Commentationes Mathematicae Universitatis Carolinae

Nobuyuki Kemoto

Countable compactness of lexicographic products of GO-spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 60 (2019), No. 3, 421–439

Persistent URL: `http://dml.cz/dmlcz/147853`

Comment.Math.Univ.Carolin. 60,3 (2019) 421–439

Countable compactness of lexicographic products of GO-spaces

Nobuyuki Kemoto

Abstract. We characterize the countable compactness of lexicographic products of GO-spaces. Applying this characterization about lexicographic products, we see:

- the lexicographic product X^2 of a countably compact GO-space X need not be countably compact,
- ω_1^2, $\omega_1 \times \omega$, $(\omega+1) \times (\omega_1+1) \times \omega_1 \times \omega$, $\omega_1 \times \omega \times \omega_1$, $\omega_1 \times \omega \times \omega_1 \times \omega \times \cdots$, $\omega_1 \times \omega^\omega$, $\omega_1 \times \omega^\omega \times (\omega+1)$, ω_1^ω, $\omega_1^\omega \times (\omega_1+1)$ and $\prod_{n\in\omega} \omega_{n+1}$ are countably compact,
- $\omega \times \omega_1$, $(\omega+1) \times (\omega_1+1) \times \omega \times \omega_1$, $\omega \times \omega_1 \times \omega \times \omega_1 \times \cdots$, $\omega \times \omega_1^\omega$, $\omega_1 \times \omega^\omega \times \omega_1$, $\omega_1^\omega \times \omega$, $\prod_{n\in\omega} \omega_n$ and $\prod_{n\leq\omega} \omega_{n+1}$ are not countably compact,
- $[0,1)_\mathbb{R} \times \omega_1$, where $[0,1)_\mathbb{R}$ denotes the half open interval in the real line $\mathbb{R}$, is not countably compact,
- $\omega_1 \times [0,1)_\mathbb{R}$ is countably compact,
- both $\mathbb{S} \times \omega_1$ and $\omega_1 \times \mathbb{S}$ are not countably compact,
- $\omega_1 \times (-\omega_1)$ is not countably compact, where for a GO-space $X = \langle X, <_X, \tau_X\rangle$, $-X$ denotes the GO-space $\langle X, >_X, \tau_X\rangle$.

Keywords: lexicographic product; GO-space; LOTS; countably compact product

Classification: 54F05, 54B10, 54B05, 54C05

1. Introduction

Lexicographic products of LOTS's were studied in [2] and it was proved:

- a lexicographic product of LOTS's is compact if and only if all factors are compact;
- a lexicographic products of paracompact LOTS's is also paracompact.

Recently, the author defined the notion of the lexicographic product of GO-spaces and extended the results above for GO-spaces, see [6], [7]. It is also known:

- the usual Tychonoff product of GO-spaces is countably compact if and only if all factors are countably compact, therefore the usual Tychonoff product ω_1^γ is countably compact for every ordinal γ;
- the lexicographic product ω_1^ω is countably compact, but the lexicographic product $\omega_1^{\omega+1}$ is not countably compact, see [4].

DOI 10.14712/1213-7243.2019.020

In this paper, we will characterize the countable compactness of lexicographic products of GO-spaces, further we give some applications.

When we consider a product $\prod_{\alpha<\gamma} X_\alpha$, all X_α are assumed to have cardinality at least 2 with $\gamma \geq 2$. Set theoretical and topological terminology follow [9] and [1].

A linearly ordered set $\langle L, <_L \rangle$ has a natural topology λ_L, which is called an interval topology, generated by $\{(\leftarrow, x)_L \colon x \in L\} \cup \{(x, \rightarrow)_L \colon x \in L\}$ as a subbase, where $(x, \rightarrow)_L = \{z \in L \colon x <_L z\}$, $(x, y)_L = \{z \in L \colon x <_L z <_L y\}$, $(x, y]_L = \{z \in L \colon x <_L z \leq_L y\}$ and so on. The triple $\langle L, <_L, \lambda_L \rangle$, which is simply denoted by L, is called a *LOTS*.

A triple $\langle X, <_X, \tau_X \rangle$ is said to be a *GO-space*, which is also simply denoted by X, if $\langle X, <_X \rangle$ is a linearly ordered set and τ_X is a T_2-topology on X having a base consisting of convex sets, where a subset C of X is *convex* if for every $x, y \in C$ with $x <_X y$, $[x, y]_X \subset C$ holds. For more information on LOTS's or GO-spaces, see [10]. Usually $<_L$, $(x, y)_L$, λ_L or τ_X are written simply $<$, (x, y), λ or τ if contexts are clear.

The symbols ω and ω_1 denote the first infinite ordinal and the first uncountable ordinal, respectively. Ordinals, which are usually denoted by Greek letters $\alpha, \beta, \gamma, \cdots$, are considered to be LOTS's with the usual interval topologies.

The cofinality of α is denoted by $\mathrm{cf}\,\alpha$.

For GO-spaces $X = \langle X, <_X, \tau_X \rangle$ and $Y = \langle Y, <_Y, \tau_Y \rangle$, X is said to be a *subspace* of Y if $X \subset Y$, the linear order "$<_X$" is the restriction $<_Y \restriction X$ of the order "$<_Y$" and the topology τ_X is the subspace topology $\tau_Y \restriction X$ $(= \{U \cap X \colon U \in \tau_Y\})$ on X of the topology τ_Y. So a subset of a GO-space is naturally considered as a GO-space. For every GO-space X, there is a LOTS X^* such that X is a dense subspace of X^* and X^* has the property that if L is a LOTS containing X as a dense subspace, then L also contains the LOTS X^* as a subspace, see [11]. Such a X^* is called the *minimal d-extension of a GO-space* X. The construction of X^* is also shown in [6]. Obviously, we can see:

- ∘ if X is a LOTS, then $X^* = X$;
- ∘ the space X has a maximal element $\max X$ if and only if X^* has a maximal element $\max X^*$, in this case, $\max X = \max X^*$ (similarly for minimal elements).

For every $\alpha < \gamma$, let X_α be a LOTS and $X = \prod_{\alpha<\gamma} X_\alpha$. Every element $x \in X$ is identified with the sequence $\langle x(\alpha) \colon \alpha < \gamma \rangle$. For notational convenience, $\prod_{\alpha<\gamma} X_\alpha$ is considered as the trivial one point LOTS $\{\emptyset\}$ whenever $\gamma = 0$, where $\emptyset$ is considered to be a function whose domain is 0 $(= \emptyset)$. When $0 \leq \beta < \gamma$, $y_0 \in \prod_{\alpha<\beta} X_\alpha$ and $y_1 \in \prod_{\beta\leq\alpha} X_\alpha$, $y_0{}^\wedge y_1$ denotes the sequence $y \in \prod_{\alpha<\gamma} X_\alpha$ defined by

$$y(\alpha) = \begin{cases} y_0(\alpha) & \text{if } \alpha < \beta, \\ y_1(\alpha) & \text{if } \beta \leq \alpha. \end{cases}$$

In this case, whenever $\beta = 0$, $\emptyset^\wedge y_1$ is considered as y_1. In case $0 \le \beta < \gamma$, $y_0 \in \prod_{\alpha<\beta} X_\alpha$, $u \in X_\beta$ and $y_1 \in \prod_{\beta<\alpha} X_\alpha$, $y_0{}^\wedge\langle u\rangle^\wedge y_1$ denotes the sequence $y \in \prod_{\alpha<\gamma} X_\alpha$ defined by

$$y(\alpha) = \begin{cases} y_0(\alpha) & \text{if } \alpha < \beta, \\ u & \text{if } \alpha = \beta, \\ y_1(\alpha) & \text{if } \beta < \alpha. \end{cases}$$

More general cases are similarly defined. The lexicographic order "$<_X$" on X is defined as follows: for every $x, x' \in X$,

$$x <_X x' \text{ if and only if for some } \alpha < \gamma, \quad x \upharpoonright \alpha = x' \upharpoonright \alpha \text{ and } x(\alpha) <_{X_\alpha} x'(\alpha),$$

where $x \upharpoonright \alpha = \langle x(\beta) \colon \beta < \alpha \rangle$ (in particular $x \upharpoonright 0 = \emptyset$) and "$<_{X_\alpha}$" is the order on X_α. Now for every $\alpha < \gamma$, let X_α be a GO-space and $X = \prod_{\alpha<\gamma} X_\alpha$. The subspace X of the lexicographic product $\widehat{X} = \prod_{\alpha<\gamma} X_\alpha^*$ is said to be the *lexicographic product of GO-spaces* X_α's, for more details see [6]. Product $\prod_{i\in\omega} X_i$ $\left(\prod_{i\le n} X_i \text{ where } n \in \omega\right)$ is denoted by $X_0 \times X_1 \times X_2 \times \cdots$ ($X_0 \times X_1 \times X_2 \times \cdots \times X_n$, respectively). Product $\prod_{\alpha<\gamma} X_\alpha$ is also denoted by X^γ whenever $X_\alpha = X$ for all $\alpha < \gamma$.

Let X and Y be LOTS's. A map $f \colon X \to Y$ is said to be *order preserving* or 0-*order preserving* if $f(x) <_Y f(x')$ whenever $x <_X x'$. Similarly a map $f \colon X \to Y$ is said to be *order reversing* or 1-*order preserving* if $f(x) >_Y f(x')$ whenever $x <_X x'$. Obviously a 0-order preserving map (also 1-order preserving map) $f \colon X \to Y$ between LOTS's X and Y, which is onto, is a homeomorphism, i.e., both f and f^{-1} are continuous. Now let X and Y be GO-spaces. A 0-order preserving map $f \colon X \to Y$ is said to be a 0-*order preserving embedding* if f is a homeomorphism between X and $f[X]$, where $f[X]$ is the subspace of the GO-space Y. In this case, we identify X with $f[X]$ as a GO-space and write $X = f[X]$ and $X \subset Y$.

Let X be a GO-space. A subset A of X is called a 0-*segment* of X if for every $x, x' \in X$ with $x \le x'$, if $x' \in A$, then $x \in A$. A 0-*segment* A is said to be *bounded* if $X \setminus A$ is nonempty. Similarly the notion of (bounded) 1-segment can be defined. Both $\emptyset$ and X are 0-segments and 1-segments. Obviously if A is a 0-segment of X, then $X \setminus A$ is a 1-segment of X.

Let A be a 0-segment of a GO-space X. A subset U of A is *unbounded in* A if for every $x \in A$, there is $x' \in U$ such that $x \le x'$. Let

$$0\text{-}\operatorname{cf}_X A = \min\{|U| \colon U \text{ is unbounded in } A\}.$$

A set $0\text{-}\operatorname{cf}_X A$ can be $0, 1$ or regular infinite cardinals. $0\text{-}\operatorname{cf}_X A = 0$ means $A = \emptyset$ and $0\text{-}\operatorname{cf}_X A = 1$ means that A has a maximal element. If contexts are clear, $0\text{-}\operatorname{cf}_X A$ is denoted by $0\text{-}\operatorname{cf} A$. For cofinality in compact LOTS and linearly ordered compactifications, see also [3], [8].

Remember that a topological space is said to be countably compact if every infinite subset has a cluster point.

Definition 1.1. A GO-space X is (*boundedly*) *countably* 0-*compact* if for every (bounded) closed 0-segment A of X, $0\text{-}\operatorname{cf}_X A \neq \omega$ holds. The term "(*boundedly*) *countably* 1-*compact*" is analogously defined.

Obviously a GO-space X is countably 0-compact if and only if it is boundedly countably 0-compact and $0\text{-}\operatorname{cf} X \neq \omega$. Note that subspaces of ordinals are always countably 1-compact because they are well-ordered. Also note that ordinals are boundedly countably 0-compact but in general not countably 0-compact, e.g., ω, $\aleph_\omega$ etc.

We first check:

Lemma 1.2. *A GO-space X is countably 0-compact if and only if every 0-order preserving sequence $\{x_n : n \in \omega\}$ (i.e., $m < n \to x_m < x_n$) has a cluster point.*

Proof: Assuming the existence of a 0-order preserving sequence $\{x_n : n \in \omega\}$ with no cluster points, set $A = \{x \in X : \exists n \in \omega(x \leq x_n)\}$. Then A is closed 0-segment with $0\text{-}\operatorname{cf} A = \omega$.

To see the other direction, assuming the existence a closed 0-segment A with $0\text{-}\operatorname{cf} A = \omega$, by induction, we can construct a 0-order preserving sequence with no cluster points. □

Using the lemma, we can see that a GO-space is countably compact if and only if it is both countably 0-compact and countably 1-compact, see also [5].

2. A simple case

In this section, we characterize countable 0-compactness of lexicographic products of two GO-spaces. The following is easy to prove, see also [7, Lemma 3.6 (3a)].

Lemma 2.1. *Let $X = X_0 \times X_1$ be a lexicographic product of two GO-spaces and A_0 a 0-segment of X_0 with $0\text{-}\operatorname{cf}_{X_0} A_0 \geq \omega$. Then $A = A_0 \times X_1$ is also a 0-segment of X with $0\text{-}\operatorname{cf}_X A = 0\text{-}\operatorname{cf}_{X_0} A_0$.*

The following lemma will be a useful tool for handling general cases.

Lemma 2.2. *Let $X = X_0 \times X_1$ be a lexicographic product of two GO-spaces. Then the following are equivalent:*

(1) *the product X is countably 0-compact;*
(2) *the following clauses hold:*
   (a) *the space X_0 is countably 0-compact;*
   (b) *the space X_1 is boundedly countably 0-compact;*
   (c) *if X_1 has no minimal element or $(u, \to)_{X_0}$ has no minimal element (that is, $1\text{-}\operatorname{cf}_{X_0}(u, \to) \neq 1$) for some $u \in X_0$, then $0\text{-}\operatorname{cf}_{X_1} X_1 \neq \omega$;*
   (d) *if X_1 has no minimal element, then $0\text{-}\operatorname{cf}_{X_0}(\leftarrow, u) \neq \omega$ for every $u \in X_0$.*

Proof: Set $\widehat{X} = X_0^* \times X_1^*$.

(1) $\Rightarrow$ (2) Let X be countably 0-compact.

(a) Assuming that X_0 is not countably 0-compact, take a closed 0-segment A_0 of X_0 with $0\text{-}\mathrm{cf}_{X_0} A_0 = \omega$. By the lemma above, $A = A_0 \times X_1$ is a 0-segment of X with $0\text{-}\mathrm{cf}_X A = \omega$. It suffices to see that A is closed, which contradicts countable 0-compactness of X. So let $x \notin A$, then $x(0) \notin A_0$. Since A_0 is closed in X_0, there is $u^* \in X_0^*$ such that $u^* <_{X_0^*} x(0)$ and $((u^*, \rightarrow)_{X_0^*} \cap X_0) \cap A_0 = \emptyset$ (this means $(u^*, x(0))_{X_0^*} = \emptyset$). Fix $w \in X_1$ and let $x^* = \langle u^*, w\rangle \in \widehat{X}$. Let $U = (x^*, \rightarrow)_{\widehat{X}} \cap X$, then U is a neighborhood of x. To see $U \cap A = \emptyset$, assume $a \in U \cap A$ for some a. By $a(0) \in A_0$, we can take $u \in A_0$ with $a(0) < u$. Now $u^* \le a(0) < u$ shows $u \in ((u^*, \rightarrow) \cap X_0) \cap A_0$, a contradiction.

(b) Assuming that X_1 is not boundedly countably 0-compact, take a bounded closed 0-segment A_1 of X_1 with $0\text{-}\mathrm{cf}_{X_1} A_1 = \omega$. Fix $u \in X_0$ and let $A = \{x \in X\colon \exists\, v \in A_1 (x \le_X \langle u, v\rangle)\}$. Obviously A is a 0-segment of X and $\{u\} \times A_1$ is unbounded in the 0-segment A, so we see $0\text{-}\mathrm{cf}_X A = 0\text{-}\mathrm{cf}_{X_1} A_1 = \omega$. It suffices to see that A is closed, so let $x \in X \setminus A$. Note $u \le x(0)$. Since A_1 is bounded, fix $v \in X_1 \setminus A_1$ and let $y = \langle u, v\rangle$. When $y < x$, $U = (y, \rightarrow)_X$ is a neighborhood of x disjoint from A. So let $x \le y$, then we have $x(0) = u$ and $x(1) \notin A_1$. Since A_1 is closed in X_1, take $v^* \in X_1^*$ such that $v^* < x(1)$ and $((v^*, \rightarrow) \cap X_1) \cap A_1 = \emptyset$. Then $U = (\langle u, v^*\rangle, \rightarrow)_{\widehat{X}} \cap X$ is a neighborhood of x disjoint from A.

(c) First assume that X_1 has no minimal element. Fix $u \in X_0$. Then $A = (\leftarrow, u] \times X_1$ is a closed 0-segment of X and $\{u\} \times X_1$ is unbounded in the 0-segment A, therefore $0\text{-}\mathrm{cf}_{X_1} X_1 = 0\text{-}\mathrm{cf}_X A \neq \omega$.

Next assume that $(u, \rightarrow)_{X_0}$ has no minimal element. Then putting $A = (\leftarrow, u] \times X_1$, similarly we see $0\text{-}\mathrm{cf}_{X_1} X_1 \neq \omega$.

(d) Assuming that X_1 has no minimal element and $0\text{-}\mathrm{cf}_{X_0}(\leftarrow, u) = \omega$ for some $u \in X_0$, let $A = (\leftarrow, u) \times X_1$. Then A is a closed 0-segment of X with $0\text{-}\mathrm{cf}_X A = 0\text{-}\mathrm{cf}_{X_0}(\leftarrow, u)$ by Lemma 2.1. This contradicts countable 0-compactness of X.

(2) $\Rightarrow$ (1) Assuming (2) and that X is not countably 0-compact, take a closed 0-segment A of X with $0\text{-}\mathrm{cf}_X A = \omega$. Let $A_0 = \{u \in X_0\colon \exists\, v \in X_1(\langle u, v\rangle \in A)\}$. Since A is a nonempty 0-segment of X, A_0 is also a nonempty 0-segment of X_0. We consider two cases, and in each cases, we will derive a contradiction.

Case 1. *The* 0*-segment* A_0 *has no maximal element, i.e.,* $0\text{-}\mathrm{cf}\, A_0 \ge \omega$.

In this case, we have:

Claim 1. *The equality* $A = A_0 \times X_1$ *holds.*

Proof: The inclusion "$\subset$" is obvious. Let $\langle u, v\rangle \in A_0 \times X_1$. Since $u \in A_0$ and A_0 has no maximal element, we can take $u' \in A_0$ with $u < u'$. By $u' \in A_0$, there is $v' \in X_1$ with $\langle u', v'\rangle \in A$. Then from $\langle u, v\rangle < \langle u', v'\rangle \in A$, we see $\langle u, v\rangle \in A$, because A is a 0-segment. $\square$

Lemma 2.1 shows $0\text{-}\mathrm{cf}\, A_0 = 0\text{-}\mathrm{cf}\, A = \omega$. The following claim contradicts the condition (2a).

Claim 2. *The* 0*-segment* A_0 *is closed in* X_0.

PROOF: Let $u \in X_0 \setminus A_0$. Whenever $u' < u$ for some $u' \in X_0 \setminus A_0$, $(u', \rightarrow)$ is a neighborhood of u disjoint from A_0. So assume the other case, that is, $u = \min(X_0 \setminus A_0)$. Note $A_0 = (\leftarrow, u)$. If X_1 has no minimal element, then by (2d), we have $0\text{-}\operatorname{cf}(\leftarrow, u) \neq \omega$, a contradiction. Thus X_1 has a minimal element, therefore $\langle u, \min X_1 \rangle = \min(X \setminus A) \notin A$. Since A is closed, there are $u^* \in X_0^*$ and $v^* \in X_1^*$ such that $\langle u^*, v^* \rangle < \langle u, \min X_1 \rangle$ and $((\langle u^*, v^* \rangle, \rightarrow)_{\widehat{X}} \cap X) \cap A = \emptyset$. The inequality $\langle u^*, v^* \rangle < \langle u, \min X_1 \rangle$ shows $u^* < u$, so $(u^*, \rightarrow) \cap X_0$ is a neighborhood of u disjoint from A_0. $\square$

Case 2. *The* 0*-segment* A_0 *has a maximal element.*

In this case, let $A_1 = \{v \in X_1 \colon \langle \max A_0, v \rangle \in A\}$. Then A_1 is a nonempty 0-segment of X_1. Since $\{\max A_0\} \times A_1$ is unbounded in the 0-segment A, we see $0\text{-}\operatorname{cf}_{X_1} A_1 = 0\text{-}\operatorname{cf}_X A = \omega$.

Claim 3. *The* 0*-segment* A_1 *is closed in* X_1.

PROOF: Let $v \in X_1 \setminus A_1$. Since $\langle \max A_0, v \rangle \notin A$ and A is closed, there are $u^* \in X_0^*$ and $v^* \in X_1^*$ such that $\langle u^*, v^* \rangle < \langle \max A_0, v \rangle$ and $((\langle u^*, v^* \rangle, \rightarrow)_{\widehat{X}} \cap X) \cap A = \emptyset$. It follows from $A_1 \neq \emptyset$ that $u^* = \max A_0$ and so $v^* < v$. Then we see that $(v^*, \rightarrow)_{X_1^*} \cap X_1$ is a neighborhood of v disjoint from A_1. $\square$

This claim with the condition (2b) shows $A_1 = X_1$, which says

$$A = (\leftarrow, \max A_0] \times X_1,$$

in particular, we see that X_1 has no maximal element.

Claim 4. *The interval* $(\max A_0, \rightarrow)$ *has no minimal element or* X_1 *has no minimal element.*

PROOF: Assume that $(\max A_0, \rightarrow)$ has a minimal element u_0 and X_1 has a minimal element, then note $\langle u_0, \min X_1 \rangle = \min(X \setminus A)$. Since A is closed in X, there are $u^* \in X_0^*$ and $v^* \in X_1^*$ such that $\langle u^*, v^* \rangle < \langle u_0, \min X_1 \rangle$ and $((\langle u^*, v^* \rangle, \rightarrow)_{\widehat{X}} \cap X) \cap A = \emptyset$. Then we have $u^* = \max A_0$. Since X_1 has no maximal element, pick $v \in X_1$ with $v^* < v$. Then we see $\langle \max A_0, v \rangle \in ((\langle u^*, v^* \rangle, \rightarrow)_{\widehat{X}} \cap X) \cap A$, a contradiction. $\square$

Now the condition (2c) shows $0\text{-}\operatorname{cf}_{X_1} X_1 \neq \omega$, a contradiction. This completes the proof of the lemma. $\square$

3. A general case

In this section, using the results in the previous section, we characterize the countable compactness of lexicographic products of any length of GO-spaces. We use the following notations.

Definition 3.1. Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Define:

$$J^+ = \{\alpha < \gamma\colon\ X_\alpha \text{ has no maximal element}\};$$
$$J^- = \{\alpha < \gamma\colon\ X_\alpha \text{ has no minimal element}\};$$
$$K^+ = \{\alpha < \gamma\colon\ \text{there is } x \in X_\alpha \text{ such that } (x, \rightarrow)_{X_\alpha} \text{ is nonempty and has no minimal element}\};$$
$$K^- = \{\alpha < \gamma\colon\ \text{there is } x \in X_\alpha \text{ such that } (\leftarrow, x)_{X_\alpha} \text{ is nonempty and has no maximal element}\};$$
$$L^+ = \left\{\alpha \le \gamma\colon\ \text{there is } u \in \prod_{\beta<\alpha} X_\beta \text{ with } 0\text{-}\mathrm{cf}_{\prod_{\beta<\alpha} X_\beta}(\leftarrow, u) = \omega\right\};$$
$$L^- = \left\{\alpha \le \gamma\colon\ \text{there is } u \in \prod_{\beta<\alpha} X_\beta \text{ with } 1\text{-}\mathrm{cf}_{\prod_{\beta<\alpha} X_\beta}(u, \rightarrow) = \omega\right\}.$$

For an ordinal α, let

$$l(\alpha) = \begin{cases} 0 & \text{if } \ \alpha < \omega, \\ \sup\{\beta \le \alpha\colon \beta \text{ is limit}\} & \text{if } \ \alpha \ge \omega. \end{cases}$$

Some of the definitions above are introduced in [7]. Note that $0 \notin L^+ \cup L^-$ and for an ordinal $\alpha \ge \omega$, $l(\alpha)$ is the largest limit ordinal less than or equal to α, therefore the half open interval $[l(\alpha), \alpha)$ of ordinals is finite.

We also remark:

Lemma 3.2. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If $\omega \le \gamma$, then $\omega \in L^+ \cap L^-$ holds.*

Proof: Assume $\omega \le \gamma$. For each $n \in \omega$, fix $u_0(n), u_1(n) \in X_n$ with $u_0(n) < u_1(n)$. Set $y = \langle u_1(n)\colon n \in \omega\rangle$. Moreover for each $n \in \omega$, set $y_n = \langle u_1(i)\colon i < n\rangle^\wedge\langle u_0(i)\colon n \le i\rangle$. Then $\{y_n\colon n \in \omega\}$ is a 0-order preserving unbounded sequence in $(\leftarrow, y)$ in $\prod_{n\in\omega} X_n$, therefore $\omega \in L^+$. The statement $\omega \in L^-$ is similar. □

Theorem 3.3. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then the following are equivalent:*

(1) *the product X is countably 0-compact;*
(2) *the following clauses hold:*
  (a) *space X_α is boundedly countably 0-compact for every $\alpha < \gamma$;*
  (b) *if $L^+ \ne \emptyset$, then $J^- \subset \min L^+$;*
  (c) *for every $\alpha < \gamma$, if any one of the following cases* (i)–(iii) *holds, then $0\text{-}\mathrm{cf}_{X_\alpha} X_\alpha \ne \omega$ holds:*
    (i) *$J^+ \cap [l(\alpha), \alpha) = \emptyset$;*
    (ii) *$J^+ \cap [l(\alpha), \alpha) \ne \emptyset$ and $(\alpha_0, \alpha] \cap J^- \ne \emptyset$, where $\alpha_0 = \max(J^+ \cap [l(\alpha), \alpha))$;*

(iii) $J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ *and* $[\alpha_0, \alpha) \cap K^+ \neq \emptyset$, *where* $\alpha_0 = \max(J^+ \cap [l(\alpha), \alpha))$.

Proof: Note that (2a)+(2ci) implies that X_0 is countably 0-compact. Let $\widehat{X} = \prod_{\alpha<\gamma} X^*_\alpha$.

(1) $\Rightarrow$ (2) Assume that X is countably 0-compact.

(a) Let $\alpha_0 < \gamma$. Since $X = \prod_{\alpha \leq \alpha_0} X_\alpha \times \prod_{\alpha_0 < \alpha} X_\alpha$, see [6, Lemma 1.5], and X is countably 0-compact, Lemma 2.2 shows that $\prod_{\alpha \leq \alpha_0} X_\alpha$ is countably 0-compact. Now by $\prod_{\alpha \leq \alpha_0} X_\alpha = \prod_{\alpha < \alpha_0} X_\alpha \times X_{\alpha_0}$ and Lemma 2.2 again, we see that X_{α_0} is boundedly countably 0-compact.

(b) Assume $L^+ \neq \emptyset$ and $\alpha_0 = \min L^+$. Then Lemma 3.2 shows $\alpha_0 \leq \omega$. From $\alpha_0 \in L^+$ one can take $u \in \prod_{\alpha < \alpha_0} X_\alpha$ such that $0\text{-}\operatorname{cf}_{\prod_{\alpha<\alpha_0} X_\alpha}(\leftarrow, u) = \omega$. Now since $X = \prod_{\alpha < \alpha_0} X_\alpha \times \prod_{\alpha_0 \leq \alpha} X_\alpha$ is countably 0-compact, Lemma 2.2 (d) shows that $\prod_{\alpha_0 \leq \alpha} X_\alpha$ has a minimal element. Therefore X_α has a minimal element for every $\alpha \geq \alpha_0$, which shows $J^- \subset \alpha_0$.

(c) Let $\alpha_0 < \gamma$. We will see $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$ in each case of (i), (ii) and (iii).

Case (i). *I.e.,* $J^+ \cap [l(\alpha_0), \alpha_0) = \emptyset$.

Since X is countably 0-compact and $X = \prod_{\alpha \leq \alpha_0} X_\alpha \times \prod_{\alpha_0 < \alpha} X_\alpha$, Lemma 2.2 shows that $\prod_{\alpha \leq \alpha_0} X_\alpha$ is also countably 0-compact. When $\alpha_0 = 0$, by countable 0-compactness of $\prod_{\alpha \leq \alpha_0} X_\alpha = X_{\alpha_0}$, we see $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$. So let $\alpha_0 > 0$. We divide into two cases.

Case (i)-1. $l(\alpha_0) = 0$, *i.e.,* $\alpha_0 < \omega$.

In this case, since $\prod_{\alpha<\alpha_0} X_\alpha$ has a maximal element, which implies that $\left(\max \prod_{\alpha<\alpha_0} X_\alpha, \rightarrow\right)$ has no minimal element, and $\prod_{\alpha<\alpha_0} X_\alpha \times X_{\alpha_0}$ is countably 0-compact, Lemma 2.2 (2c) shows $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$.

Case (i)-2. $l(\alpha_0) \geq \omega$, *i.e.,* $\alpha_0 \geq \omega$.

In this case, note that for every $\alpha \in [l(\alpha_0), \alpha_0)$, X_α has a maximal element. For every $\alpha < l(\alpha_0)$, fix $x_0(\alpha), x_1(\alpha) \in X_\alpha$ with $x_0(\alpha) < x_1(\alpha)$, and let $y = \langle x_0(\alpha) \colon \alpha < l(\alpha_0)\rangle^\wedge\langle \max X_\alpha \colon l(\alpha_0) \leq \alpha < \alpha_0\rangle$. Moreover for every $\beta < l(\alpha_0)$, let $y_\beta = \langle x_0(\alpha) \colon \alpha < \beta\rangle^\wedge\langle x_1(\alpha) \colon \beta \leq \alpha < l(\alpha_0))\rangle^\wedge\langle \max X_\alpha \colon l(\alpha_0) \leq \alpha < \alpha_0\rangle$. Then $\{y_\beta \colon \beta < l(\alpha_0)\}$ is 1-order preserving and unbounded in $(y, \rightarrow)$, in particular, the interval $(y, \rightarrow)$ in $\prod_{\alpha<\alpha_0} X_\alpha$ has no minimal element. Now Lemma 2.2 (2c) shows $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$.

Case (ii). *I.e.,* $J^+ \cap [l(\alpha_0), \alpha_0) \neq \emptyset$ *and* $(\alpha_1, \alpha_0] \cap J^- \neq \emptyset$, *where* $\alpha_1 = \max(J^+ \cap [l(\alpha_0), \alpha_0))$.

Note that α_1 is well-defined insomuch as $[l(\alpha_0), \alpha_0)$ is finite. Also let $\alpha_2 = \max((\alpha_1, \alpha_0] \cap J^-)$, then note $0 \leq l(\alpha_0) \leq \alpha_1 < \alpha_2 \leq \alpha_0$, in particular $[0, \alpha_2) \neq \emptyset$.

Case (ii)-1. $\alpha_2 = \alpha_0$.

Since $\prod_{\alpha<\alpha_0} X_\alpha \times X_{\alpha_0}$ $\left(= \prod_{\alpha\le\alpha_0} X_\alpha\right)$ is countably 0-compact, Lemma 2.2 (2c) shows $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$.

Case (ii)-2. $\alpha_2 < \alpha_0$.

Note that by the definition of α_2, X_α has a minimal element for every $\alpha \in (\alpha_2, \alpha_0]$. Fixing $z \in \prod_{\alpha<\alpha_2} X_\alpha$, let $y = z^\wedge\langle \max X_\alpha \colon \alpha_2 \le \alpha < \alpha_0\rangle$, then $y \in \prod_{\alpha<\alpha_0} X_\alpha$.

Claim 1. *$(y, \to)_{\prod_{\alpha<\alpha_0} X_\alpha}$ is nonempty and has no minimal element.*

Proof: Because X_{α_1} has no maximal element, fix $u \in X_{\alpha_1}$ with $y(\alpha_1) < u$. Then $(y \restriction \alpha_1)^\wedge\langle u\rangle^\wedge(y \restriction (\alpha_1, \alpha_0)) \in (y, \to)$, which shows $(y, \to) \neq \emptyset$. Next assume $y < y' \in \prod_{\alpha<\alpha_0} X_\alpha$. Since $y(\alpha) = \max X_\alpha$ for every $\alpha \in [\alpha_2, \alpha_0)$, we have $y \restriction \alpha_2 < y' \restriction \alpha_2$. Since X_{α_2} has no minimal element, fix $u \in X_{\alpha_2}$ with $u < y'(\alpha_2)$. Then we have $y < (y' \restriction \alpha_2)^\wedge\langle u\rangle^\wedge((y' \restriction (\alpha_2, \alpha_0)) < y'$, which shows that $(y, \to)$ has no minimal element. $\square$

Now because $\prod_{\alpha<\alpha_0} X_\alpha \times X_{\alpha_0}$ is countably 0-compact, Lemma 2.2 (2c) and the claim above shows $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$.

Case (iii). *I.e., $J^+ \cap [l(\alpha_0), \alpha_0) \neq \emptyset$ and $[\alpha_1, \alpha_0) \cap K^+ \neq \emptyset$, where $\alpha_1 = \max(J^+ \cap [l(\alpha_0), \alpha_0))$.*

Let $\alpha_2 = \max([\alpha_1, \alpha_0) \cap K^+)$, then note $l(\alpha_0) \le \alpha_1 \le \alpha_2 < \alpha_0$. Fixing $z \in \prod_{\alpha<\alpha_2} X_\alpha$ and $u \in X_{\alpha_2}$ satisfying that $(u, \to)$ is nonempty and has no minimal element, let $y = z^\wedge\langle u\rangle^\wedge\langle \max X_\alpha \colon \alpha_2 < \alpha < \alpha_0\rangle$. Then obviously $y \in \prod_{\alpha<\alpha_0} X_\alpha$ and $(y, \to)$ has no minimal element. Since $\prod_{\alpha<\alpha_0} X_\alpha \times X_{\alpha_0}$ is countable 0-compact, Lemma 2.2 (2c) shows $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$.

(2) $\Rightarrow$ (1) Assuming (2) and the negation of (1), take a closed 0-segment A of X with $0\text{-}\operatorname{cf}_X A = \omega$. Modifying the proof of Theorem 4.8 in [7], we consider 3 cases and their subcases. In each case, we will derive a contradiction.

Case 1. $A = X$.

In this case, since X has no maximal element, we have $J^+ \neq \emptyset$, so let $\alpha_0 = \min J^+$. Then $J^+ \cap [l(\alpha_0), \alpha_0) \subset J^+ \cap [0, \alpha_0) = \emptyset$ and the condition (2ci) shows $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \ge \omega_1$. Since $\{\langle \max X_\alpha \colon \alpha < \alpha_0\rangle\} \times X_{\alpha_0}$ is unbounded in $\prod_{\alpha\le\alpha_0} X_\alpha$, we have $0\text{-}\operatorname{cf}_{\prod_{\alpha\le\alpha_0} X_\alpha} \prod_{\alpha\le\alpha_0} X_\alpha = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \ge \omega_1$. Now by $X = \prod_{\alpha\le\alpha_0} X_\alpha \times \prod_{\alpha_0<\alpha} X_\alpha$, Lemma 2.1 shows $0\text{-}\operatorname{cf}_X A = 0\text{-}\operatorname{cf}_X X = 0\text{-}\operatorname{cf}_{\prod_{\alpha\le\alpha_0} X_\alpha} \prod_{\alpha\le\alpha_0} X_\alpha = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \ge \omega_1$, a contradiction.

Case 2. *$A \neq X$ and $X \setminus A$ has a minimal element.*

Let $B = X \setminus A$ and $b = \min B$. Since A is nonempty closed and $B = [b, \to)$, there is $b^* \in \widehat{X}$ with $b^* < b$ and $((b^*, \to)_{\widehat{X}} \cap X) \cap A = \emptyset$, equivalently $(b^*, b)_{\widehat{X}} = \emptyset$. Note $b^* \notin X$ because A has no maximal element. Let $\alpha_0 = \min\{\alpha < \gamma \colon b^*(\alpha) \neq b(\alpha)\}$.

Claim 2. *For every $\alpha > \alpha_0$, X_α has a minimal element and $b(\alpha) = \min X_\alpha$.*

Proof: Assuming $b(\alpha) > u$ for some $\alpha > \alpha_0$ and $u \in X_\alpha$, let $\alpha_1 = \min\{\alpha > \alpha_0 \colon \exists\, u \in X_\alpha(b(\alpha) > u)\}$ and fix $u \in X_{\alpha_1}$ with $b(\alpha_1) > u$. Then we have $b^* < (b \restriction \alpha_1)^\wedge\langle u\rangle^\wedge(b \restriction (\alpha_1, \gamma)) < b$, a contradiction. □

Claim 3. $(b^*(\alpha_0), b(\alpha_0))_{X^*_{\alpha_0}} \cap X_{\alpha_0} = \emptyset$.

Proof: Assume $u \in (b^*(\alpha_0), b(\alpha_0))_{X^*_{\alpha_0}} \cap X_{\alpha_0}$ for some u. Then we have $b^* < (b \restriction \alpha_0)^\wedge\langle u\rangle^\wedge(b \restriction (\alpha_0, \gamma)) < b$, a contradiction. □

Claim 4. $[b(\alpha_0), \rightarrow)_{X_{\alpha_0}} \notin \lambda_{X_{\alpha_0}}$, *therefore* $b^*(\alpha_0) \notin X_{\alpha_0}$.

Proof: It follows from $b^*(\alpha_0) \in (\leftarrow, b(\alpha_0))_{X^*_{\alpha_0}}$ that $(\leftarrow, b(\alpha_0))_{X_{\alpha_0}} \neq \emptyset$. Assume $[b(\alpha_0), \rightarrow)_{X_{\alpha_0}} \in \lambda_{X_{\alpha_0}}$, then for some $u \in X_{\alpha_0}$ with $u < b(\alpha_0)$, $(u, b(\alpha_0)) = \emptyset$ holds. Claim 3 shows $b^*(\alpha_0) = u \in X_{\alpha_0}$. If there were $\alpha > \alpha_0$ and $v \in X_\alpha$ with $v > b^*(\alpha)$, then by letting $\alpha_1 = \min\{\alpha > \alpha_0 \colon \exists\, v \in X_\alpha(v > b^*(\alpha))\}$ and taking $v \in X_{\alpha_1}$ with $v > b^*(\alpha_1)$, we have $b^* < (b^* \restriction \alpha_1)^\wedge\langle v\rangle^\wedge(b^* \restriction (\alpha_1, \gamma)) < b$, a contradiction. Therefore for every $\alpha > \alpha_0$, $\max X_\alpha$ exists and $b^*(\alpha) = \max X_\alpha$. Thus we have $b^* = (b \restriction \alpha_0)^\wedge\langle u\rangle^\wedge\langle \max X_\alpha \colon \alpha_0 < \alpha\rangle \in X$, a contradiction. □

Claims 3 and 4 show that $A_0 := (\leftarrow, b(\alpha_0))$ is a bounded closed 0-segment of X_{α_0} without a maximal element. Now the condition (2a) shows $0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0 \geq \omega_1$. Since $\{b \restriction \alpha_0\} \times A_0 \times \{b \restriction (\alpha_0, \gamma)\}$ is unbounded in the 0-segment in A $(= (\leftarrow, b)_X)$, we have $\omega = 0\text{-}\operatorname{cf}_X A = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0 \geq \omega_1$, a contradiction. This completes Case 2.

Case 3. *$A \neq X$ and $X \setminus A$ has no minimal element.*

Let $B = X \setminus A$ and

$$I = \{\alpha < \gamma \colon \exists\, a \in A \ \ \exists\, b \in B \ \ (a \restriction (\alpha + 1) = b \restriction (\alpha + 1))\}.$$

Obviously I is a 0-segment of γ, so $I = \alpha_0$ for some $\alpha_0 \leq \gamma$. For each $\alpha < \alpha_0$, fix $a_\alpha \in A$ and $b_\alpha \in B$ with $a_\alpha \restriction (\alpha + 1) = b_\alpha \restriction (\alpha + 1)$. By letting $Y_0 = \prod_{\alpha < \alpha_0} X_\alpha$ and $Y_1 = \prod_{\alpha_0 \leq \alpha} X_\alpha$, define $y_0 \in Y_0$ by $y_0(\alpha) = a_\alpha(\alpha)$ for every $\alpha < \alpha_0$. The ordinal α_0 can be 0, then in this case, $Y_0 = \{\emptyset\}$ and $y_0 = \emptyset$.

Claim 5. *For every $\alpha < \alpha_0$, $y_0 \restriction (\alpha + 1) = a_\alpha \restriction (\alpha + 1) = b_\alpha \restriction (\alpha + 1)$ holds.*

Proof: The second equality is obvious. To see the first equality, assuming $y_0 \restriction (\alpha + 1) \neq a_\alpha \restriction (\alpha + 1)$ for some $\alpha < \alpha_0$, let $\alpha_1 = \min\{\alpha < \alpha_0 \colon y_0 \restriction (\alpha + 1) \neq a_\alpha \restriction (\alpha + 1)\}$. Moreover let $\alpha_2 = \min\{\alpha \leq \alpha_1 \colon y_0(\alpha) \neq a_{\alpha_1}(\alpha)\}$. It follows from $y_0(\alpha_1) = a_{\alpha_1}(\alpha_1)$ that $\alpha_2 < \alpha_1$. Since $y_0 \restriction \alpha_2 = a_{\alpha_1} \restriction \alpha_2$ and $y_0(\alpha_2) \neq a_{\alpha_1}(\alpha_2)$ hold, by the minimality of α_1, we have $y_0 \restriction (\alpha_2 + 1) = a_{\alpha_2} \restriction (\alpha_2 + 1) = b_{\alpha_2} \restriction (\alpha_2 + 1)$. When $y_0(\alpha_2) < a_{\alpha_1}(\alpha_2)$, we have $B \ni b_{\alpha_2} < a_{\alpha_1} \in A$, a contradiction. When $y_0(\alpha_2) > a_{\alpha_1}(\alpha_2)$, we have $B \ni b_{\alpha_1} < a_{\alpha_2} \in A$, a contradiction. □

Claim 5 remains true when $\alpha_0 = 0$, because there is no ordinal α with $\alpha < \alpha_0$.

Claim 6. $\alpha_0 < \gamma$.

PROOF: Assume $\alpha_0 = \gamma$, then note $y_0 \in Y_0 = X = A \cup B$. Assume $y_0 \in A$. Since A has no maximal element, one can take $a \in A$ with $y_0 < a$. Letting $\beta_0 = \min\{\beta < \gamma \colon y_0(\beta) \neq a(\beta)\}$, we see $A \ni a > b_{\beta_0} \in B$, a contradiction. The remaining case is similar. $\square$

Let $A_0 = \{a(\alpha_0) \colon a \in A,\ a \restriction \alpha_0 = y_0\}$ and $B_0 = \{b(\alpha_0) \colon b \in B,\ b \restriction \alpha_0 = y_0\}$.

Claim 7. *The following hold:*

(1) *for every $a \in A$, $a \restriction \alpha_0 \leq y_0$ holds;*
(2) *for every $x \in X$, if $x \restriction \alpha_0 < y_0$, then $x \in A$.*

PROOF: (1) Assume $a \restriction \alpha_0 > y_0$ for some $a \in A$. Letting $\beta_0 = \min\{\beta < \alpha_0 \colon a(\beta) \neq y_0(\beta)\}$, we see $B \ni b_{\beta_0} < a \in A$, a contradiction.

(2) Assume $x \restriction \alpha_0 < y_0$. Letting $\beta_0 = \min\{\beta < \alpha_0 \colon x(\beta) \neq y_0(\beta)\}$, we see $x < a_{\beta_0} \in A$. Since A is a 0-segment, we have $x \in A$. $\square$

Similarly we have:

Claim 8. *The following hold:*

(1) *for every $b \in B$, $b \restriction \alpha_0 \geq y_0$ holds;*
(2) *for every $x \in X$, if $x \restriction \alpha_0 > y_0$, then $x \in B$.*

Claim 9. *A_0 is a 0-segment of X_{α_0} and $B_0 = X_{\alpha_0} \setminus A_0$.*

PROOF: To see that A_0 is a 0-segment, let $u' < u \in A_0$. Pick $a \in A$ with $a \restriction \alpha_0 = y_0$ and $u = a(\alpha_0)$. Let $a' = (a \restriction \alpha_0)^\wedge\langle u'\rangle^\wedge(a \restriction (\alpha_0, \gamma))$. Since A is a 0-segment and $a' < a \in A$, we have $a' \in A$. Now we see $u' = a'(\alpha_0) \in A_0$ because of $a' \restriction \alpha_0 = y_0$.

To see $B_0 = X_{\alpha_0} \setminus A_0$, first let $u \in B_0$. Take $b \in B$ with $b \restriction \alpha_0 = y_0$ and $b(\alpha_0) = u$. If $u \in A_0$ were true, then by taking $a \in A$ with $a \restriction \alpha_0 = y_0$ and $a(\alpha_0) = u$, we see $a \restriction (\alpha_0 + 1) = b \restriction (\alpha_0 + 1)$, therefore $\alpha_0 \in I = \alpha_0$, a contradiction. So we have $u \in X_{\alpha_0} \setminus A_0$. To see the remaining inclusion, let $u \in X_{\alpha_0} \setminus A_0$. Take $x \in X$ with $x \restriction (\alpha_0 + 1) = {y_0}^\wedge\langle u\rangle$. If $x \in A$ were true, then by $x \restriction \alpha_0 = y_0$, we have $u = x(\alpha_0) \in A_0$, a contradiction. So we have $x \in B$, therefore $u \in B_0$. $\square$

Claim 10. $A_0 \neq \emptyset$.

PROOF: Assume $A_0 = \emptyset$. We prove the following facts.

Fact 1. $(\leftarrow, y_0)_{Y_0} \times Y_1 = A$.

PROOF: One inclusion follows from Claim 7 (2). To see the other inclusion, let $a \in A$. Claim 7 (1) shows $a \restriction \alpha_0 \leq y_0$. If $a \restriction \alpha_0 = y_0$ were true, then we have $a(\alpha_0) \in A_0$, a contradiction. So we have $a \restriction \alpha_0 < y_0$ therefore $a \in (\leftarrow, y_0) \times Y_1$. $\square$

Fact 2. *$\alpha_0 > 0$ and α_0 is limit.*

PROOF: If $\alpha_0 = 0$ were true, then by taking $a \in A$, we have $a(\alpha_0) \in A_0$, a contradiction. Therefore we have $\alpha_0 > 0$. Next if $\alpha_0 = \beta_0 + 1$ were true for some ordinal β_0, then by $\beta_0 \in \alpha_0$ and Claim 5, we have $y_0 \restriction \alpha_0 = y_0 \restriction (\beta_0 + 1) = a_{\beta_0} \restriction (\beta_0 + 1) = a_{\beta_0} \restriction \alpha_0$, thus we have $a_{\beta_0}(\alpha_0) \in A_0$, a contradiction. Thus α_0 is limit. $\Box$

Now Claim 6 and Fact 2 show $\omega \leq \alpha_0 < \gamma$, so Lemma 3.2 shows $\omega \in L^+$. Moreover the condition (2b) shows $J^- \subset \min L^+ \leq \omega \leq \alpha_0$, in particular, X_α has a minimal element for every $\alpha \geq \alpha_0$. This means Y_1 $\left(= \prod_{\alpha_0 \leq \alpha} X_\alpha\right)$ has a minimal element. Now by Fact 1, we see $y_0{}^\wedge \min Y_1 = \min(X \setminus A)$, which contradicts our case. $\Box$

Next, let $Z_0 = \prod_{\alpha \leq \alpha_0} X_\alpha$, $Z_1 = \prod_{\alpha_0 < \alpha} X_\alpha$ and

$$A^* = \{z \in Z_0 \colon z \restriction \alpha_0 < y_0 \text{ or } (z \restriction \alpha_0 = y_0, z(\alpha_0) \in A_0)\}.$$

Note $A^* = ((\leftarrow, y_0) \times X_{\alpha_0}) \cup (\{y_0\} \times A_0)$.

Claim 11. *A^* is a 0-segment of Z_0 and $A = A^* \times Z_1$.*

PROOF: Since A_0 is a 0-segment of X_{α_0}, A^* is obviously a 0-segment of Z_0. To see $A \subset A^* \times Z_1$, let $a \in A$. Claim 7 (1) shows $a \restriction \alpha_0 \leq y_0$. When $a \restriction \alpha_0 < y_0$, obviously we have $a \restriction (\alpha_0 + 1) \in A^*$. When $a \restriction \alpha_0 = y_0$, $a \in A$ shows $a(\alpha_0) \in A_0$ thus $a \restriction (\alpha_0 + 1) \in A^*$. To see $A \supset A^* \times Z_1$, let $a \in A^* \times Z_1$. Then note $a \restriction (\alpha_0 + 1) \in A^*$. When $a \restriction \alpha_0 < y_0$, letting $\beta_0 = \min\{\beta < \alpha_0 \colon a(\beta) \neq y_0(\beta)\}$, we see $a < a_{\beta_0} \in A$ thus $a \in A$. When $a \restriction \alpha_0 = y_0$ and $a(\alpha_0) \in A_0$, Claim 9 shows $a \in A$. $\Box$

Since $\{y_0\} \times A_0$ is unbounded in the 0-segment A^*, we see $1 \leq 0\text{-}\operatorname{cf}_{Z_0} A^* = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0$. We divide Case 3 into two subcases.

Case 3-1. *$0\text{-}\operatorname{cf}_{Z_0} A^* \geq \omega$.*

In this case, Claim 11 and Lemma 2.1 show $\omega = 0\text{-}\operatorname{cf}_X A = 0\text{-}\operatorname{cf}_{Z_0} A^* = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0$.

Claim 12. *$A_0 \neq X_{\alpha_0}$.*

PROOF: Assume $A_0 = X_{\alpha_0}$. Then $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0 = \omega$ shows $\alpha_0 \in J^+$. Assume $\alpha_0 = \beta_0 + 1$ for some ordinal β_0. Then $\beta_0 < \alpha_0 = I$ shows $b_{\beta_0} \in B$. Now from $b_{\beta_0} \restriction \alpha_0 = b_{\beta_0} \restriction (\beta_0 + 1) = y_0 \restriction (\beta_0 + 1) = y_0 \restriction \alpha_0$, we have $b_{\beta_0}(\alpha_0) \in B_0 = X_{\alpha_0} \setminus A_0$, a contradiction. Thus we see that $\alpha_0 = 0$ or α_0 is limit, that is, $[l(\alpha_0), \alpha_0) = \emptyset$. Now the condition (2ci) shows $0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0} \neq \omega$, a contradiction. $\Box$

Claim 13. *A_0 is closed in X_{α_0}.*

Proof: When B_0 has no minimal element, obviously A_0 is closed. So assume that B_0 has a minimal element, say $u = \min B_0$. It suffices to find a neighborhood of u disjoint from A_0. The facts $A^* = (\leftarrow, y_0{}^\wedge\langle u\rangle)_{Z_0}$ and $0\text{-}\operatorname{cf}_{Z_0} A^* = \omega$ show $\alpha_0 + 1 \in L^+$, therefore $\min L^+ \leq \alpha_0 + 1$. The condition (2b) ensures $J^- \subset \min L^+ \leq \alpha_0 + 1$, so $J^- \subset [0, \alpha_0]$. Therefore X_α has a minimal element for every $\alpha > \alpha_0$. Let $b = y_0{}^\wedge\langle u\rangle{}^\wedge\langle \min X_\alpha \colon \alpha_0 < \alpha\rangle$. Since $b \in B$ $(= X \setminus A)$ and A is closed in X, there is $b^* \in \widehat{X}$ such that $b^* < b$ and $(b^*, b)_{\widehat{X}} \cap A = \emptyset$. Set $\beta_0 = \min\{\beta < \gamma \colon b^*(\beta) \neq b(\beta)\}$, then obviously $\beta_0 \leq \alpha_0$. If $\beta_0 < \alpha_0$ were true, we have $a_{\beta_0} \in (b^*, b)_{\widehat{X}} \cap A$, a contradiction. Thus we have $\beta_0 = \alpha_0$, so $b^* \restriction \alpha_0 = y_0$ and $b^*(\alpha_0) < u$. If there were $v \in (b^*(\alpha_0), \rightarrow)_{X^*_{\alpha_0}} \cap A_0$, then $v < u$ shows $y_0{}^\wedge\langle v\rangle{}^\wedge\langle \min X_\alpha \colon \alpha_0 < \alpha\rangle \in (b^*, b) \cap A$, a contradiction. Therefore $(b^*(\alpha_0), \rightarrow)_{X^*_{\alpha_0}} \cap X_{\alpha_0}$ is a neighborhood of u disjoint from A_0. $\square$

These claims above show that A_0 is a bounded closed 0-segment of X_{α_0}. Now the condition (2a) shows $0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0 \neq \omega$, a contradiction.

Case 3-1. $0\text{-}\operatorname{cf}_{Z_0} A^* = 1$.

Since $A = A^* \times Z_1$, A^* has a maximal element but A has no maximal element, we see that Z_1 has no maximal element. Therefore X_α has no maximal element for some $\alpha > \alpha_0$, in particular $(\alpha_0, \gamma) \neq \emptyset$. Let $\alpha_1 = \min\{\alpha > \alpha_0 \colon X_\alpha$ has no maximal element$\}$. Then we have $\alpha_0 < \alpha_1 \in J^+$ and $(\alpha_0, \alpha_1) \cap J^+ = \emptyset$. As $A = A^* \times Z_1 = A^* \times \left(\prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha \times \prod_{\alpha_1 < \alpha} X_\alpha\right) = \left(A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha\right) \times \prod_{\alpha_1 < \alpha} X_\alpha$ and $A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha$ is a 0-segment in $\prod_{\alpha \leq \alpha_1} X_\alpha$ with no maximal element, Lemma 2.1 shows $\omega = 0\text{-}\operatorname{cf}_X A = 0\text{-}\operatorname{cf}\left(A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha\right) = 0\text{-}\operatorname{cf}_{X_{\alpha_1}} X_{\alpha_1}$ (that $\{y_0{}^\wedge\langle \max A_0\rangle{}^\wedge\langle \max X_\alpha \colon \alpha_0 < \alpha < \alpha_1\rangle\} \times X_{\alpha_1}$ is unbounded in the 0-segment $A^* \times \prod_{\alpha_0 < \alpha \leq \alpha_1} X_\alpha$ witnesses the last equality).

Claim 14. *Let $l(\alpha_1) \leq \alpha_0$ and $J^+ \cap [l(\alpha_1), \alpha_0] \neq \emptyset$ hold, in particular $J^+ \cap [l(\alpha_1), \alpha_1) \neq \emptyset$.*

Proof: First assume $\alpha_0 < l(\alpha_1)$. Then $J^+ \cap [l(\alpha_1), \alpha_1) \subset J^+ \cap (\alpha_0, \alpha_1) = \emptyset$ and the condition (2ci) show $0\text{-}\operatorname{cf}_{X_{\alpha_1}} X_{\alpha_1} \neq \omega$, a contradiction. Thus we have $l(\alpha_1) \leq \alpha_0$.

Next assume $J^+ \cap [l(\alpha_1), \alpha_0] = \emptyset$, then we have $J^+ \cap [l(\alpha_1), \alpha_1) = \emptyset$ because of $J^+ \cap (\alpha_0, \alpha_1) = \emptyset$. Therefore the condition (2ci) shows $0\text{-}\operatorname{cf}_{X_{\alpha_1}} X_{\alpha_1} \neq \omega$, a contradiction. Thus $J^+ \cap [l(\alpha_1), \alpha_0] \neq \emptyset$. $\square$

Using the above claim, set $\alpha_2 = \max(J^+ \cap [l(\alpha_1), \alpha_1))$. Note $0 \leq l(\alpha_1) \leq \alpha_2 \leq \alpha_0 < \alpha_1$ and $J^+ \cap (\alpha_2, \alpha_1) = \emptyset$.

Claim 15. *B_0 has a minimal element.*

Proof: First we check $B_0 \neq \emptyset$, so assume $B_0 = \emptyset$, i.e., $A_0 = X_{\alpha_0}$. The equations $1 = 0\text{-}\operatorname{cf}_{Z_0} A^* = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} A_0 = 0\text{-}\operatorname{cf}_{X_{\alpha_0}} X_{\alpha_0}$ show $\alpha_0 \notin J^+$. Also $\alpha_2 \leq \alpha_0$ and $\alpha_2 \in J^+$ show $0 \leq \alpha_2 < \alpha_0$. Assume that $\alpha_0 = \beta_0 + 1$ for some ordinal β_0, then by $\beta_0 < \alpha_0 = I$, we have $b_{\beta_0} \in B$ and $b_{\beta_0} \restriction \alpha_0 = b_{\beta_0} \restriction (\beta_0 + 1) = y_0 \restriction (\beta_0 + 1) = y_0 \restriction \alpha_0$. Therefore we have $b_{\beta_0}(\alpha_0) \in B_0$, a contradiction. So we have $0 < \alpha_0$ and α_0 is limit, therefore $\alpha_0 \leq l(\alpha_1) \leq \alpha_2$, which contradicts $\alpha_2 < \alpha_0$. We have seen $B_0 \neq \emptyset$.

Next we check that B_0 has a minimal element. Assume that B_0 has no minimal element, then $\max A_0$ witnesses $\alpha_0 \in [\alpha_2, \alpha_1) \cap K^+$. The definition of α_2 and the condition (2ciii) show $0\text{-}\operatorname{cf}_{X_{\alpha_1}} X_{\alpha_1} \neq \omega$, a contradiction. □

Now since B has no minimal element, by the claim above, there is $\alpha > \alpha_0$ such that X_α has no minimal element. So let $\alpha_3 = \min\{\alpha > \alpha_0 \colon X_\alpha$ has no minimal element$\}$. Then we have $\alpha_0 < \alpha_3 \in J^-$. When $\omega \leq \gamma$, Lemma 3.2 and the condition (2b) show $J^- \subset \min L^+ \leq \omega$. When $\gamma < \omega$, obviously $J^- \subset \omega$. So in any case we have $J^- \subset \omega$. Therefore $l(\alpha_1) \leq \alpha_0 < \alpha_3 \in \omega$ so we have $\alpha_1 \in \omega$.

Claim 16. $\alpha_3 \leq \alpha_1$.

Proof: Assume $\alpha_1 < \alpha_3$, then X_α has a minimal element for every $\alpha \in (\alpha_0, \alpha_1]$. So let $y = y_0{}^\wedge\langle \min B_0\rangle{}^\wedge\langle \min X_\alpha \colon \alpha_0 < \alpha \leq \alpha_1\rangle$. Note $y \in \prod_{\alpha \leq \alpha_1} X_\alpha$ and consider the interval $(\leftarrow, y)$ in $\prod_{\alpha \leq \alpha_1} X_\alpha$. The definition of α_2 and $\alpha_2 \leq \alpha_0$ show that X_α has a maximal element for every $\alpha \in (\alpha_0, \alpha_1)$. Since $\{y_0{}^\wedge\langle \max A_0\rangle{}^\wedge\langle \max X_\alpha \colon \alpha_0 < \alpha < \alpha_1\rangle\} \times X_{\alpha_1}$ is unbounded in $(\leftarrow, y)$, we have $0\text{-}\operatorname{cf}(\leftarrow, y) = 0\text{-}\operatorname{cf}_{X_{\alpha_1}} X_{\alpha_1} = \omega$. Thus y witnesses $\alpha_1 + 1 \in L^+$. The condition (2b) ensures $J^- \subset \min L^+ \leq \alpha_1 + 1$, thus $\alpha_3 \in J^- \subset [0, \alpha_1]$, a contradiction. Now we have $\alpha_3 \leq \alpha_1$. □

Now $\alpha_3 \in (\alpha_0, \alpha_1] \cap J^- \subset (\alpha_2, \alpha_1] \cap J^-$, $\alpha_2 = \max(J^+ \cap [l(\alpha_1), \alpha_1))$ and the condition (2cii) show $0\text{-}\operatorname{cf}_{X_{\alpha_1}} X_{\alpha_1} \neq \omega$, a contradiction. This completes the proof of the theorem. □

Analogously we can see:

Theorem 3.4. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then the following are equivalent:*

(1) *the product X is countably 1-compact;*
(2) *the following clauses hold:*
  (a) *X_α is boundedly countably 1-compact for every $\alpha < \gamma$;*
  (b) *if $L^- \neq \emptyset$, then $J^+ \subset \min L^-$;*
  (c) *for every $\alpha < \gamma$, if any one of the following cases bellow holds; then $1\text{-}\operatorname{cf}_{X_\alpha} X_\alpha \neq \omega$ holds;*
    (i) *$J^- \cap [l(\alpha), \alpha) = \emptyset$;*
    (ii) *$J^- \cap [l(\alpha), \alpha) \neq \emptyset$ and $(\alpha_0, \alpha] \cap J^+ \neq \emptyset$, where $\alpha_0 = \max(J^- \cap [l(\alpha), \alpha))$;*
    (iii) *$J^- \cap [l(\alpha), \alpha) \neq \emptyset$ and $[\alpha_0, \alpha) \cap K^- \neq \emptyset$, where $\alpha_0 = \max(J^- \cap [l(\alpha), \alpha))$.*

4. Applications

In this section, we apply the theorems in the previous section

Corollary 4.1. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. Then the following hold:*

(1) *if X is countably 0-compact, then $J^- \subset \omega$;*
(2) *if X is countably 1-compact, then $J^+ \subset \omega$;*
(3) *if X is countably 0-compact, then for every $\delta < \gamma$, the lexicographic product $\prod_{\alpha<\delta} X_\alpha$ is countably 0-compact, in particular X_0 is countably 0-compact;*
(4) *if X is countably 1-compact, then for every $\delta < \gamma$, the lexicographic product $\prod_{\alpha<\delta} X_\alpha$ is countably 1-compact, in particular X_0 is countably 1-compact.*

Proof: Lemma 3.2 and the condition (2b) in Theorem 3.3 show (1). (3) obviously follows from Theorem 3.3 or Lemma 2.2 directly. The remaining is similar. □

Corollary 4.2. *Let X be a GO-space. Then the lexicographic product $X^{\omega+1}$ is countably compact if and only if X is countably compact and has both a minimal and a maximal element.*

Proof: That $X^{\omega+1}$ is countably compact implies that X is countably compact and has both a minimal and a maximal element follows from the corollary above. The other implication follows from the theorems in the previous section because of $J^+ = J^- = \emptyset$. □

Corollary 4.3. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of countably compact GO-spaces. Then the following are equivalent:*

(1) *the product X is countably compact;*
(2) *the following clauses hold:*
    (a) *if $L^+ \neq \emptyset$, then $J^- \subset \min L^+$;*
    (b) *if $L^- \neq \emptyset$, then $J^+ \subset \min L^-$.*

Proof: Since all X_α's are countably compact, (2a)+(2c) in Theorems 3.3 and 3.4 of the previous section are true. □

Example 4.4. Let $[0,1)_{\mathbb{R}}$ denote the unit half open interval in the real line $\mathbb{R}$ with the usual order. Let X be the lexicographic product $[0,1)_{\mathbb{R}} \times \omega_1$. Since $[0,1)_{\mathbb{R}}$ is not countably 0-compact, Corollary 4.1 shows that X is not countably 0-compact. Both $[0,1)_{\mathbb{R}}$ and ω_1 are countably 1-compact. Considering $X_0 = [0,1)_{\mathbb{R}}$ and $X_1 = \omega_1$, we see $1 \in L^-$ (0 in $[0,1)_{\mathbb{R}}$ witnesses this) therefore $1 = \min L^-$. Moreover by $1 \in J^+$, (2b) in Theorem 3.4 does not hold. Therefore X is neither countably 0-compact nor countably 1-compact. Note that X is not paracompact, see [7, Example 4.6].

Example 4.5. Let X be the lexicographic product $\omega_1 \times [0,1)_{\mathbb{R}}$. Checking all clauses in the theorems in the previous section, we see that X is countably compact. Since it is not compact, it is not paracompact. The lexicographic product $\omega_1 \times [0,1)_{\mathbb{R}}$ is called the long line of length ω_1 and denoted by $\mathbb{L}(\omega_1)$.

Example 4.6. Let $\mathbb{S}$ be the Sorgenfrey line, where half open intervals $[a,b)_{\mathbb{R}}$'s are declared to be open. Then it is known that $\omega_1 \times \mathbb{S}$ is paracompact but $\mathbb{S} \times \omega_1$ is not paracompact, see [7]. On the other hand, both lexicographic products $\omega_1 \times \mathbb{S}$ and $\mathbb{S} \times \omega_1$ are not countably compact, because $\mathbb{S}$ is not boundedly countably 0-compact.

Example 4.7. Let X be the lexicographic product $\omega_1 \times [0,1)_{\mathbb{R}} \times \omega_1$, and consider as $X_0 = \omega_1$, $X_1 = [0,1)_{\mathbb{R}}$ and $X_2 = \omega_1$. Then $1\text{-}\operatorname{cf}_{\omega_1 \times [0,1)_{\mathbb{R}}}(\langle 0,0\rangle, \rightarrow) = \omega$ shows $2 \in L^-$. Since $0, 1 \notin L^-$, we have $\min L^- = 2$. Now $2 \in J^+$ implies $J^+ \not\subset \min L^-$. Thus Theorem 3.4 shows that X is not countably (1-)compact. On the other hand, we will later see that the lexicographic product $\omega_1 \times \omega \times \omega_1$ is countably compact.

Corollary 4.8. *There is a countably compact LOTS X whose lexicographic square X^2 is not countably compact.*

Proof: $X = \mathbb{L}(\omega_1)$ is such an example, because $\mathbb{L}(\omega_1)^2 = (\omega_1 \times [0,1)_{\mathbb{R}} \times \omega_1) \times [0,1)_{\mathbb{R}}$ (use Example 4.7). We will later see that the lexicographic product $X = \omega_1^{\omega}$ is also such an example. $\square$

In the rest of the paper, we consider countable compactness of lexicographic products whose all factors have minimal elements. In the following, apply theorems with $J^- = \emptyset$.

Corollary 4.9. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If all X_α's have minimal elements, then the following are equivalent:*

(1) *the product X is countably 0-compact;*
(2) *the following clauses hold:*
  (a) *X_α is boundedly countably 0-compact for every $\alpha < \gamma$;*
  (b) *for every $\alpha < \gamma$, if either one of the following cases holds, then $0\text{-}\operatorname{cf}_{X_\alpha} X_\alpha \neq \omega$ holds:*
    (i) *$J^+ \cap [l(\alpha), \alpha) = \emptyset$;*
    (ii) *$J^+ \cap [l(\alpha), \alpha) \neq \emptyset$ and $[\alpha_0, \alpha) \cap K^+ \neq \emptyset$, where $\alpha_0 = \max(J^+ \cap [l(\alpha), \alpha))$.*

Corollary 4.10. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of GO-spaces. If all X_α's have minimal elements, then the following are equivalent:*

(1) *the product X is countably 1-compact;*
(2) *the following clauses hold:*
  (a) *X_α is (boundedly) countably 1-compact for every $\alpha < \gamma$;*
  (b) *if $L^- \neq \emptyset$, then $J^+ \subset \min L^-$.*

Now we consider the case that all factors are subspaces of ordinals. First let X be a subspace of an ordinal. Since X is well-ordered, the following hold:

- the GO-space X is countably 1-compact;
- the GO-space X has a minimal element;
- for every $u \in X$ with $(u, \rightarrow) \neq \emptyset$, $(u, \rightarrow)$ has a minimal element;
- there is $u \in X$ such that $(\leftarrow, u)$ is nonempty and has no maximal element if and only if the order type of X is greater than ω.

Note that a subspace X of ω_1 is countably compact if and only if it is closed in ω_1, and also note that the subspace $X = \{\alpha < \omega_2 \colon \operatorname{cf} \alpha \leq \omega\}$ is countably compact but not closed in ω_2.

Next let X_α be a subspace of an ordinal for every $\alpha < \gamma$ and $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product. Then using the notation in Section 3, we see:

- $J^- = \emptyset$;
- $K^+ = \emptyset$;
- $\alpha \in K^-$ if and only if the order type of X_α is greater than ω.

Remarking these facts with corollaries above, we see:

Corollary 4.11. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product. If all X_α's are subspaces of ordinals, then the following are equivalent:*

(1) *the product X is countably 0-compact;*
(2) *the following clauses hold:*
  (a) *X_α is boundedly countably 0-compact for every $\alpha < \gamma$;*
  (b) *for every $\alpha < \gamma$ with $J^+ \cap [l(\alpha), \alpha) = \emptyset$, $0\text{-}\operatorname{cf}_{X_\alpha} X_\alpha \neq \omega$ holds.*

Corollary 4.12. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product. If all X_α's are subspaces of ordinals, then the following are equivalent:*

(1) *the product X is countably 1-compact;*
(2) *$J^+ \subset \omega$.*

Proof: (1) $\Rightarrow$ (2) Assume that X is countably 1-compact. By Corollary 4.10, if $L^- \neq \emptyset$, then $J^+ \subset \min L^-$. When $\gamma \geq \omega$, because of $\omega \in L^-$, we see $J^+ \subset \min L^- \leq \omega$. When $\gamma < \omega$, obviously we see $J^+ \subset \gamma < \omega$.

(2) $\Rightarrow$ (1) Assume $J^+ \subset \omega$. It suffices to check (2a) and (2b) in Corollary 4.10. (2a) is obvious. To see (2b), let $L^- \neq \emptyset$. Now assume $\omega \cap L^- \neq \emptyset$, and take $n \in \omega \cap L^-$. Then we can take $u \in \prod_{m<n} X_m$ with $1\text{-}\operatorname{cf}(u, \rightarrow) = \omega$. But this is a contradiction, because a lexicographic product of finite length of subspaces of ordinals are also a subspace of ordinal, see [7, Lemma 4.3]. Therefore we have $\omega \cap L^- = \emptyset$. $L^- \neq \emptyset$ and Lemma 3.2 show $J^+ \subset \omega = \min L^-$. □

If X is an ordinal, then it is boundedly countably 0-compact and $0\text{-}\operatorname{cf}_X X = \operatorname{cf} X$. Therefore we have:

Corollary 4.13. *Let $X = \prod_{\alpha<\gamma} X_\alpha$ be a lexicographic product of ordinals. Then the following are equivalent:*

(1) *the product X is countably compact;*
(2) *the following clauses hold:*
  (a) *if $J^+ \neq \emptyset$, then* $\mathrm{cf}\, X_{\min J^+} \geq \omega_1$*;*
  (b) $J^+ \subset \omega$.

Corollary 4.14 ([4])**.** *The following clauses hold:*

(1) *the lexicographic product ω_1^γ is countably* 0*-compact for every ordinal γ;*
(2) *the lexicographic product ω_1^γ is countably* (1-)*compact if and only if $\gamma \leq \omega$.*

Example 4.15. Using Corollary 4.13, we see:

(1) lexicographic products ω_1^2, $\omega_1 \times \omega$, $(\omega+1)\times(\omega_1+1)\times\omega_1\times\omega$, $\omega_1\times\omega\times\omega_1$, $\omega_1\times\omega\times\omega_1\times\omega\times\cdots$, $\omega_1\times\omega^\omega$, $\omega_1\times\omega^\omega\times(\omega+1)$, ω_1^ω, $\omega_1^\omega\times(\omega_1+1)$ and $\prod_{n\in\omega}\omega_{n+1}$ are countably compact;
(2) lexicographic products $\omega\times\omega_1$, $(\omega+1)\times(\omega_1+1)\times\omega\times\omega_1$, $\omega\times\omega_1\times\omega\times\omega_1\times\cdots$, $\omega\times\omega_1^\omega$, $\omega_1\times\omega^\omega\times\omega_1$, $\omega_1^\omega\times\omega$, $\prod_{n\in\omega}\omega_n$ and $\prod_{n\leq\omega}\omega_{n+1}$ are not countably compact;
(3) let $X=\omega_1^\omega$, then the lexicographic product X^2 is not countably compact because of $X^2=\omega_1^\omega\times\omega_1^\omega=\omega_1^{\omega+\omega}$, so this shows also Corollary 4.8.

For a GO-space $X=\langle X,<_X,\tau_X\rangle$, $-X$ denotes the reverse of X, that is, the GO-space $\langle X,>_X,\tau_X\rangle$, see [7]. Note that X and $-X$ are topologically homeomorphic.

Example 4.16. As above, the lexicographic product ω_1^2 was countably compact. But the lexicographic product $\omega_1\times(-\omega_1)$ is not countably compact. Indeed, let $X=\omega_1\times(-\omega_1)$, $X_0=\omega_1$ and $X_1=-\omega_1$. The element $\omega\in X_0$ with $0\text{-}\mathrm{cf}_{X_0}(\leftarrow,\omega)=\mathrm{cf}\,\omega=\omega$ witnesses $1\in L^+$, therefore $\min L^+=1$. On the other hand $-\omega_1$ has no minimal element, so we have $1\in J^-$. Therefore (2b) of Theorem 3.3 does not hold, thus X is not countably (0-)compact.

Also note that $(-\omega_1)\times(-\omega_1)$ is countably compact but $(-\omega_1)\times\omega_1$ is not countably compact, because $(-\omega_1)\times(-\omega_1)$ and $(-\omega_1)\times\omega_1$ are topologically homeomorphic to ω_1^2 and $\omega_1\times(-\omega_1)$, respectively, see [7].

Moreover $\omega_1\times(-\omega)$ is directly shown not to be countably (1-)compact, because the 1-order preserving sequence $\{\langle 0,n\rangle : n\in\omega\}$ has no cluster point in $\omega_1\times(-\omega)$.

Acknowledgment. The author thanks the reviewer for careful reading the manuscript and for giving useful comments.

References

[1] Engelking R., *General Topology*, Sigma Series in Pure Mathematics, 6, Herdermann Verlag, Berlin, 1989.
[2] Faber M. J., *Metrizability in Generalized Ordered Spaces*, Mathematical Centre Tracts, 53, Mathematisch Centrum, Amsterdam, 1974.
[3] Kemoto N., *Normality of products of GO-spaces and cardinals*, Topology Proc. **18** (1993), 133–142.

[4] Kemoto N., *The lexicographic ordered products and the usual Tychonoff products*, Topology Appl. **162** (2014), 20–33.
[5] Kemoto N., *Normality, orthocompactness and countable paracompactness of products of GO-spaces*, Topology Appl. **231** (2017), 276–291.
[6] Kemoto N., *Lexicographic products of GO-spaces*, Topology Appl. **232** (2017), 267–280.
[7] Kemoto N., *Paracompactness of lexicographic products of GO-spaces*, Topology Appl. **240** (2018), 35–58.
[8] Kemoto N., *The structure of the linearly ordered compactifications of GO-spaces*, Topology Proc. **52** (2018), 189–204.
[9] Kunen K., *Set Theory. An Introduction to Independence Proofs*, Studies in Logic and the Foundations of Mathematics, 102, North-Holland Publishing, Amsterdam, 1980.
[10] Lutzer D. J., *On generalized ordered spaces*, Dissertationes Math. Rozprawy Mat. **89** (1971), 32 pages.
[11] Miwa T., Kemoto N., *Linearly ordered extensions of GO-spaces*, Topology Appl. **54** (1993), no. 1–3, 133–140.

N. Kemoto:
Department of Mathematics, Oita University, 700 Dannoharu, Oita, 870-1192, Japan

E-mail: nkemoto@cc.oita-u.ac.jp

(*Received* July 23, 2018, *revised* December 1, 2018)