_1, \ldots, \pi_k$, whose centers are ellipses written in the form of M-circles. Hence, it will be necessary to know how to solve the following problems:

(i) For the cluster π_j, we should know how to determine its representative (center) in the form of an M-circle.

(ii) By using the minimal distance principle we should know how to partition the set $\mathcal{A}$ into k nonempty disjoint clusters whose centers are M-circles.

For solving both of the aforementioned problems it is necessary to define well the distance from the point $a \in \mathcal{A}$ to the M-circle E. There are many ways of defining the distance from the point $a \in \mathcal{A}$ to a circle that can be found in the literature (see e.g. [5], [18], [6]). These definitions can be easily adjusted for the case of M-circles. In our paper we use the *algebraic distance* from the point $a \in \mathcal{A}$ to the M-circle $E(S, r, \Sigma)$

$$(3.1) \qquad \mathfrak{D}(E, a) = \mathfrak{D}(E(S, r, \Sigma), a) = (\|S - a\|_\Sigma^2 - r^2)^2,$$

because this possibility occurs most frequently in applications.

Problem (i): Searching for the best representative of the cluster π_j in the form of an M-circle can be defined as the following GOP:

$$(3.2) \qquad E_j = \underset{\substack{S \in \mathbb{R}^2,\ r \in \mathbb{R},\\ \Sigma \in \mathbb{R}^{2 \times 2}}}{\arg\min} \sum_{a \in \pi_j} \mathfrak{D}(E(S, r, \Sigma), a).$$

R e m a r k 3.1. A solution to GOP (3.2) can be found by using some locally optimization method (Nelder-Meade, Quasi-Newton), since for every $j = 1, \ldots, k$ in the cluster π_j we are able to determine a very favorable initial approximation $\widehat{E}_j(\widehat{S}_j, \hat{r}_j, \widehat{\Sigma}_j)$. Namely, for $\widehat{S}_j$ we can choose a centroid $\widehat{S}_j = \mathrm{Mean}[\pi_j]$ of the cluster π_j, because the Mahalanobis center of the cluster π_j is also an ordinary mean of the set π_j, (see [11])

$$\underset{u \in \mathbb{R}^2}{\arg\min} \sum_{a \in \pi_j} d_M(u, a; \Sigma_j) = \frac{1}{|\pi_j|} \sum_{a \in \pi_j} a.$$

Furthermore, for the matrix $\widehat{\Sigma}_j$ we can choose

$$\widehat{\Sigma}_j = \frac{1}{|\pi_j|} \sum_{a \in \pi_j} (\widehat{S}_j - a)(\widehat{S}_j - a)^\top$$

(Kronecker product). Also, $\hat{r}_j$ is determined from

$$(3.3) \qquad \hat{r}_j^2 = \frac{1}{|\pi_j|} \sum_{a \in \pi_j} \|\widehat{S}_j - a\|_2^2,$$

because

$$\sum_{a \in \pi_j} (\|\widehat{S}_j - a\|_2^2 - r_j^2)^2 \geqslant \sum_{a \in \pi_j} (\|\widehat{S}_j - a\|_2^2 - \hat{r}_j^2)^2 \quad \forall r_j \in \mathbb{R}.$$

Problem (ii): Determining the cluster π_j of the partition $\Pi = \{\pi_1, \ldots, \pi_k\}$ by applying the minimal distance principle can be written as follows:

$$(3.4) \qquad \pi_j := \pi_j(E_j) = \{a \in \mathcal{A}:\ \mathfrak{D}(E_j(S_j, r_j, \Sigma_j), a)$$
$$\leqslant \mathfrak{D}(E_s(S_s, r_s, \Sigma_s), a)\ \forall s \neq j\}, \quad j = 1, \ldots, k.$$

3.1. Searching for an optimal k-partition. Searching for an optimal k-partition $\Pi^* = \{\pi_1^*, \ldots, \pi_k^*\}$ with M-circle-centers $E_j(S_j, r_j, \Sigma_j)$, where $S_j = (p_j, q_j)^\top$ and $\Sigma_j = \begin{bmatrix} u_j & v_j \\ v_j & t_j \end{bmatrix}$, $j = 1, \ldots, k$ boils down to searching for optimal

parameters $(p_j^*, q_j^*, r_j^*, u_j^*, v_j^*, t_j^*)$, $j = 1, \ldots, k$, which give a solution to the GOP (cf. (2.3))

$$(3.5) \qquad \operatorname*{arg\,min}_{\substack{(p_j, q_j) \in [\alpha, \beta],\ r_j \in [0, R], \\ \Sigma_j \in M_2}} \sum_{i=1}^{m} \min_{1 \leqslant j \leqslant k} \{\mathfrak{D}(E_j(S_j, r_j, \Sigma_j), a^i)\},$$

where $R = \frac{1}{2} \min\{\beta_1 - \alpha_1, \beta_2 - \alpha_2\}$, M_2 is the set of positive definite symmetric matrices of second order and $\mathfrak{D}(E_j(S_j, r_j, \Sigma_j), a^i)$ represents the distance from the point $a^i \in \mathcal{A}$ to the M-circle $E_j(S_j, r_j, \Sigma_j)$.

Note that (3.5) is a GOP with $6k$ independent variables. If we apply the global optimization algorithm DIRECT to this GOP, we will see that the necessary CPU-time will be unreasonably long, since the algorithm will search for all $k!$ solutions. Thus for solving this GOP we propose the following procedure:

(1) Find a good initial approximation for GOP (3.5).
(2) Apply a modification of the k-means algorithm for M-circle-centers to this initial approximation.

3.1.1. The adaptive Mahalanobis k-closest M-circle-centers algorithm. As predicted by the new algorithm, a globally optimal solution of (3.5) will be obtained by applying the *Adaptive Mahalanobis k-closest M-circle-centers algorithm* (KMCC), where an initial approximation is chosen in accordance with the previous subsection. This algorithm is the well-known k-means algorithm (see e.g. [9], [15]) adapted for searching for a locally optimal partition with M-circles as cluster-centers (see [6], [10]). The algorithm can be described in two steps which are repeated iteratively:

Algorithm 3.1 (The adaptive Mahalanobis k-closest M-circle-centers algorithm (KMCC))

Step A. (Assignment step) For each set of mutually different M-circles $E_1(S_1, r_1, \Sigma_1), \ldots, E_k(S_k, r_k, \Sigma_k)$, the set $\mathcal{A}$ should be divided into k disjoint unempty clusters $\pi_1, \ldots, \pi_k$ by using the minimal distance principle (3.4);

Step B. (Update step) Given a partition $\Pi = \{\pi_1, \ldots, \pi_k\}$ of the set $\mathcal{A}$, one can define the corresponding M-circle-centers $\widehat{E}_j(\widehat{S}_j, \hat{r}_j, \widehat{\Sigma}_j)$, $j = 1, \ldots, k$, by solving GOP (3.2) for each $j = 1, \ldots, k$.

R e m a r k 3.2. The k-means algorithm for searching for a locally optimal partition whose cluster-centers are ordinary circles is developed similarly. The algorithm can be found in [18] as the *k-closest circles algorithm* (KCC).

3.1.2. The new algorithm for searching for an optimal k-partition.
For the purpose of constructing a better initial approximation for the solution of
GOP (3.5), we will conduct a numerical experiment described in the following
example.

E x a m p l e 3.1. Let $\mathcal{B} = \{b^i \in [\alpha, \beta] \subset \mathbb{R}^2 : i = 1, \ldots, m_B\}$ be a set of data
points coming from a known ellipse in the plane (see Fig. 1 (a)). The best circle-
representative $C^*(S^*, r^*)$ of the set $\mathcal{B}$ is obtained by solving GOP

$$(3.6) \qquad \underset{S \in [\alpha, \beta], r \in [0, R]}{\arg\min} F(S, r), \quad F(S, r) = \sum_{i=1}^{m_B} (\|S - b^i\|_2^2 - r^2)^2.$$

This problem is easy to solve by using some locally optimization method using a very
good initial approximation $C_0(S_0, r_0)$, where (see also Remark 2.1)

$$(3.7) \qquad S_0 = \text{Mean}[\mathcal{B}], \quad r_0 = \frac{1}{m_B} \sum_{b^i \in \mathcal{B}} \|S_0 - b^i\|^2.$$

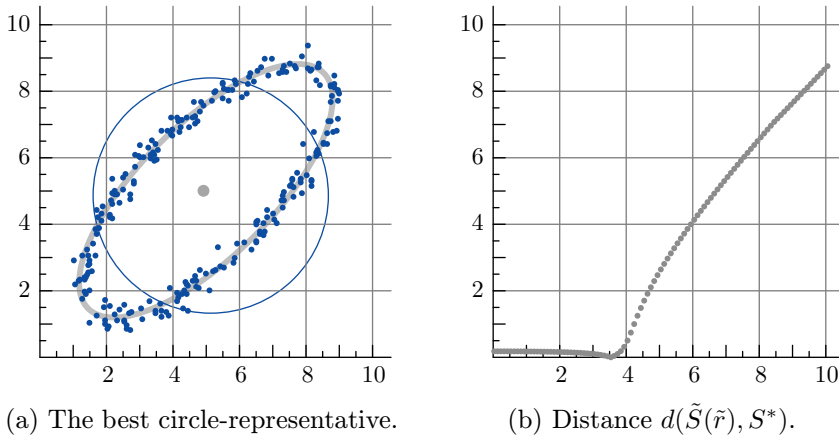For the example given in Fig. 1 (a) we obtain $C_0((4.923, 5.006), 3.489)$ and $C^*((5.135, 4.864), 3.499)$.

(a) The best circle-representative.   (b) Distance $d(\tilde{S}(\tilde{r}), S^*)$.

Figure 1. Data originating from a known ellipse and the center of the best circle-
representative.

Furthermore, the best circle-representative of the given radius $\tilde{r} > 0$ for the set $\mathcal{B}$
can be obtained by solving the GOP

$$(3.8) \qquad \underset{S \in [\alpha, \beta]}{\arg\min} \widetilde{F}(S), \quad \widetilde{F}(S) = \sum_{i=1}^{m_B} (\|S - b^i\|_2^2 - \tilde{r}^2)^2,$$

with the initial approximation S_0 as in (3.7).

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In order to monitor the movement of the optimal center $\widetilde{S}(\tilde{r})$ of the best circle-representative $\widetilde{C}(\widetilde{C}, \tilde{r})$ for different values of $\tilde{r} > 0$, GOP (3.8) will be solved for $\tilde{r} = 10 - 0.1(i-1)$, $i = 1, \ldots, 100$, and for every $\widetilde{S}(\tilde{r})$ obtained, $d(\widetilde{S}(\tilde{r}), S^*) = \|\widetilde{S}(\tilde{r}) - S^*\|_2$ will be calculated (see Fig. 1 (b)).

It can be seen that $\widetilde{S}(\tilde{r}) \approx S^*$ for each $\tilde{r} < r^*$ (see Fig. 1 (b)), which means that the position of the center of the requested ellipse can be approximated well by simply using the arithmetic mean of the set $\mathcal{B}$.

The numerical experiment from Example 3.1 indicates that a good approximation of the positions of centers $S_1, \ldots, S_k$ of ellipses with multiple ellipses detection problem (3.5) can be searched for by solving a simple GOP with $2k$ independent variables

$$(3.9) \qquad \arg\min_{\mathbf{p}, \mathbf{q} \in [\alpha, \beta]^k} F(\mathbf{p}, \mathbf{q}), \quad F(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^m \min_{1 \leqslant j \leqslant k} \|(p_j, q_j) - a^i\|_2^2,$$

where $S_j = (p_j, q_j)^\top$.

Since the function from (3.9) is a Lipschitz-continuous function [16], [19], by applying a smaller number of iterations (say 10 to 20) of the DIRECT algorithm to GOP (3.9) we will obtain a sufficiently good initial approximation for ellipse centers $\widehat{S}_1, \ldots, \widehat{S}_k$.

In order to be able to apply the DIRECT algorithm to GOP (3.9), the objective function $F: [\alpha, \beta]^{2k} \to \mathbb{R}$ will be transformed to function $f: [0, 1]^{2k} \to \mathbb{R}$, $f(x) = (F \circ T^{-1})(x)$, where the mapping $T: [\alpha_1, \beta_1]^k \times [\alpha_2, \beta_2]^k \to [0, 1]^{2k}$ is given by

$$(3.10) \qquad T(x) = D(x - u),$$
$$D = \mathrm{diag}\Big(\frac{1}{\beta_1 - \alpha_1}, \frac{1}{\beta_2 - \alpha_2}, \ldots, \frac{1}{\beta_1 - \alpha_1}, \frac{1}{\beta_2 - \alpha_2}\Big) \in \mathbb{R}^{2k \times 2k},$$
$$u = (\alpha_1, \beta_1, \ldots, \alpha_1, \beta_1)^\top \in \mathbb{R}^{2k}.$$

An initial approximation $\hat{\mathbf{x}} \in [0, 1]^{2k}$ for the GOP

$$(3.11) \qquad \arg\min_{x \in [0,1]^{2k}} f(x), \quad f(x) = (F \circ T^{-1})(x)$$

will be determined by using a smaller number of iterations (say, 10 to 20) of the DIRECT algorithm. The criterion for determining the number of iterations of the DIRECT algorithm can be an absolute relative value of the objective function f. Since $(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = T^{-1}(\hat{\mathbf{x}})$, in this way we obtain a good initial approximation for ellipse centers $\widehat{S}_1, \ldots, \widehat{S}_k$.

Searching for an optimal k-partition will be conducted by using Algorithm 3.2 which was constructed in the following way. After transforming problem (3.9) into a unit hypersquare, in the second step of Algorithm 3.2 we look for a good initial approximation of ellipse centers $\widehat{S}_1, \ldots, \widehat{S}_k$ by means of the DIRECT algorithm.

Algorithm 3.2 (Searching for an optimal k-partition)

Input: $\mathcal{A} \subset [\alpha, \beta]^2$ {Data set}; $\quad k \geqslant 2$;

1: Define the mapping $T^{-1}\colon [0,1]^{2k} \to [\alpha, \beta]^k$, $T^{-1}(x) = D^{-1}x + u$ and the objective function $f = F \circ T^{-1}$, where T and F are given by (3.10) and (3.9), respectively;

2: By using the DIRECT algorithm find the initial approximation of vector centers $\widehat{S} = (\widehat{S}_1, \ldots, \widehat{S}_k) \in [0,1]^{2k}$ by solving GOP (3.11);

3: By using the minimal distance principle find the clusters $\hat{\pi}_1, \ldots, \hat{\pi}_k$;

4: For each cluster $\hat{\pi}_j$ and the corresponding center $\widehat{S}_j$ determine $\hat{r}_j$ from (3.3);

5: By using the KMCC algorithm with the initial approximation $(\widehat{S}_j, \hat{r}_j/2, \mathbf{I}_2)$, $j = 1, \ldots, k$, determine M-circles $C_j^*(S_j^*, r_j^*, \Sigma_j^*)$, $j = 1, \ldots, k$;

Output: $\{k, C_j^*(S_j^*, r_j^*, \Sigma_j^*), \ j = 1, \ldots, k\}$.

In *Step 3* of Algorithm 3.2, we apply the minimum distance principle to the obtained centers in order to obtain the corresponding clusters $\hat{\pi}_1, \ldots, \hat{\pi}_k$, and in *Step 4*, for every cluster we determine the approximation of the radius $\hat{r}_j$ from (3.3) (see Remark 2.1).

After that, in *Step 5*, we apply the KMCC algorithm with the initial approximation $(\widehat{S}_j, \hat{r}_j/2, \mathbf{I}_2)$, $j = 1, \ldots, k$, where $\mathbf{I} \in \mathbb{R}^{2 \times 2}$ is the identity matrix of the second order. In the implementation of Algorithm 3.2, it was shown that the KMCC algorithm converges quickly to the solution if the initial circles $\widehat{C}_j$, $j = 1, \ldots, k$, have a slightly smaller radius than that given by (3.3). This is consistent with the remarks mentioned in Example 3.1.

Fig. 2 shows initial approximations for sets of data points given in Fig. 3, obtained by using the DIRECT algorithm in *Step 2* of Algorithm 3.2. Figures of the obtained optimal partitions are not given as they correspond to Fig. 3. Table 1 gives an average CPU-time used for carrying out *Step 2* (the DIRECT algorithm) and *Step 5* (the KMCC algorithm) for searching for $2, 3, 4$, and 5-partitions.

R e m a r k 3.3. Due to the symmetry property of function (3.9), in *Step 2* of Algorithm 3.2, the DIRECT algorithm searches for all $k!$ solutions, but since in this case the minimizing function is simple, this procedure does not require a lot of CPU-time.
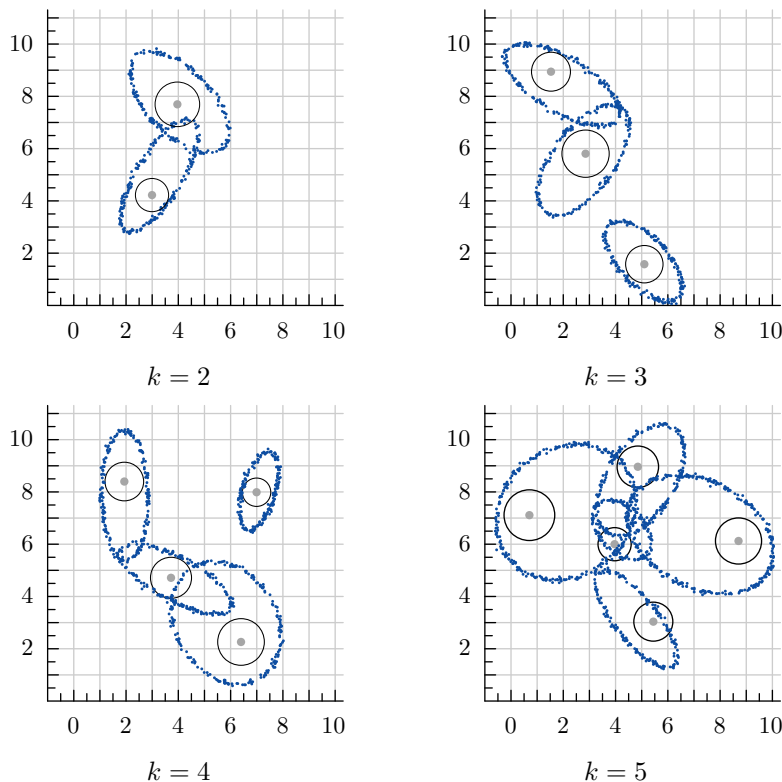
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Figure 2. Initial approximations for four selected examples obtained by the DIRECT algorithm.

3.2. Determining an optimal partition with the most appropriate number of clusters. For searching for an optimal partition with the most appropriate number of clusters we will use a modified Davies-Bouldin index [11], [22].

A specially modified variant of this index for an optimal k-partition $\Pi_k^* = (\pi_1^*, \ldots, \pi_k^*)$ with M-circle-centers $E_j^*(S_j^*, r_j^*, \Sigma_j^*)$, $j = 1, \ldots, k$, can be seen in [6]:

$$(3.12) \qquad \mathrm{DBE}(\Pi_k^*) = \frac{1}{k} \sum_{j=1}^{k} \max_{\substack{s=1,\ldots,k \\ s \neq j}} \frac{V(\pi_j^*) + V(\pi_s^*)}{\delta^4(E_j^*, E_s^*)},$$

where

$$V(\pi_j^*) = \frac{1}{|\pi_j^*|} \sum_{a \in \pi_j^*} \mathfrak{D}(a, E_j^*(S_j^*, r_j^*, \Sigma_j^*)),$$

and $\delta(E_j^*, E_s^*)$ is the Hausdorff distance between M-circle-centers E_j^* and E_s^*. The Davies-Bouldin index for an optimal k-partition $\Pi_k^* = (\pi_1^*, \ldots, \pi_k^*)$ with ordinary circle-centers $C_j^*(S_j^*, r_j^*)$, $j = 1, \ldots, k$, is proposed in [18], where a special formula

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is derived for the distance between two circles, which has eliminated a complicated procedure of calculating the Hausdorff distance between two circles.

In our paper, by observing an ellipse as an M-circle (see Section 2.2) analogously to [18] for an optimal k-partition $\Pi_k^* = (\pi_1^*, \ldots, \pi_k^*)$ with M-circle-centers $E_j^*(S_j^*, r_j^*, \Sigma_j^*)$, $j = 1, \ldots, k$, we define the variance $V(\pi_j^*)$, the distance $\delta(E_j^*, E_s^*)$ between two ellipses and the Davies-Bouldin index as follows:

$$(3.13) \quad V(\pi_j^*) = \frac{1}{|\pi_j^*|} \sum_{a \in \pi_j^*} \sqrt{\mathfrak{D}(E_j^*, a)}, \quad \delta(E_j^*, E_s^*) = \|S_j^* - S_s^*\|_2 + |r_j^* - r_s^*|,$$

$$(3.14) \quad \mathrm{DBE}(\Pi_k^*) = \frac{1}{k} \sum_{j=1}^{k} \max_{\substack{s=1,\ldots,k \\ s \neq j}} \frac{V(\pi_j^*) + V(\pi_s^*)}{\delta^2(E_j^*, E_s^*)}.$$

Note that the distance $\delta(E_j^*, E_s^*)$ between two ellipses was viewed as the distance between two M-circles. Let us also note that in the definition of index (3.14) the property of non-dimensionality has been retained, since the nominators and denominators have the same dimension (of squared length). The most appropriate number of M-circle-centers corresponds to the minimal DBE-index.

4. Numerical experiments

The described method for solving the multiple ellipses detection problem will be implemented through Algorithm 4.1 and Algorithm 3.2 given below and tested on 100 randomly generated data sets[1].

4.1. Generating sets of data points. A data set $\mathcal{A}$ originating from k ellipses will be defined in the following way. First, we choose $k \in \mathcal{U}(2, 5)$ and k centers $S_1, \ldots, S_k \in [1.5, 8.5]^2$ whose mutual distance is at least 2.5. Semi-axes ξ_j, η_j of every ellipse will be taken from $\mathcal{U}(0.5, 2.5)$, and the angle of rotation ϑ_j from $\in \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$. This yields k ellipses $E_j(S_j, \xi_j, \eta_j, \vartheta_j)$, $j = 1, \ldots, k$, given parametrically by (2.5).

On the ellipse E_j we choose $m_j \in \mathcal{U}(180, 220)$ uniformly distributed points (see [6]), and at any point in the direction of the normal a random point $a^i \in \mathcal{N}(0, \sigma^2)$, $\sigma^2 = 0.05$ (see [1], [6]). In this way we have defined the data set $\mathcal{A}$ and its partition $\Pi = \{\pi_1, \ldots, \pi_k\}$, where π_j is a set of data coming from ellipse E_j.

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[1] All evaluations were done on the basis of our own *Mathematica*-modules freely available at `https://www.mathos.unios.hr/images/homepages/scitowsk/ELLIPSES-AM.rar`, and were performed on the computer with a 2.90 GHz Intel(R) Core(TM)i7-75000 CPU with 16 GB of RAM.

Note that the ellipse E_j need not be the center of the cluster π_j. Therefore, to the set $\mathcal{A}$ we will apply the KMCC-algorithm (see Section 3.1.1) with initial M-circle-centers $(S_j, \sqrt{\xi_j \eta_j}, \mathbf{I})$, where $\mathbf{I}_2$ is the identity matrix of second order. This yields the ellipse $\widehat{E}_j$ as the center of the cluster π_j, which is slightly different from the original ellipses.

In this way, we will define 100 sets of such ellipses and the corresponding data on which our method will be tested. The selected four examples with $2, 3, 4, 5$ ellipses are shown in Fig. 3.
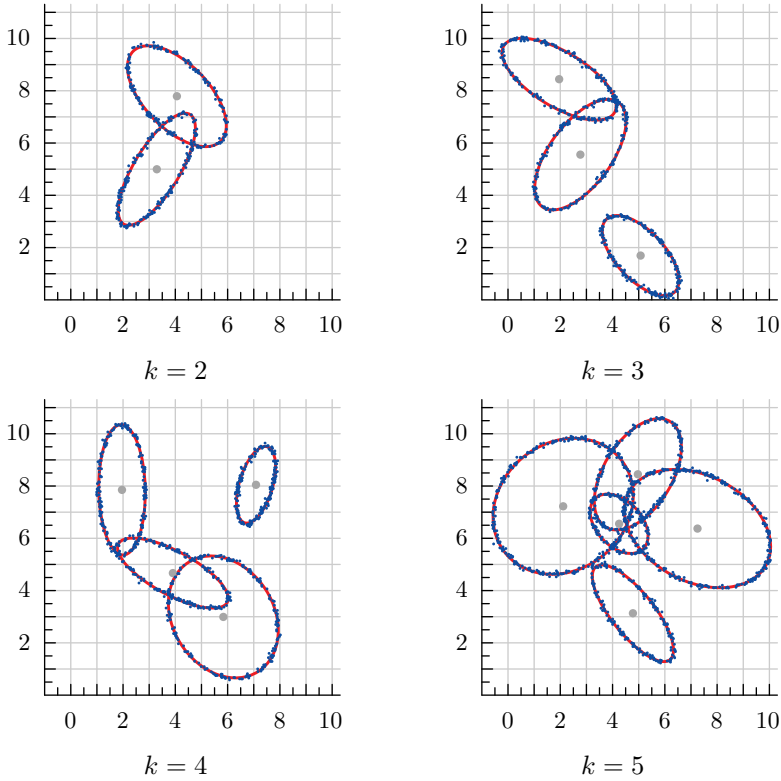

Figure 3. Four selected examples with 2, 3, 4, and 5 ellipses.

4.2. Algorithm for searching for an optimal partition with the most appropriate number of clusters. Assuming that the number of ellipses searched for is not greater than $k_{\max}$, searching for an optimal partition with the most appropriate number of clusters will be conducted by using Algorithm 4.1. The algorithm is run as long as the value of the DBE index is decreased and the number of clusters in the partition is less than $k_{\max}$. This practically means Algorithm 3.2 will be invoked

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at most $k_{\max}$ times. It is clear that it may happen that the DBE index does not decrease monotonically and that this will not yield a globally optimal partition. In order to minimize the CPU-time, we take this risk consciously. A possible alternative is always to find all $k_{\max}$ optimal partitions and, among them, select the one with the least value of the DBE index.

Finally, the obtained M-circles will be transformed into ellipses by Lemma 2.1.

Algorithm 4.1 (Searching for an optimal partition with the most appropriate number of clusters)

Input: $\mathcal{A} \subset [\alpha, \beta]^2$ {Set of data points}; $\quad k \geqslant 2$; $\quad k_{\max} = 5$;

1: Define the mapping $T^{-1}\colon [0,1]^{2k} \to [\alpha, \beta]^k$, $T^{-1}(x) = D^{-1}x + u$, and the objective function $f = F \circ T^{-1}$, where T and F are given by (3.10) and (3.9), respectively;

2: Set $k = 2$;

3: Call Algorithm 3.2 and denote the obtained partition by Π^0;

4: According to (3.14), calculate the Davies-Bouldin Index $vdb_0 = \mathrm{DBE}(\Pi^0)$;

5: **while**

         Set $k = k + 1$;

         Call Algorithm 3.2 and denote the obtained partition by Π^1;

         According to (3.14), calculate the Davies-Bouldin Index $vdb_1 = \mathrm{DBE}(\Pi^1)$;

         $vdb_1 < vdb_0$ & $k < k_{\max}$,

6: $vdb_0 = vdb_1$; $\Pi^0 = \Pi^1$;

7: **end while**

8: Set $\Pi^* = (\pi_1^*, \ldots, \pi_\kappa^*) = \Pi^0$, where the center of π_j^* is the ellipse $E_j^*(S_j^*, r_j^*, \Sigma_j^*)$;

9: Compute eigenvalue decomposition of the matrices $\left(\frac{r_j^{*2}}{\sqrt{\det \Sigma_j^*}} \Sigma_j^* \right)$, $j = 1, \ldots, \kappa$, denote eigenvalues by ξ_j^*, η_j^*, $j = 1, \ldots, \kappa$, the corresponding orthogonal matrix by U_j and calculate $\vartheta_j = \arctan u_{21}^{(j)} / u_{11}^{(j)}$, $j = 1, \ldots, \kappa$;

Output: $\{E_j^*(S_j^*, \xi_j^*, \eta_j^*, \vartheta_j^*), \ j = 1, \ldots, \kappa\}$.

Table 1 shows results of Algorithm 4.1 applied to 100 data sets generated as described previously. It can be noticed that Algorithm 4.1 fully recognized 2-partitions and 3-partitions, but somewhat less 4-partitions and 5-partitions. The total percentage of recognition is 92%. This happened, because the DBE index did not always monotonically decrease. As can be seen in Table 1, if the smallest value of the DBE index is searched for, the total percentage of recognition is better (95%).

CPU-time required for running the algorithm is very short (see Table 1), and this raises hope that, with appropriate software optimization, the algorithm could be run in real time.

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	$k = 2$	$k = 3$	$k = 4$	$k = 5$	Σ
Number of examples	25	22	28	25	100
Detected partitions (Algorithm 4.1)	25	22	24	21	92
Detected partitions (smallest DBE)	25	22	26	22	95
Average DIRECT-CPU-time (sec)	0.03	0.075	0.22	0.95	–
Average KMCC-CPU-time (sec)	0.678	1.36	2.66	5.90	–

Table 1. Number of detected partitions and necessary CPU-time in seconds.

5. CONCLUSIONS

The multiple ellipses detection problem plays an important role in different areas of application. It is therefore important to have a method that would recognize ellipses in the figure well, but it is almost equally important to have the possibility of running the corresponding algorithm as close to real time as possible.

Our method recognizes ellipses with clear edges very well, but also ellipses with noisy edges, and since CPU-time required for running the algorithm is very short, this raises hope that, with appropriate software optimization, the algorithm could be run in real time.

It has been shown that in order to better recognize a partition with the most appropriate number of clusters, it would be necessary to carry out an additional analysis of the application of the DBE index.

Compared to the method given in [6], our method achieves approximately the same degree of recognition, but the necessary CPU-time is considerably shorter.

The advantage of our method in relation to the EDCircles Algorithm [2] is that it recognizes the ellipses with noisy edges very well, which EDCircles is not capable of.

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Authors' address: *Rudolf Scitovski* (corresponding author), *Kristian Sabo*, Department of Mathematics, University of Osijek, Trg Ljudevita Gaja 6, 31 000 Osijek, Croatia, `scitowsk @mathos.hr`, `ksabo@mathos.hr`.